Physics 4E Final exam solutions Spring 2016

$P = \sigma T^4$ per unit area, $\sigma = 5.67\times10^{-8}\,W/m^2K^4$

area $A = 4\pi r^2 = 12.57\,cm^2 = 12.57\times10^{-4}\,m^2$

$P_{tot} = A\sigma T^4 = 5.67\times10^{-8}\times12.57\times10^{-4}\times1300^4 = \boxed{203.6\,\frac{J}{s}}$ (a)

(b) Since $T \gg T_E$, heat capacity is

$C_V = 3R$ for 1 mole. $\rho = 19.3\,gr/cm^3$, $R = 8.31\,J/K\,mole$

Volume is $V = \frac{4}{3}\pi r^3$, mass is $M = \rho V = \frac{4}{3}\pi r^3 \rho$

$\Rightarrow M = 80.8\,gr \Rightarrow$ # of moles is $n = \frac{80.8}{197} = 0.41$

$\Rightarrow$ heat capacity is $C = 3\times8.31\times0.41\,\frac{J}{K} = \boxed{10.22\,\frac{J}{K}}$

So to cool from 1300 to 1295 is 5K, $\Rightarrow$

$\Rightarrow \Delta U = C_V\Delta T = 10.22\times5\,J = \boxed{51.1\,J}$

radiates $203.6\,\frac{J}{s}$ $\Rightarrow$ time is $\boxed{\Delta t = \frac{51.1}{203.6}\,s = 0.25\,s}$ (b)

(c) At 20 K, power emitted is : $\boxed{P_{tot} = 203.6\times\left(\frac{20}{1300}\right)^4 = 1.14\times10^{-5}\,\frac{J}{s}}$

Heat capacity at $T = 20K$ is not $3R.n$, because $T \ll T_E$. $\boxed{T_E/T = 5}$

$C_V = 3Rn\left(\frac{T_E}{T}\right)^2\frac{e^{T_E/T}}{(e^{T_E/T}-1)^2} \simeq 3Rn\left(\frac{T_E}{T}\right)^2 e^{-T_E/T} = \boxed{1.72\,J/K}$

So to cool by 1 K, time is $\Delta t = \frac{1.72}{1.14\times10^{-5}}\,s = 1.51\times10^5\,s = \boxed{41.9\text{ hours}}$ (c)

(d) $T_E^{Pb} = 50K$. So (a), (b) stay same. For (c), C_V is higher $\Rightarrow$ $\boxed{\text{it takes longer}}$ (d)

Problem 2

$T = 100\,K$.

$$\langle E \rangle = \frac{3}{2} k_B T = \frac{3}{2} \times \frac{100}{11{,}600}\, eV = \boxed{0.0129\, eV} \text{ (a)}$$

(b) $n(E) = C E^{1/2} e^{-E/k_BT}$

most probable E:

$$n'(E) = 0 \Rightarrow \frac{1}{2E^{1/2}} - \frac{E^{1/2}}{k_BT} = 0 \Rightarrow \boxed{E = \frac{k_BT}{2} = 0.0043\, eV} \text{ (b)}$$

(c) twice the energy

$$\frac{n(2E)}{n(E)} = 2^{1/2} \frac{e^{-2E/k_BT}}{e^{-E/k_BT}} = 2^{1/2} e^{-0.5} = \boxed{0.86} \text{ (c)}$$

half the energy

$$\frac{n(E/2)}{n(E)} = \frac{1}{2^{1/2}} \frac{e^{-E/2k_BT}}{e^{-E/k_BT}} = \frac{1}{2^{1/2}} e^{E/2k_BT} = \frac{1}{2^{1/2}} e^{0.25} = \boxed{0.91} \text{ (c)}$$

blem 3

$\lambda = 3.1 \text{Å}$. Photon energy is

$$E = \frac{hc}{\lambda} = 4000\,eV$$

it takes $4 \times 13.6\,eV = 54.4\,eV$ to ionize He^+ $\Rightarrow$ kinetic

energy of electron is $4000 - 54.4 = \boxed{3945.6\,eV}$ (a)

(b) Photon loses most energy when $\theta = 180°$

$$\lambda' - \lambda = \lambda_c (1 - \cos\theta) = 2\lambda_c \Rightarrow$$

$$\lambda' = \lambda + 2\lambda_c = 3.1 + 2 \times 0.0243 = 3.1486\,\text{Å}$$

energy lost by photon is

$$\Delta E = \frac{hc}{\lambda} - \frac{hc}{\lambda'} = 4000\,eV - 3938.3\,eV = 61.7\,eV$$

$\Rightarrow$ kinetic energy of electron $= 61.7 - 54.4 = \boxed{7.3\,eV}$ (b)

(c) For ionization, need $\Delta E \geq 54.4\,eV$

$\Rightarrow \lambda' = 3.1427 \Rightarrow 0.0427 = \lambda_c(1-\cos\theta) \Rightarrow \cos\theta = 0.12 \Rightarrow \theta = 139°$

$\Rightarrow$ $\boxed{\theta \geq 139°}$ (c)

(d) For $\theta \leq 139°$, electron will be excited to another state.

For example, for $n = 1 \to n = 2$ $\Rightarrow$ $\Delta E = \frac{3}{4} \times 54.4\,eV = 40.8\,eV$

So scattered photon has energy $E' = 4000\,eV - 40.8\,eV = 3959.2\,eV$

$\Rightarrow \lambda' = 3.132\,\text{Å} \Rightarrow \lambda' - \lambda = 0.032 = \lambda_c(1-\cos\theta) \Rightarrow \boxed{\theta = 108.5°}$ (d)

The photon emitted has wavelength $\boxed{\lambda = \frac{hc}{40.8\,eV} = 304\,\text{Å}}$ (d)

Problem 4

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{3}{2}\hbar\omega$$

$$\Rightarrow \frac{\langle p^2\rangle}{2m} = \frac{3}{4}\hbar\omega \Rightarrow \langle p^2\rangle = \frac{3}{2}\hbar\omega m$$

$$\Rightarrow \frac{1}{2} m\omega^2 \langle x^2\rangle = \frac{3}{4}\hbar\omega \Rightarrow \langle x^2\rangle = \frac{3}{2}\frac{\hbar}{m\omega}$$

since $\langle p\rangle = 0$, $\langle x\rangle = 0 \Rightarrow \langle p^2\rangle = \Delta p^2$, $\langle x^2\rangle = \Delta x^2 \Rightarrow$

$$\Rightarrow \Delta x^2 \Delta p^2 = \frac{3}{2}\frac{\hbar}{\cancel{m}\cancel{\omega}} \cdot \frac{3}{2}\hbar\cancel{\omega}\cancel{m} = \left(\frac{3}{2}\hbar\right)^2$$

$$\Rightarrow \boxed{\Delta x \Delta p = \frac{3}{2}\hbar}$$

$\boxed{\text{agrees with Heisenberg} \quad \Delta x \Delta p \geq \frac{\hbar}{2}}$

Problem 5

$E_0 = \frac{\hbar\omega}{2} = 1eV \Rightarrow \hbar\omega = 2eV$

$\frac{1}{2} m\omega^2 a^2 = E_0 \Rightarrow a^2 = \frac{\cancel{2}}{m\omega^2} \cdot \frac{\hbar\omega}{\cancel{2}} = \frac{\hbar}{m\omega} = \frac{\hbar^2}{m(\hbar\omega)} =$

$= \frac{\cancel{2} \times 3.81}{\cancel{2}} \mathring{A}^2 \Rightarrow \boxed{a = 1.95\mathring{A}}$ (a)

$\frac{1}{2} m\omega^2 b^2 = 9eV \Rightarrow b^2 = \frac{9eV \times 2}{m\omega^2} = \frac{\hbar^2}{m} \frac{\times 18eV}{(\hbar\omega)^2} =$

$= \cancel{2} \times 3.81 \times \frac{\cancel{18}^{\,9}}{\cancel{4}\,2} \mathring{A}^2 = 34.29\mathring{A}^2 \Rightarrow \boxed{b = 5.9\mathring{A}}$ (a)

Tunneling probability

$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(V_0 - E)}\,\Delta x} = T_1 T_2$

First region: $V_0 - E \cong \frac{8eV}{2} = 4eV$, $\Delta x = b - a = 3.9\mathring{A}$

$\sqrt{\frac{2m}{\hbar^2}(V_0 - E)}\,\Delta x = \sqrt{\frac{4}{3.81}} \times 3.9 = \boxed{4}$

Second region: $V_0 - E = 8eV$, $\Delta x = b = 5.9\mathring{A}$

$\sqrt{\frac{2m}{\hbar^2}(V_0 - E)} = \sqrt{\frac{8}{3.81}} \times 5.9 = \boxed{8.55}$

total exponent $= 2 \times (4 + 8.55) = 25.1$

$\boxed{T = e^{-25.1} = 1.257 \times 10^{-11}}$ (b)

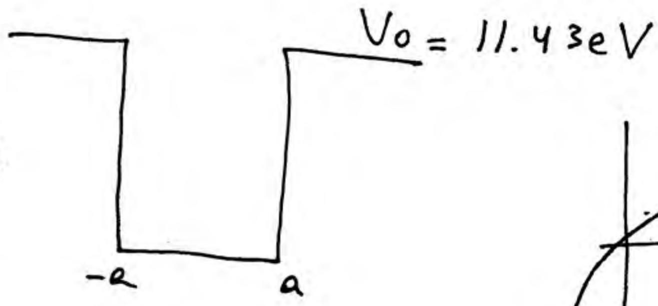
(c) Frequency is ω, so $\tau = \frac{2\pi}{\omega}$ is time for each attempt to tunnel

So the total time it will take is τ / T

$\tau = \frac{2\pi}{\hbar\omega} \cdot \hbar = \frac{3.14}{eV} \hbar = 2.07 \times 10^{-15} s$

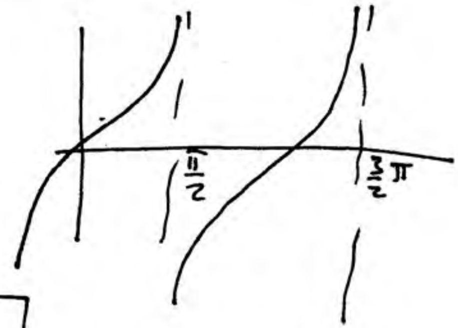
$\Rightarrow$ time it will take $= \frac{\tau}{1.26 \times 10^{-11}} = \boxed{1.6 \times 10^{-4} s}$ (c)

Problem 6



$\psi(x) = A\cos kx \quad x<a$

$\psi(x) = Be^{-\alpha x} \quad x>a$



Continuity:

$$A\cos(ka) = Be^{-\alpha a}$$

$$kA\sin(ka) = \alpha Be^{-\alpha a} \Rightarrow \boxed{\tan(ka) = \frac{\alpha}{k}}$$

$$k = \sqrt{\frac{2m}{\hbar^2}E}\ , \quad \alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}\ , \quad E = 3.81\,eV,\ V_0 = 11.43\,eV$$

$$\Rightarrow k = 1\,Å^{-1}\ , \quad \alpha = 1.41\,Å^{-1}$$

$$\Rightarrow \tan(ka) = 1.41 \Rightarrow ka = 0.95 + n\pi$$

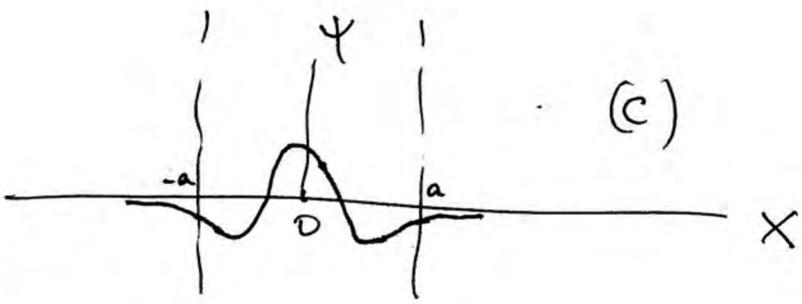
If it is the third state, $n = 1 \Rightarrow$

$$ka = 4.096 \Rightarrow \boxed{a = 4.096\,Å} \Rightarrow \boxed{L = 8.19\,Å}\ (a)$$

(b) The relative probability is

$$\frac{(\cos ka)^2}{1} = 0.58^2 = 0.338 \Rightarrow \boxed{2.96 \sim 3 \text{ times}}\ (b)$$

(c)

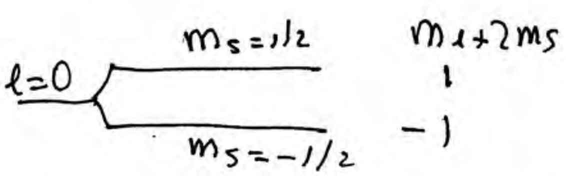


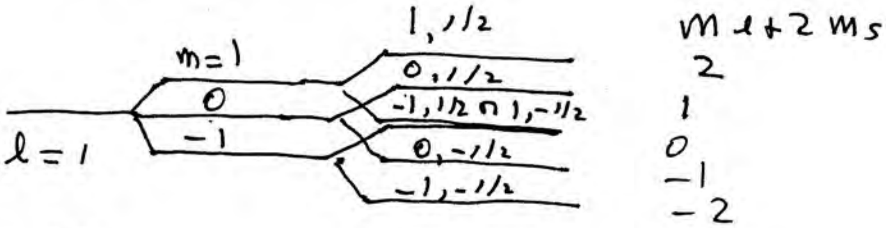
(c)

Problem 7

energy in magnetic field (no spin-orbit)

$$E = \mu_B B(m_\ell + 2m_s) \qquad \mu_B = 5.79 \times 10^{-5}\, eV$$





$$\mu_B B = 5.79 \times 10^{-3}\, eV = E_0$$

$$\frac{E}{k_B T} = \frac{E_0}{k_B T}(m_\ell + 2m_s) \equiv x(m_\ell + 2m_s)$$

$$x = \frac{E_0}{k_B T} = \frac{5.79 \times 10^{-3}}{\frac{1}{11{,}600} \times 300} = 0.224$$

$$\ell = 0: \quad \frac{N(m_s = 1/2)}{N(m_s = -\frac{1}{2})} = e^{-2x} = 0.639$$

$\Rightarrow$ For each 1000 with $\ell=0$, $m_s = -\frac{1}{2}$, $\boxed{639 \text{ have } \ell=0, m_s = \tfrac{1}{2}}$ (a)

(b) same energy as $\ell=0$, $m_s=\frac{1}{2}$ for $\ell=1$, $m=0$, $m_s=\frac{1}{2}$
$\Rightarrow$ $\boxed{639 \text{ atoms}}$ (b)

(c) $$\frac{N(m=1, m_s = 1/2)}{N(m=0, m_s = -\frac{1}{2})} = \frac{e^{-2x}}{e^{x}} = e^{-3x} = 0.511$$

$\Rightarrow$ $\boxed{511 \text{ atoms}}$ (c)

(d) For extra credit: how many have $\ell = 1$?

$m_\ell + 2m_s = 2$: $\boxed{511}$ $m_\ell + 2m_s = 1$: $\boxed{639}$ $m_\ell + 2m_s = 0$: $2 \times 799 = \boxed{1598}$

$m_\ell + 2m_s = -1$: $\boxed{1000}$ $m_\ell + 2m_s = -2$: $\boxed{1251}$

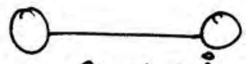
$$\frac{N(m_\ell + 2m_s = 0)}{N(m + 2m_s = -1)} = e^{-x} = 0.799$$

$$\frac{N(m + 2m_s = -2)}{N(m + 2m_s = -1)} = e^{x} = 1.251$$

total for $\ell = 1$:

$511 + 639 + 1598 + 1000 + 1251 = \boxed{5000}$ (d)

Problem 8



$R = 1.2\,\text{Å}$

$$E_R = \frac{\hbar^2}{2I}\,l(l+1)\ ; \quad I = \frac{1}{2}MR^2 \ ; \quad E_R = E_{0R}\,l(l+1)$$

$$E_{0R} = \frac{\hbar^2}{2\cdot\frac{1}{2}MR^2} = \frac{\hbar^2}{MR^2} = \frac{(\hbar c)^2}{Mc^2R^2}$$

$$Mc^2 = 16\,u = 931.5\times 16\ \text{MeV} = 14{,}904\ \text{MeV}$$

$$\Rightarrow E_{0R} = \frac{1973^2}{14{,}904\times 10^6\times 1.2^2}\ \text{eV} = \boxed{1.81\times 10^{-4}\,\text{eV}} = k_B T_R$$

$$\Rightarrow \boxed{T_R = 2.10\,\text{K}}\ \text{(a)}$$

Separation between lowest and next lowest

$$E_R(l=1) - E_R(l=0) = 2E_{0R} = \boxed{3.62\times 10^{-4}\,\text{eV}}\ \text{(a)}$$

(b) $$n(l) \propto g_l\, e^{-E_R(l)/k_BT} \qquad g_l = 2l+1$$

$$n(l=3) = n(l=2) \Rightarrow 7e^{-12E_{0R}/k_BT} = 5e^{-6E_{0R}/k_BT} \Rightarrow$$

$$\Rightarrow e^{-6E_{0R}/k_BT} = \frac{5}{7} \Rightarrow T = \frac{6\,T_E}{\ln 7/5} = 17.8\,K\cdot T_E = 37.4\,K$$

$$n(l=4) = n(l=3) \Rightarrow 7e^{-12E_{0R}/k_BT} = 9e^{-20E_{0R}/k_BT} \Rightarrow$$

$$\Rightarrow e^{-8E_{0R}/k_BT} = \frac{7}{9} \Rightarrow T = \frac{8\,T_E}{\ln 9/7} = 66.8\,K$$

$$\Rightarrow \boxed{37.4\,K < T < 66.8\,K}\ \text{(b)}$$