PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #7 SOLUTIONS

(1) Consider the equation of state

$$p\sqrt{v^2 - b^2} = RT\,\exp\bigg(-\frac{a}{RTv^2}\bigg)\ .$$

(a) Find the critical point (v_c, T_c, p_c).

(b) Defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$, and $\bar{T} = T/T_c$, write the equation of state in dimensionless form $\bar{p} = \bar{p}(\bar{v}, \bar{T})$.

(c) Expanding $\bar{p} = 1 + \pi$, $\bar{v} = 1 + \epsilon$, and $\bar{T} = 1 + t$, find $\epsilon_{\rm liq}(t)$ and $\epsilon_{\rm gas}(t)$ for $-1 \ll t < 0$.

(2) Consider a nearest neighbor two-state Ising *antiferromagnet* on a triangular lattice. The Hamiltonian is

$$\hat{H} = J\sum_{\langle ij\rangle} \sigma_i\sigma_j - \mathsf{H}\sum_i \sigma_i\ ,$$

with $J > 0$.

(a) Show graphically that the triangular lattice is *tripartite*, *i.e.* that it may be decomposed into three component sublattices A, B, and C such that every neighbor of A is either B or C, *etc.*

(b) Use a variational density matrix which is a product over single site factors, where

$$\begin{aligned}\rho(\sigma_i) &= \frac{1+m}{2}\,\delta_{\sigma_i,+1} + \frac{1-m}{2}\,\delta_{\sigma_i,-1} \qquad \text{if } i \in \text{A or } i \in \text{B} \\ &= \frac{1+m_{\rm C}}{2}\,\delta_{\sigma_i,+1} + \frac{1-m_{\rm C}}{2}\,\delta_{\sigma_i,-1} \qquad \text{if } i \in \text{C}\ .\end{aligned}$$

Compute the variational free energy $F(m, m_{\rm C}, T, \mathsf{H}, N)$.

(c) Find the mean field equations.

(d) Find the mean field phase diagram.

(e) While your mean field analysis predicts the existence of an ordered phase, it turns out that $T_c = 0$ for this model because it is so highly frustrated when $h = 0$. The ground state is highly degenerate. Show that for any ground state, no triangle can be completely ferromagnetically aligned. What is the ground state energy? Find a lower bound for the ground state entropy per spin.

(3) Consider a spin-S magnet on a cubic lattice system with mixed ferromagnetic and antiferromagnetic interactions:

$$J_{ij} = \begin{cases} +J_1 > 0 & \text{6 nearest neighbors} \\ -J_2 < 0 & \text{12 next-nearest neighbors} \\ 0 & \text{otherwise .} \end{cases}$$

(a) Find $\hat{J}(\boldsymbol{q})$. Show that the ordering wavevector $\boldsymbol{Q}$ depends on the ratio $r = J_2/J_1$, with $\boldsymbol{Q} = 0$ for $r < r_\mathrm{c}$ and $\boldsymbol{Q} \neq 0$ for $r > r_\mathrm{c}$. Find r_c and $\boldsymbol{Q}$ in the latter regime. In general $\boldsymbol{Q}$ is incommensurate with the lattice. Such a system is called a *helimagnet*. *Hint : Assume $\boldsymbol{Q} = Q(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})$, which is consistent with the cubic symmetry.*

(b) Find the critical temperature T_c where order sets in for the cases $r < r_\mathrm{c}$ and $r > r_\mathrm{c}$.

(c) Find the uniform susceptibility $\chi(T) \equiv \hat{\chi}(\boldsymbol{q} = 0, T)$. Over what range of r is it resembling that of a Curie-Weiss ferromagnet, *i.e.* with a positive T-axis intercept for $\chi^{-1}(T)$, and over what range is it resembling that of a Curie-Weiss antiferromagnet, *i.e.* with a negative T-axis intercept for $\chi^{-1}(T)$?