PHYSICS 200B : CLASSICAL MECHANICS
PROBLEM SET #4

(1) Consider two coupled nonlinear oscillators, with

$$\frac{d\boldsymbol{\varphi}_1}{dt} = \boldsymbol{V}_1(\boldsymbol{\varphi}_1) + \epsilon\, \boldsymbol{F}_1(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$$
$$\frac{d\boldsymbol{\varphi}_2}{dt} = \boldsymbol{V}_2(\boldsymbol{\varphi}_2) + \epsilon\, \boldsymbol{F}_2(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \ .$$

Assume that for $\epsilon = 0$ each of the oscillators exhibits at least one stable limit cycle. Assume further that the natural frequencies $\omega_{1,2}$ of their respective limit cycles are close to resonance. That is, assume that the detuning

$$\nu = m\omega_1 - n\omega_2$$

is small for some integer values of m and n.

(a) Using the phase representation and isochrones for each oscillator, show that to lowest order one may write

$$\frac{d\phi_1}{dt} = \omega_1 + \epsilon\, Q_1(\phi_1, \phi_2)$$
$$\frac{d\phi_2}{dt} = \omega_2 + \epsilon\, Q_2(\phi_1, \phi_2) \ .$$

(b) Expand the functions $Q_{1,2}(\phi_1, \phi_2)$ in a double Fourier series in their arguments. What terms satisfy the (near) resonance condition?

(c) Keeping only the terms which are nearly resonant, define $\psi = m\phi_1 - n\phi_2$ and derive an ODE describing the behavior of ψ. Analyze this ODE and classify its fixed points. What is the condition for synchronization of the two nonlinear oscillators?

(2) Consider the function $F(x)$ defined by

$$F(x) = \begin{cases} -x & \text{if } 0 \le 1 \le 1 \\ 3x - 4 & \text{if } 1 \le x \le 2 \\ -5x + 12 & \text{if } 2 \le x \le 3 \\ 7x - 24 & \text{if } x \ge 3 \end{cases}$$

with $F(-x) \equiv -F(x)$.

(a) Sketch $F(x)$ over the interval $x \in [-4, 4]$.

(b) Consider the nonlinear oscillator $\ddot{x} + \mu F'(x)\, \dot{x} + x = 0$. Find all the stable limit cycles and their periods for $\mu \gg 1$.

(3) Use the method of characteristics to solve the quasilinear PDE

$$\phi_t + \gamma\, x \phi_x = -B x^2 \phi \quad ,$$

subject to the initial condition $\phi(x,0) = f(x)$.

(4) Consider shock formation in the equation $c_t + c\, c_x = 0$ with initial conditions $c(\zeta) = c(x=\zeta, t=0)$ an odd function of its argument, *i.e.* $c(-\zeta) = -c(\zeta)$. Suppose further that $c(\zeta)$ is monotonically decreasing.

(a) Find an expression for the time the wave first breaks, t_{B}.

(b) Show that $\zeta_- = -\zeta_+$ and find an equation relating ζ_+ and t for $t > t_{\mathrm{B}}$.

(c) Show that the position of the shock remains at $x_{\mathrm{s}}(t) = 0$ for all $t > t_{\mathrm{B}}$.

(d) Find t_{B}, $\zeta_+(t)$, and the shock discontinuity $\Delta c(t)$ for the case

$$c(\zeta) = -\frac{c_0\, \zeta}{\sqrt{a^2 + \zeta^2}} \quad .$$

(e) Sketch the time evolution of $c(x,t)$ with and without the shock fitting. (Without shock fitting, the function will eventually become multivalued.)

(5) Consider shock formation in the equation $c_t + c\, c_x = 0$ with initial conditions

$$c(x, t=0) = \frac{c_0}{1+x^2} \quad .$$

(a) Find the position and time $(x_{\mathrm{B}}, t_{\mathrm{B}})$ where the wave first breaks.

(b) Find the two equations relating ζ_+, ζ_-, and t.

(c) For general t, your equations from part (b) cannot be solved analytically. However, as $t \to \infty$, one can make progress. Show in the late time limit that $\zeta_+ \propto \zeta_-^2$ and use the first shock fitting equation to obtain a relation between ζ_+ and ζ_- valid for $c_0 t \gg 1$.

(d) Invoking the second shock equation, obtain expressions for $\zeta_\pm(t)$ and $c\big(\zeta_\pm(t)\big)$.

(e) Find the motion of the shock $x_{\mathrm{s}}(t)$ at late times $c_0 t \gg 1$. Show that it agrees with the late time results derived in §4.5 of the Lecture Notes.