so natural to define: $\boxed{\psi = \phi - \omega t}$ phase variable

deviation from rotation with forcing

$$\frac{d\psi}{dt} = \frac{d\phi}{dt} - \omega = \omega_0 + \epsilon g(\psi) - \omega$$

so

$$\boxed{\frac{d\psi}{dt} = -\nu + \epsilon g(\psi)}$$

Simple phase Dynamics Equation

$\nu \equiv \omega - \omega_0 \equiv$ Frequency mismatch

$\epsilon g(\psi) \equiv$ forcing

(*)

N.B. - Competition between frequency mismatch of oscillator with entraining forcing (more generally: entrainee vs. entrainer) is essence of synchronization problem.

- more generally, mis-match vs. interaction strength generic to any nonlinear mode coupling problem: *

i.e. – parametric oscillator instability $\Leftrightarrow$

$$\gamma^2 = \frac{1}{4}\left[\underbrace{\left(\frac{1}{2}\,\Delta\omega_0\right)^2}_{\text{osc amplitude ampl}} - \underbrace{\epsilon^2}_{\omega-\omega_0 \text{ mismatch}}\right]$$

– forced Duffing bifurcation

$$b^2 = \frac{f^2/4m^2\omega_0^2}{\left[\left(\underbrace{\epsilon}_{\text{mis-match}} - \underbrace{\kappa b^2}_{\text{NL shift}}\right)^2 + \lambda^2\right]}$$

– 3 wave coupling in turbulence

$$\theta_{\underline{k},\underline{p},\underline{q}} = i\Big/\Big[\underbrace{(\omega_{\underline{p}} + \omega_{\underline{q}} - \omega_{\underline{k}})}_{\text{triad mis-match}} + i\underbrace{(\mu_{\underline{k}} + \mu_{\underline{p}} + \mu_{\underline{q}})}_{\text{NL broadening}}\Big]$$

triad resonance amplitude

$$\underbrace{\mu_{\underline{k}}}_{\text{self-damping}} = \sum_{\underline{k}'} |(CC)_{\underline{k}'}|^2 \, \overbrace{|\tilde{V}_{\underline{k}'}|^2}^{\text{amplitude}} \, \underbrace{\theta_{\underline{k}+\underline{k}'}}_{\text{triad coherence time.} \Rightarrow \text{mis-match}}$$

$$\boxed{\frac{d\psi}{dt} = -\gamma + \epsilon g(\psi)}$$

1D dynamical system $\Rightarrow$ parameters $\left\{ \begin{array}{l} \epsilon \\ \gamma \end{array} \right.$

So, for synchronization / phase locking $\Rightarrow$ seek:

- stable fixed points ;

i.e. $\frac{d\psi}{dt} = 0 \Rightarrow \boxed{\gamma = \epsilon g(\psi_s)}$

$\hookrightarrow \psi_{synch}$.

and $\psi = \psi_s + \delta\psi$

$$\frac{d}{dt}\delta\psi = \epsilon g'(\psi_s)\,\delta\psi$$

$\Rightarrow g'(\psi_s) < 0 \rightarrow$ stable fixed point

$g'(\psi_s) > 0 \rightarrow$ unstable fixed point

Obviously, $g'(\psi_s) < 0 \Rightarrow$ stable fixed points
$\Rightarrow$ synchronized states

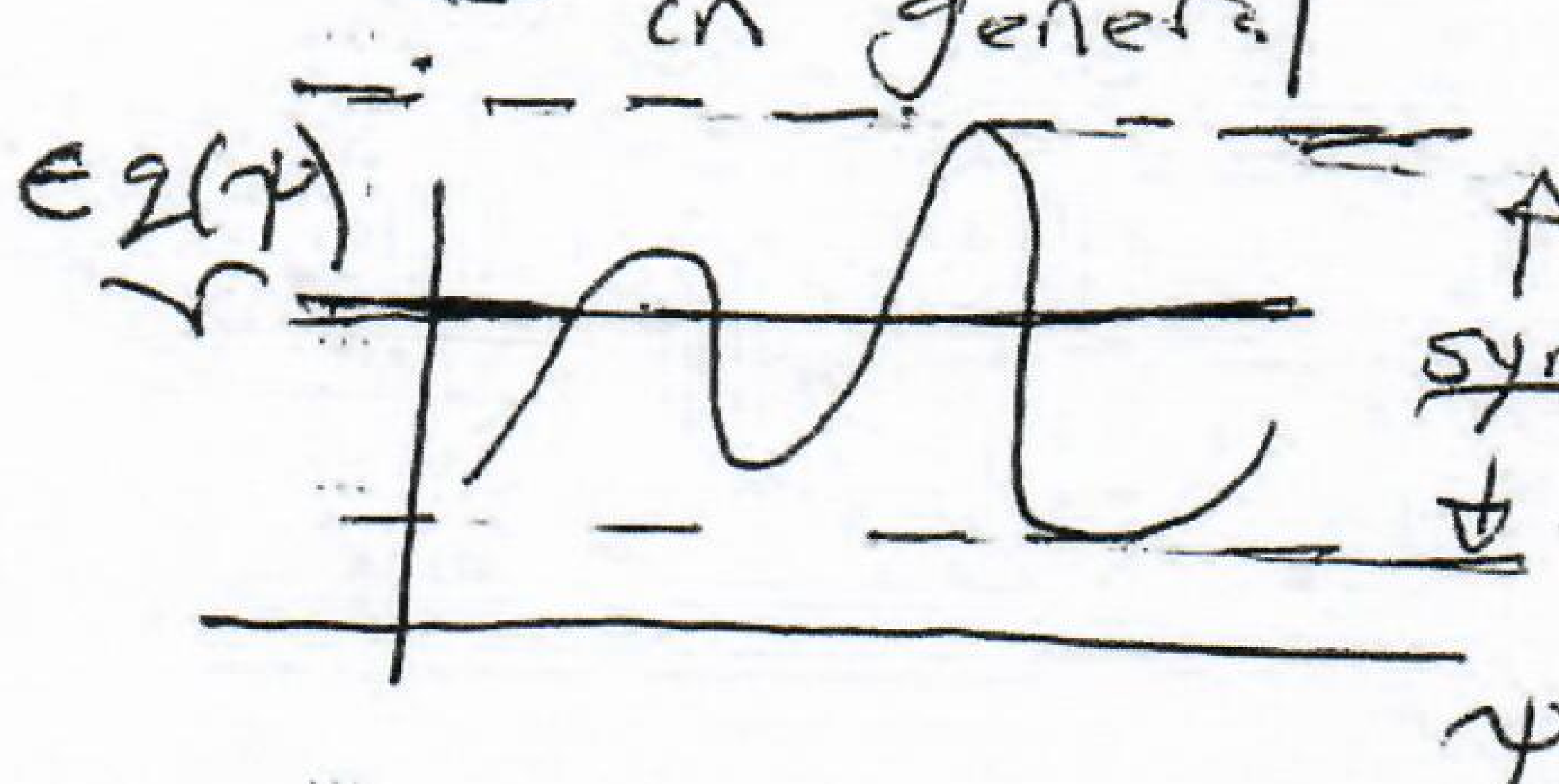
Note: – at $\psi = \psi_s$

$$\phi - \omega t = \psi_s$$
$$\phi = \psi_s + \omega t$$

↳ oscillator phase "synchs" to external force.

– in general

$\nu = \epsilon q(\psi)$



$(\epsilon q)_{min} < \nu < (\epsilon q)_{max}$

– synchronization for $\epsilon q_{min} < \nu < \epsilon q(\psi)_{max}$.

– fixed points come in stable-unstable pairs (curve crossings)

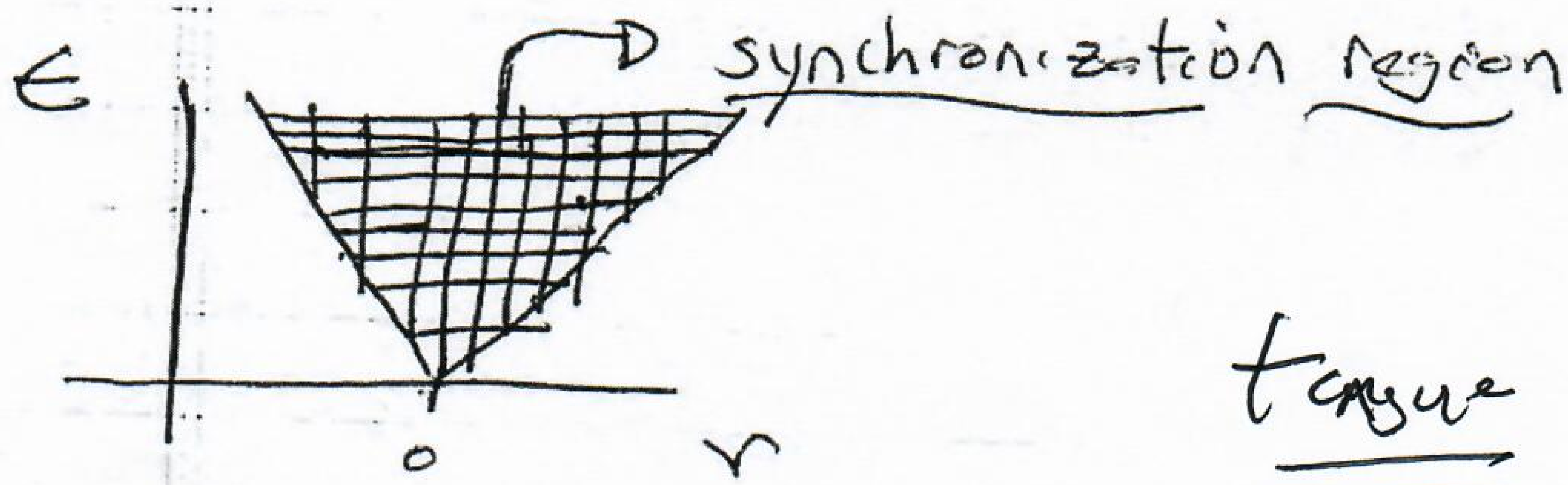
– except when pair dis-appears

i.e. synchronization transition (a bifurcation) where stable and unstable points collide

∴ onset of synchronization ⇒ bifurcation

Synchronization Region

$\epsilon \Omega_{min} < r < \epsilon \Omega_{max}$ $\Rightarrow$ boundaries are straight lines



tongue G vs r

- several pairs of fixed points can exist in synchronization region
  ⇒ several synchronized states possible, given mis-match, etc.

Now, if ν outside synchronization range: large.

$$\frac{d\psi}{dt} = -\nu + \epsilon g(\psi) \qquad t = \int^{\psi} \frac{d\theta}{[\epsilon g(\psi) - \nu]}$$

⇒ gives $\psi(t)$, so $\phi = \omega t + \psi(t)$

- case of quasi-periodic motion, with two incommensurate periods.

Quasi-Periodicity

- periods / frequencies

→ driver: ω, ω

→ "beat frequency" ≡ difference between observed oscillator frequency and external force frequency.

↓

effective frequency of $\psi = \phi - \omega t$

$$T_\psi = \left| \int_0^{2\pi} d\psi \Big/ \left(\epsilon g(\psi) - \nu\right) \right| \quad \rightarrow \text{ beat period}$$

$$\Omega_\psi = 2\pi / T_\psi \quad \rightarrow \text{ beat frequency}$$

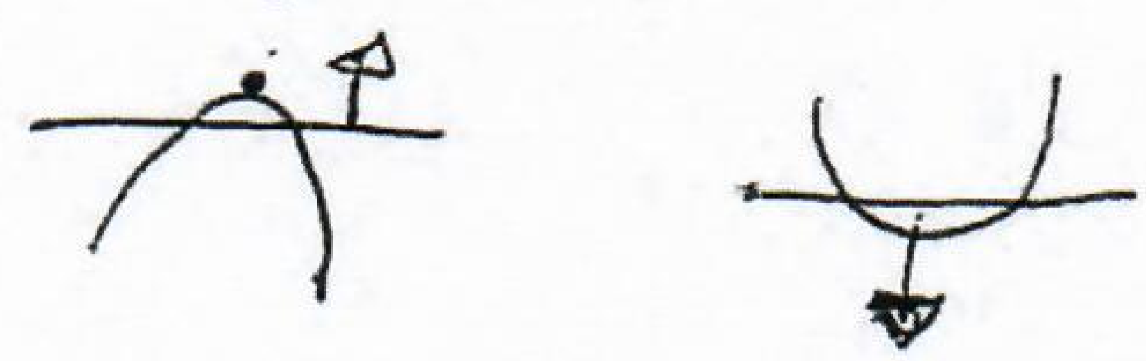
$$\text{so } \langle \dot\phi \rangle = \Omega = \omega + \Omega_\psi$$

time avg. on ω^{-1}

actual observed frequency

Now, how do T_ψ, Ω_ψ etc. behave near synch transition point?

→ ψ_m

- near synchronization 

fixed points collide....

- $\nu_{max} = \epsilon g_{max}$, then expanding:

$$\epsilon g(\psi) - \nu = \epsilon g(\psi) - \nu_{max} - (\nu - \nu_{max})$$

$$= \cancel{\epsilon g(\psi_m)} + \cancel{\epsilon g'(\psi_m)}(\psi - \psi_m) + \tfrac{1}{2}\epsilon g''(\psi_m)(\psi - \psi_m) - \cancel{\nu_{max}} - (\nu - \nu_{max})$$

so $T_\psi \cong \int_0^{2\pi} d\psi \Big/ \left[\frac{1}{2} \epsilon \xi''(\psi_M)(\psi - \psi_M)^2 - (r - r_M) \right]$

obviously, T_ψ dominated by contribution where $|\epsilon \xi(\psi) - r| \to 0$, i.e. ψ near bifurcation point!!

$$\Rightarrow T_\psi \simeq \int_{-\infty}^{+\infty} \frac{d\psi}{(r - r_M)} \left(1 \Big/ \left[\frac{\frac{1}{2} \epsilon \xi''(\psi_M)(\psi - \psi_M)^2}{(r - r_M)} - 1 \right] \right)$$

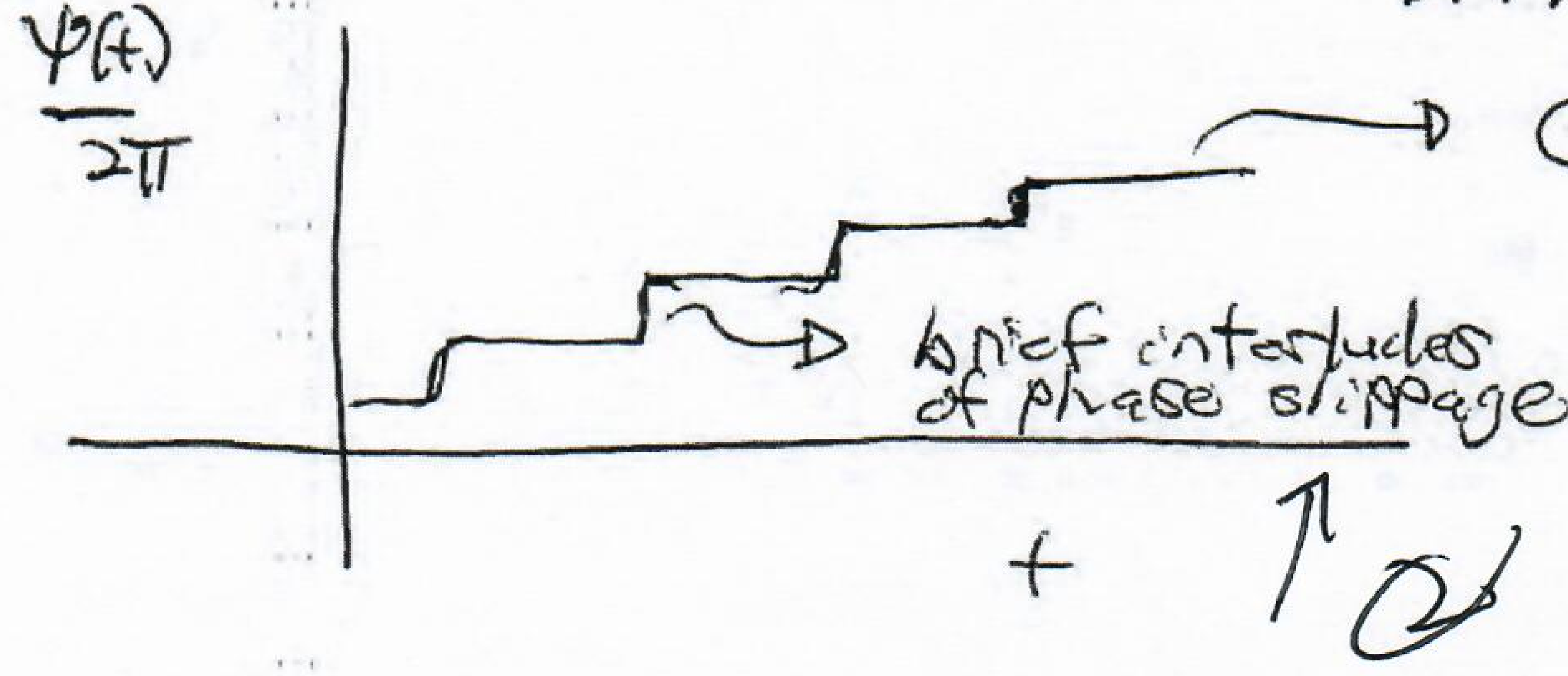
$$\cong \left[\epsilon \, \xi''(\psi_{max}) (r - r_M) \right]^{-1/2} \quad \#$$

$$\Rightarrow \boxed{\Omega_\psi \sim \left[\epsilon \xi''(\psi_{max})(r - r_M) \right]^{1/2} \sim \sqrt{\epsilon}\,(r - r_M)^{1/2}} \rightarrow \text{frequency of phase jumps (not evenly distr.)}$$

beat frequency, as $r \to r_M$ (from outside synch. region)

$\therefore$ not surprisingly, beat frequency slows near bifurcation point $\Rightarrow$ system spends long time near ψ_{max}

so ⇒ bottom line: $\psi = \phi - \omega t$ looks like: 1



1 relatively long periods, nearly constant, locked ψ

i.e. trajectory:

- long periods of near synchronization, where phase nearly locked

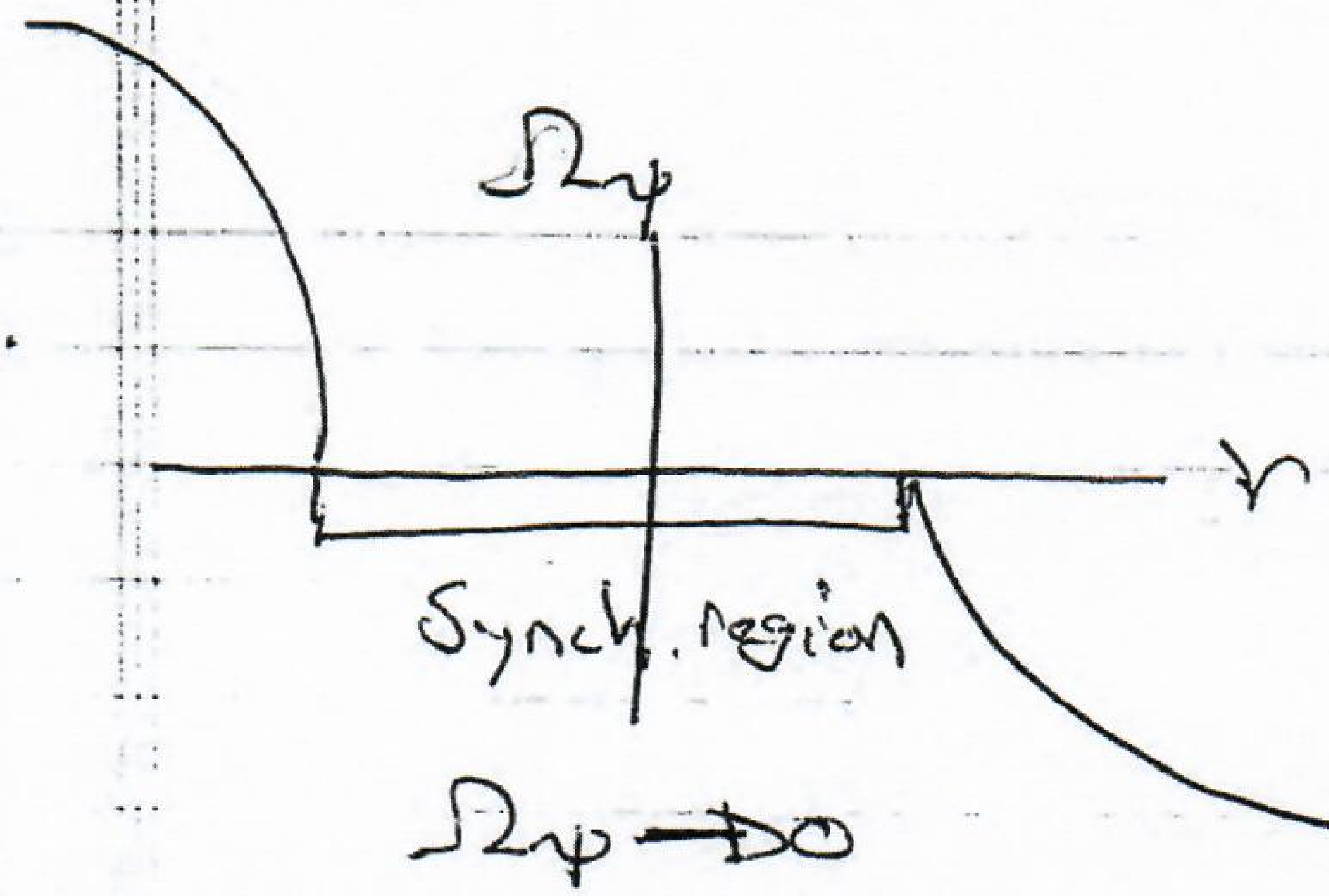
interspersed between

- short periods of rapid phase variation, or slips. Phase rotates by 2π during slip ⟺ "phase slip"

- slip is much longer in duration than ω^{-1}

- transition to synchronization ⇒ time interval between slips increases!

i.e.



$|\Omega_\psi| \sim \sqrt{r - r_M}$ away from synch region.

→ time interval between slips increases approaching bifurcation point (unless time intervals of slips diverge there).