PHYSICS 152B/232
Spring 2017
Homework Assignment #3 Solutions

[1] *Cyclotron resonance in Si and Ge* – Both Si and Ge are indirect gap semiconductors with anisotropic conduction band minima and doubly degenerate valence band maxima. In Si, the conduction band minima occur along the $\langle 100 \rangle$ ($\langle \Gamma X \rangle$) directions, and are six-fold degenerate. The equal energy surfaces are cigar-shaped, and the effective mass along the $\langle \Gamma X \rangle$ principal axes (the 'longitudinal' effective mass) is $m^*_{\rm l} \simeq 1.0\, m_{\rm e}$, while the effective mass in the plane perpendicular to this axis (the 'transverse' effective mass) is $m^*_{\rm t} \simeq 0.20\, m_{\rm e}$. The valence band maximum occurs at the unique Γ point, and there are two isotropic hole branches: a 'heavy' hole with $m^*_{\rm hh} \simeq 0.49\, m_{\rm e}$, and a 'light' hole with $m^*_{\rm lh} \simeq 0.16\, m_{\rm e}$.

In Ge, the conduction band minima occur at the fourfold degenerate L point (along the eight $\langle 111 \rangle$ directions) with effective masses $m^*_{\rm l} \simeq 1.6\, m_{\rm e}$ and $m^*_{\rm t} \simeq 0.08\, m_{\rm e}$. The valence band maximum again occurs at the Γ point, where the hole masses are $m^*_{\rm hh} \simeq 0.34\, m_{\rm e}$ and $m^*_{\rm lh} \simeq 0.044\, m_{\rm e}$. Use the following figures to interpret the cyclotron resonance data shown below. Verify whether the data corroborate the quoted values of the effective masses in Si and Ge.

Solution :

We found that $\sigma_{\alpha\beta} = ne^2\, \Gamma^{-1}_{\alpha\beta}$, with

$$\begin{aligned}\Gamma_{\alpha\beta} &\equiv (\tau^{-1} - i\omega)\, m_{\alpha\beta} \pm \frac{e}{c}\, \epsilon_{\alpha\beta\gamma}\, B^\gamma \\ &= \begin{pmatrix} (\tau^{-1} - i\omega) m^*_x & \pm eB_z/c & \mp eB_y/c \\ \mp eB_z/c & (\tau^{-1} - i\omega) m^*_y & \pm eB_x/c \\ \pm eB_y/c & \mp eB_x/c & (\tau^{-1} - i\omega) m^*_z \end{pmatrix} .\end{aligned}$$

The valence band maxima are isotropic in both cases, with

$$\begin{aligned} m^*_{\rm hh}({\rm Si}) &\simeq 0.49\, m_{\rm e} \qquad & m^*_{\rm hh}({\rm Ge}) &\simeq 0.34\, m_{\rm e} \\ m^*_{\rm lh}({\rm Si}) &\simeq 0.16\, m_{\rm e} \qquad & m^*_{\rm lh}({\rm Ge}) &\simeq 0.044\, m_{\rm e} \ . \end{aligned}$$

With isotropic bands, the absorption is peaked at $\omega = \omega_{\rm c} = eB/m^* c$, assuming

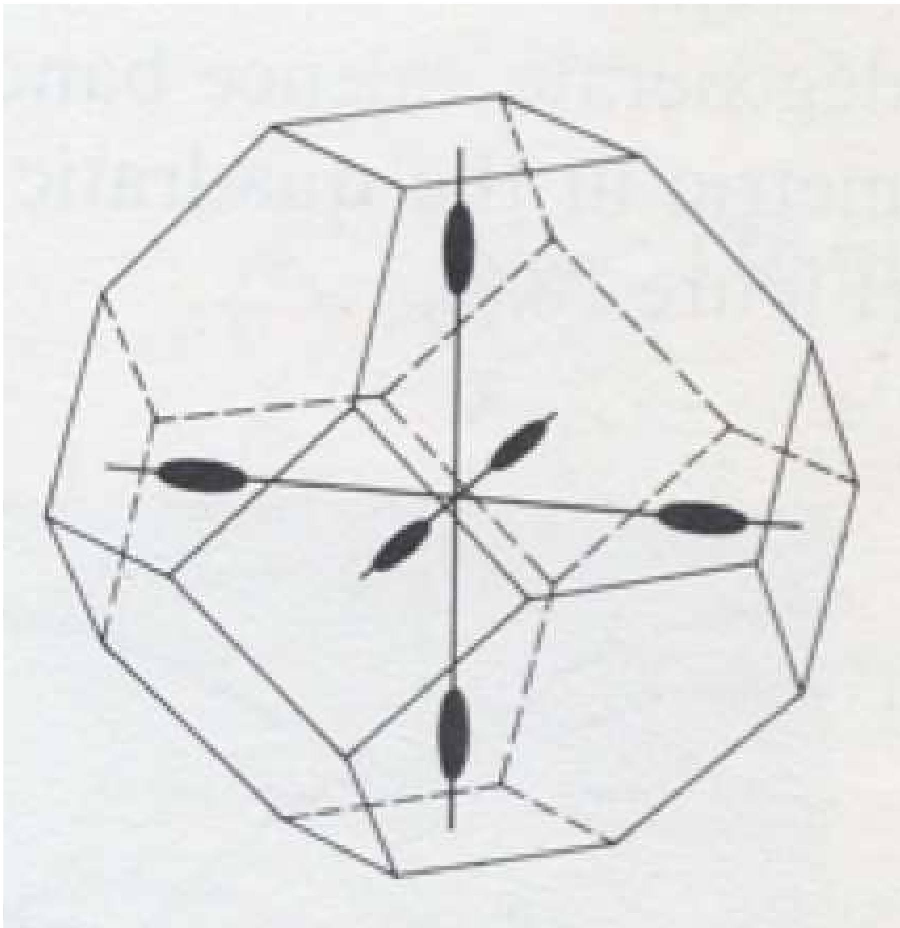

Figure 1: Constant energy surfaces near the conduction band minima in silicon. There are six symmetry-related ellipsoidal pockets whose long axes run along the $\langle 100\rangle$ directions.

$\omega_c\tau \gg 1$. Writing $\omega = 2\pi f$, the resonance occurs at a field

$$\begin{aligned} B(f) &= 2\pi f \cdot \frac{m^* c}{e} \\ &= \frac{hc}{e} \cdot \frac{m^*}{m_e} \cdot \frac{1}{2\pi a_B^2} \cdot \frac{hf}{(e^2/a_B)} \\ &= 3.58 \times 10^{-7}\,\mathrm{G} \cdot \frac{m^*}{m_e} \cdot f[\mathrm{Hz}] \\ &= 8590\,\mathrm{G} \cdot \frac{m^*}{m_e}\ , \end{aligned}$$

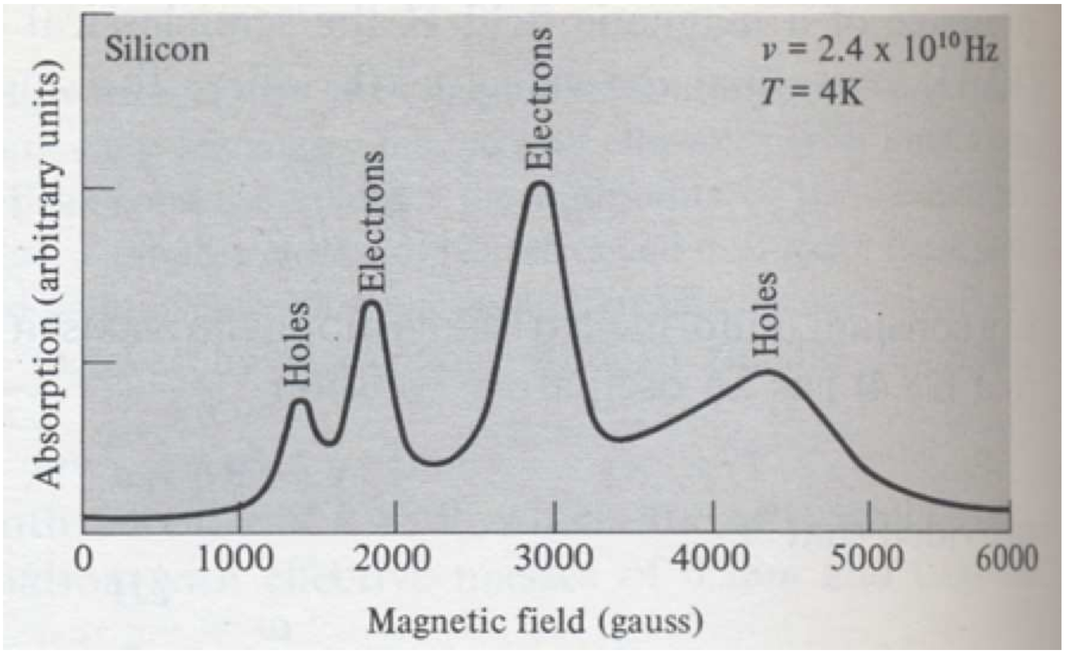

Figure 2: Cyclotron resonance data in Si (G. Dresselhaus *et al.*, *Phys*, *Rev*, **98**, 368 (1955).) The field lies in a (110) plane and makes an angle of 30° with the [001] axis.

where we have used

$$\begin{aligned}
\frac{hc}{e} &= 4.137 \times 10^{-7}\,\mathrm{G} \cdot \mathrm{cm}^2 \\
a_{\mathrm{B}} &= \frac{\hbar^2}{m_{\mathrm{e}}\, e^2} = 0.529\,\text{Å} \\
h &= 4.136 \times 10^{-15}\,\mathrm{eV} \cdot \mathrm{s} \\
\frac{e^2}{a_{\mathrm{B}}} &= 27.2\,\mathrm{eV} = 2\,\mathrm{Ry} \\
f &= 2.40 \times 10^{10}\,\mathrm{Hz}\ .
\end{aligned}$$

Thus, we predict

$$\begin{aligned}
B_{\mathrm{hh}}(\mathrm{Si}) &\simeq 4210\,\mathrm{G} \qquad & B_{\mathrm{hh}}(\mathrm{Ge}) &\simeq 2920\,\mathrm{G} \\
B_{\mathrm{lh}}(\mathrm{Si}) &\simeq 1370\,\mathrm{G} \qquad & B_{\mathrm{lh}}(\mathrm{Ge}) &\simeq 378\,\mathrm{G}\ .
\end{aligned}$$

All of these look pretty good.

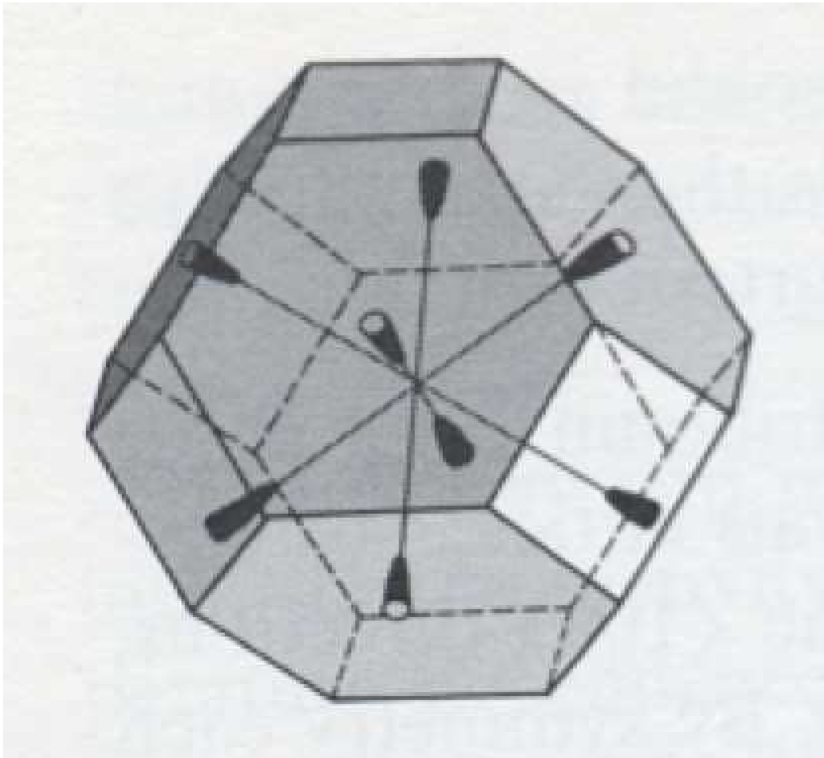

Figure 3: Constant energy surfaces near the conduction band minima in germanium. There are eight symmetry-related half-ellipsoids whose long axes run along the $\langle 111 \rangle$ directions, and are centered on the midpoints of the hexagonal zone faces. With a suitable choice of primitive cell in $\boldsymbol{k}$-space, these can be represented as four ellipsoids, the half-ellipsoids on opposite faces being joined together by translations through suitable reciprocal lattice vectors.

Now let us review the situation with electrons near the conduction band minima:

$$\begin{aligned} \text{Si}: &\ \text{6-fold degenerate minima along } \langle 100 \rangle \\ \text{Ge}: &\ \text{4-fold degenerate minima along } \langle 111 \rangle \text{ (at L point)} \end{aligned}$$

$$\begin{aligned} m^*_{\rm l}(\text{Si}) &\simeq 1.0\, m_{\rm e} \qquad & m^*_{\rm l}(\text{Ge}) &\simeq 1.6\, m_{\rm e} \\ m^*_{\rm t}(\text{Si}) &\simeq 0.20\, m_{\rm e} \qquad & m^*_{\rm t}(\text{Ge}) &\simeq 0.08\, m_{\rm e}\ . \end{aligned}$$

The resonance condition is that $\sigma_{\alpha\beta} = \infty$, which for $\tau > 0$ occurs only at complex frequencies, *i.e.* for real frequencies there are no true divergences, only resonances. The location of the resonance is determined by $\det \Gamma = 0$. Taking the determinant, one finds

$$\det \Gamma = (\tau^{-1} - i\omega)\, m^*_{\rm l} \cdot \left\{ (\tau^{-1} - i\omega)^2 m^{*2}_{\rm t} + \frac{e^2}{c^2}\, B_z^2 + \frac{m^*_{\rm t}}{m^*_{\rm l}}\, \frac{e^2}{c^2} \left(B_x^2 + B_y^2 \right) \right\} .$$

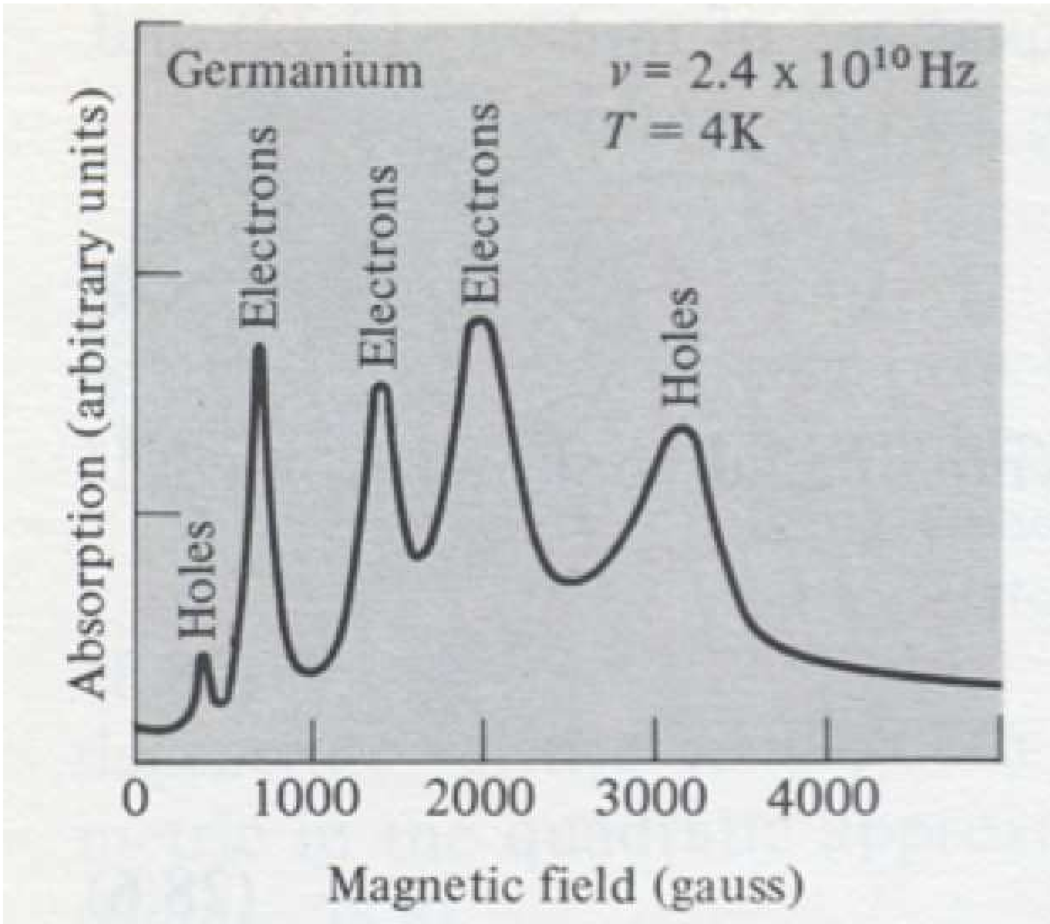

Figure 4: Cyclotron resonance data in Ge (G. Dresselhaus *et al.*, *Phys*, *Rev*, **98**, 368 (1955).) The field lies in a (110) plane and makes an angle of 60° with the [001] axis.

Assuming $\omega\tau \gg 1$, the location of the resonance is given by

$$\omega^2 = \left(\frac{eB_\parallel}{m_\mathrm{t}^* c}\right)^2 + \frac{m_\mathrm{t}^*}{m_\mathrm{l}^*}\left(\frac{e\boldsymbol{B}_\perp}{m_\mathrm{t}^* c}\right)^2 ,$$

where $B_\parallel \equiv B_z$ and $\boldsymbol{B}_\perp \equiv B_x\,\hat{\boldsymbol{x}} + B_y\,\hat{\boldsymbol{y}}$. Let the polar angle of $\boldsymbol{B}$ be θ, so $B_\parallel = B\cos\theta$ and $B_\perp = B\sin\theta$. We then have

$$\omega^2 = \left(\frac{eB}{m_\mathrm{t}^* c}\right)^2 \left\{\cos^2\theta + \frac{m_\mathrm{t}^*}{m_\mathrm{l}^*}\sin^2\theta\right\}$$

$$B(f) = 8600\,\mathrm{G}\cdot\left(\frac{m_\mathrm{t}^*}{m_\mathrm{e}}\right)\bigg/\sqrt{\cos^2\theta + \frac{m_\mathrm{t}^*}{m_\mathrm{l}^*}\sin^2\theta} ,$$

where again we take $f = \omega/2\pi = 2.4 \times 10^{10}$ Hz.

According to the diagrams, the field lies in the (110) plane, which means we can write

$$\hat{\boldsymbol{B}} = \sqrt{\tfrac{1}{2}}\,\sin\chi\,\hat{\boldsymbol{e}}_1 - \sqrt{\tfrac{1}{2}}\,\sin\chi\,\hat{\boldsymbol{e}}_2 + \cos\chi\,\hat{\boldsymbol{e}}_3 \ ,$$

where χ is the angle $\hat{\boldsymbol{B}}$ makes with $\hat{\boldsymbol{e}}_3 = [001]$.

<u>Ge</u> : We have

$$\frac{m_{\rm t}^*}{m_{\rm e}} = 0.082 \qquad\qquad \frac{m_{\rm t}^*}{m_{\rm l}^*} = 0.051 \ ,$$

and we are told $\chi = 60°$, so

$$\hat{\boldsymbol{B}} = \sqrt{\tfrac{3}{8}}\,\hat{\boldsymbol{e}}_1 - \sqrt{\tfrac{3}{8}}\,\hat{\boldsymbol{e}}_2 + \tfrac{1}{2}\,\hat{\boldsymbol{e}}_3 \ .$$

The conduction band minima lie along $\langle 111\rangle$, which denotes a *set* of directions in real space:

$$\begin{aligned}
\pm[111] &: \ \hat{\boldsymbol{n}} = \pm\tfrac{1}{\sqrt{3}}(\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3) &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{1}{12} &&\Rightarrow \ B = 1950\,{\rm G} \\
\pm[11\bar{1}] &: \ \hat{\boldsymbol{n}} = \pm\tfrac{1}{\sqrt{3}}(\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 - \hat{\boldsymbol{e}}_3) &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{1}{12} &&\Rightarrow \ B = 1950\,{\rm G} \\
\pm[\bar{1}11] &: \ \hat{\boldsymbol{n}} = \pm\tfrac{1}{\sqrt{3}}(-\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3) &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{7-2\sqrt{6}}{12} &&\Rightarrow \ B = 1510\,{\rm G} \\
\pm[1\bar{1}1] &: \ \hat{\boldsymbol{n}} = \pm\tfrac{1}{\sqrt{3}}(\hat{\boldsymbol{e}}_1 - \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3) &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{7+2\sqrt{6}}{12} &&\Rightarrow \ B = 710\,{\rm G} \ .
\end{aligned}$$

All OK!

<u>Si</u> : Again, $\boldsymbol{B}$ lies in the (110) plane, this time with $\chi = 30°$, so

$$\hat{\boldsymbol{B}} = \sqrt{\tfrac{1}{8}}\,\hat{\boldsymbol{e}}_1 - \sqrt{\tfrac{1}{8}}\,\hat{\boldsymbol{e}}_2 + \sqrt{\tfrac{3}{4}}\,\hat{\boldsymbol{e}}_3 \ .$$

The conduction band minima lie along $\langle 100\rangle$, so

$$\begin{aligned}
\pm[001] &: \ \hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{e}}_3 &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{3}{4} &&\Rightarrow \ B = 1820\,{\rm G} \\
\pm[010] &: \ \hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{e}}_2 &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{1}{8} &&\Rightarrow \ B = 2980\,{\rm G} \\
\pm[100] &: \ \hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{e}}_1 &&\Rightarrow \ \cos^2\theta = (\hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}})^2 = \tfrac{1}{8} &&\Rightarrow \ B = 2980\,{\rm G} \ .
\end{aligned}$$

These also look pretty good.