April 20, 2016

1 Fourier Analysis

There is a result in the theory of differential equations which tells us that we can construct the solution to an ODE/PDE out of a set of orthogonal (or even orthonormal) basis functions. One example of a set of basis functions are the oscillating exponential functions $\psi_n(x) = e^{ik_n x}$ (maybe a more familiar set of basis functions is the set of functions $\{x^n\}$, but they are not orthogonal). Alternatively, we could just as well use the sin and cos functions. The words 'basis function' means that for a given function $f(x)$, we can write it as a summation over these 'basis functions'

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} = \sum_{n=0}^{\infty} a_n \cos k_n x + b_n \sin k_n x$$

where a_n, b_n, c_n are coefficients, sometimes called the fourier coefficients, which depend only on their label n and not on the variable x. Fourier series like this work well for functions which are smooth and period. Periodic means $f(x + nL) = f(x)$ for some period L with $n = \pm 1, \pm 2, ...$ such as the functions sin or cos for example, which are periodic in 2π. Smooth means that the function is not too spiky.

If the function is to be periodic, then our basis functions should exhibit the same periodicity. So for a particular ψ_n, this means we require

$$e^{ik_n x} = e^{ik_n (x+L)} = e^{ik_n x} e^{ik_n L} \implies e^{ik_n L} = 1$$

so we find that k_n must be a multiple of $2\pi/L$, so in otherwords we have that $k_n = 2\pi n/L$. Now we can discuss the orthogonality property: I claim that

$$\int_0^L dx\ \psi_n^*(x)\psi_m(x) = L\delta_{nm}$$

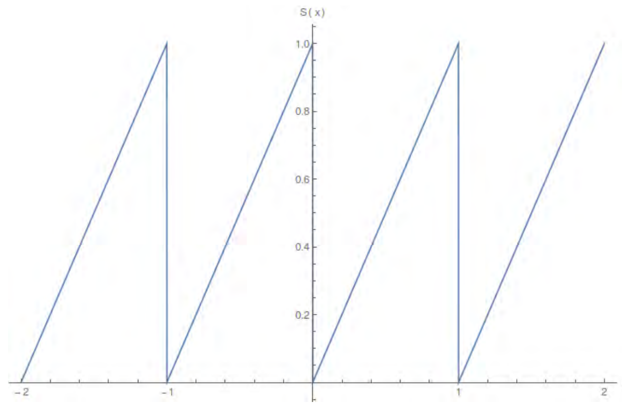
When $n = m$, $\psi^*\psi = 1$ so we are just integrating 1 from 0 to L, which gives L. I leave it to you to check that when $n \neq m$, the answer is zero for all combinations of integers n and m. This is great, because it means we can take our function $f(x)$ as defined above and integrate it against $e^{-ik_n x}$ from 0 to L to extract the fourier coefficient c_n:

$$\int_0^L e^{-ik_n x} f(x) = \int_0^L e^{-ik_n x} \sum_j c_j e^{ik_j x} = \sum_j L c_j \delta_{jn} = L c_n$$

so more directly the fourier coefficient c_n for some function $f(x)$ is given by

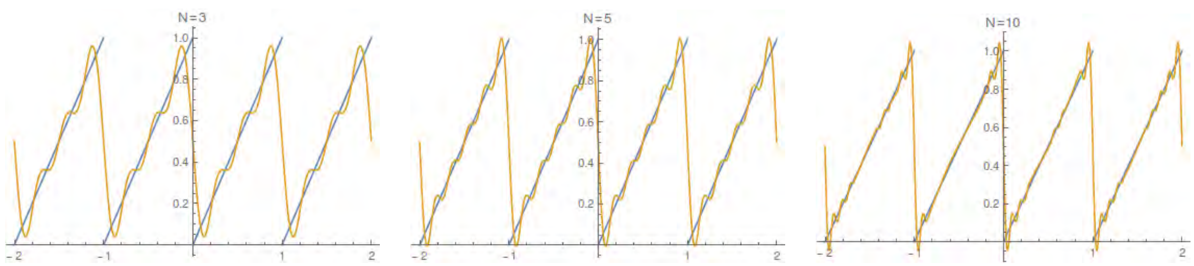
$$c_n = \frac{1}{L} \int_0^L \psi_n^*(x) f(x)$$

A couple of examples will be helpful. First, consider the sawtooth function $S(x) = x$ for $0 < x < 1$, and $S(x) = f(x-n)$ where $0 < x - n < 1$, plotted at right. We want to represent this function as a fourier series, so we need the coefficients c_n. We obtain them using the formula above (with $L = 1$)

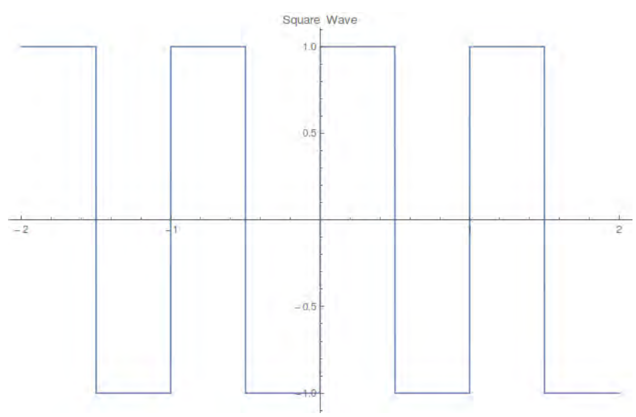


$$c_n = \int_0^1 e^{-2\pi i n x} S(x) = \frac{i}{2\pi n}$$

EXCEPT when $n = 0$ in which case the result is $c_0 = 1/2$. If we evaluated the sum from minus infinity to infinity, we would recover the exact function. Often we cannot do this though, but a partial summation suffices as an approximation. Plotted below are the results from doing the sum from $-N$ to N with $N = 3, 5, 10$, overlayed on the original function itself.



We can see that as N increases, the approximation becomes better. However there are still some sharp wiggles in the function: the sharp cusps are hard to approximate in this way, and the resulting wiggles, or 'ringing,' are known as a Gibbs phenomenon. This is a problem which one needs to confront in e.g. signal processing. Exercise for the reader: find the fourier series representation for the square wave, which switches between $+1$ and -1, shown at right. Another canonical example is the triangle wave, which I will not show here but you can easily find if interested.

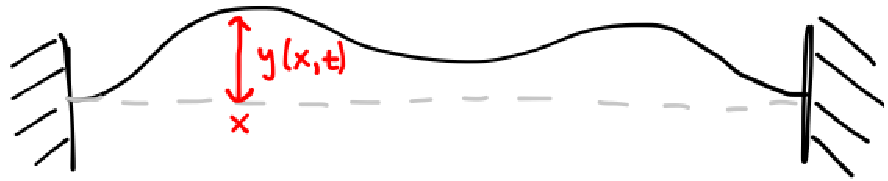


At this point it may be useful to explain what the Fourier transform is in words. The Fourier transformed function $\tilde{f}(k)$ tells you how much of *frequency* k there is in the function $f(x)$. Remember the frequency is the inverse of the period. So it's reasonable that any periodic function can be written as the sum of other functions which have definite frequencies. Then the fourier transform tells you how "much" a particular frequency is represented in the function. What's not so obvious is that the same thing can be done for functions which are not periodic (or periodic on an infinite domain).

What is the fourier representation good for? It allows us to solve partial differential equations, among other things. Consider the 1-dimensional wave equation

$$\partial_t^2 f(x,t) = c^2 \partial_x^2 f(x,t)$$

which could describe the height of a string at position x as a function of time t.



We can substitute in the fourier series representation for $f(x,t) = \sum_n f_n(t) e^{ik_n x}$ where $f_n(t)$ is the fourier coefficient of mode n, which now also depends on time. Then we can see that

$$\partial_x f(x,t) = \partial_x \sum f_n(t) e^{ik_n x} = \sum f_n(t)(ik_n) e^{ik_n x}$$

so acting with derivatives just pulls down factors of ik_n. Then when we use the orthogonality trick to isolate the coefficient of the $n'th$ mode, we will be left with an algebraic equation. This is (one of) the utility of the fourier series; it turns differential equations into algebraic ones. Explicitly then, we can reduce our PDE for $f(x,t)$ into an ODE for the coefficient $f_n(t)$:

$$\ddot{f}_n(t) = -c^2 k_n^2 f_n(t) = -\omega_n^2 f_n(t)$$

You may recognize the solutions to this equation as sines and cosines, or complex exponentials $e^{\pm i\omega t}$. We also need some boundary conditions in this case to specify the solution: suppose that the string we are describing is fixed at positions $x = 0$ and $x = L$. The b.c. at $x = 0$ implies $f_n(t) = -f_{-n}(t)$ so we can add together the $\pm n$ fourier modes to obtain a sin function. The boundary condition at $x = L$ then determines $k = n\pi/L$.

Suppose we also have some initial condition $f(x, t = 0) = f_0(x)$. Now we have a representation for our function

$$f(x,t) = \sum (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin(k_n x)$$

where $\omega_n = ck_n$. At $t = 0$, we have

$$f(x, t = 0) = \sum b_n \sin(k_n x)$$

which is supposed to be equal to $f_0(x)$. We can once again use the orthogonality of the sin functions:

$$\int_0^L dx \ \sin(k_n x)\sin(k_m x) = \frac{L}{2}\delta_{nm}$$

to extract the coefficient a_n:

$$a_n = \frac{2}{L}\int_0^L dx \ f_0(x)\sin\left(\frac{2\pi n x}{L}\right)$$

(notice that the factor in front of the integral here is $2/L$ rather than $1/L$, why is that?). To determine the coefficient b_n, we need a condition on the derivative of $f(x,t)$ at $t = 0$, we can see that this will eliminate the cosine term and leave behind the coefficient b_n to be solved for. If our initial condition on the derivative is $\dot{f}(0) = 0$, then this determines $b_n = 0$ for all n. Otherwise, we again use orthogonality of the sin functions to determine the coefficients b_n.

There is a generalization as the domain of the function becomes infinite, which is that we replace the sums by integrals. So we write the fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \ f(x)e^{-ikx} \qquad\qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}$$

where the factor of $(2\pi)^{-1}$ in front of the second integral is a matter of convention and the orthogonality of the complex exponential function in this setting is realized as

$$\int e^{-inx}e^{imx} = 2\pi\delta_{nm}$$

(notice the opposite signs in the exponentials of the fourier transform definitions).

When we study quantum mechanics, we will find an equation that is similar to the wave equation, called the Schrodinger equation. Roughly speaking, we will have an equation

$$\partial_t f \sim \hat{H} f$$

where $\hat{H}$ is an operator which acts on the function f. Some examples: if $\hat{H} = x$, then $\hat{H}f(x) = xf(x)$. But we could have something more comlicated like $\hat{H} = x + \partial_x$ in which case $\hat{H}f(x) = xf(x) + \partial_x f(x)$. So we can try to do the same thing as we did above for the wave equation, we need to find some basis functions ψ_n for which $\hat{H}\psi_n = \lambda_n\psi_n$. To make contact with what we had previously, above we had $\partial_x e^{ik_n x} = ik_n e^{ik_n x}$ so $\lambda_n = ik_n$ in that case. Then, we can construct the full solution to the PDE as a summation over the basis functions.