

# Patterns in Convection I, cont'd

1.

— recall, derived  $\left\{ \begin{array}{l} \text{Swift-Hohenberg} \\ \text{Newell-Whitehead} \end{array} \right.$   
envelope eqns. In gradients:

→ linear marginality curve  $\Rightarrow$  curvature  $\leftrightarrow$  envelope approach

→ saturation  $Ra = Ra_{crit} + \epsilon$ ; slightly above marginality; inversion + phase symmetry

→ symmetry of base state,

i.e.  $\underline{z} = \underline{z}_0 \hat{x} + \underline{k}$



key pt.  $\underline{z}_0 \partial_t w = Ra_{crit} + \delta Ra$

$-\epsilon_0^2 (z - z_0)^2$   
"parabolic" marginal curve admits restricted range of modes in dynamics

→ physics: modulation of base state. Patterns observed deviates from linear prediction.

⇒ Newell-Whitehead Eqn.:

2.

$$\gamma_0 \partial_t A = rA + \epsilon_0^2 \left( \partial_x + \frac{1}{2ik_c} \partial_y^2 \right)^2 A - g_{\text{eff}} |A|^2 A$$

for amplitude  $\rightarrow$  prototype of  
(A complex) "Amplitude Eqn."

further  $A = a e^{i\phi}$   
 $a$   $\rightarrow$  amplitude  
 $\phi$   $\rightarrow$  phase

⇒ For  $\partial_y = 0$ ;

$$\gamma_0 \partial_t a = \left( r - \epsilon_0^2 \partial_x^2 \right) a + \epsilon_0^2 \partial_x^2 a - g_{\text{eff}} a^3$$

$$\partial_t \phi = \frac{\epsilon_0^2}{\gamma_0} \left( \partial_x^2 \phi + 2 \frac{\partial_x a}{a} \partial_x \phi \right)$$

For exact, stationary solutions:

$$\partial_x a = 0$$

$$\phi = dkx + \phi_0 \rightarrow \underline{\text{phase winding}}$$

So

$$0 = (r - \epsilon_0^2 (\delta k)^2) a - g a^3$$

W

$$a = \left[ (r - \epsilon_0^2 \delta k^2) / g_{\text{eff}} \right]^{1/2}$$

needs,

$$\delta k < \sqrt{r} / \epsilon_0$$

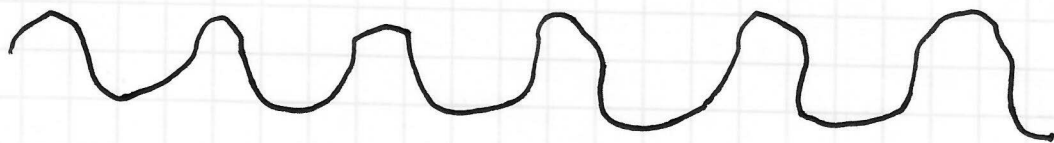
→ long wavelength, as  $r \sim O(\epsilon)$

So → termed "phase winding" solution

$$W = \frac{\pm}{2} \left( a e^{i\phi_0} e^{ik_0 x} e^{i\delta k x} + \text{c.c.} \right)$$

→ weak modulation of amplitude

Now → what type of secondary instabilities might occur?



2.7

b



cells

Consider symmetry:

4.

→ translation

(no boundary in analysis)

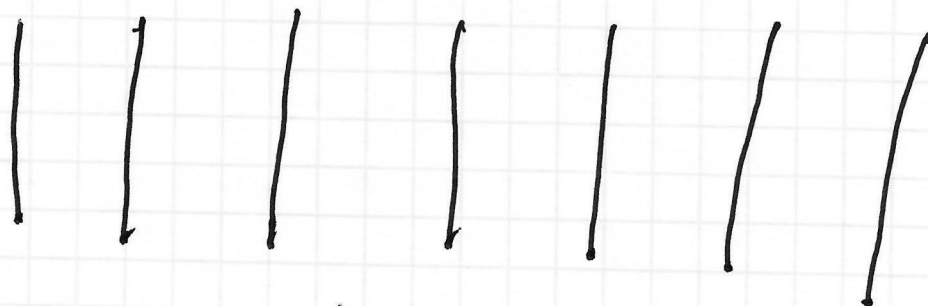
→ rotation

(instability mechanism (primary) invariant to rotation in  $x, y$ )

①

→ Breaking translation

stripe → rolls



cluster



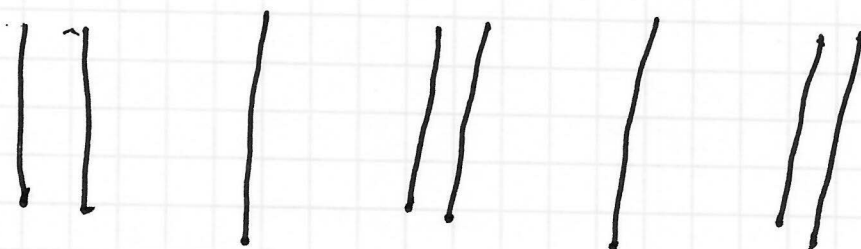
modulation

like spin vortices attract.

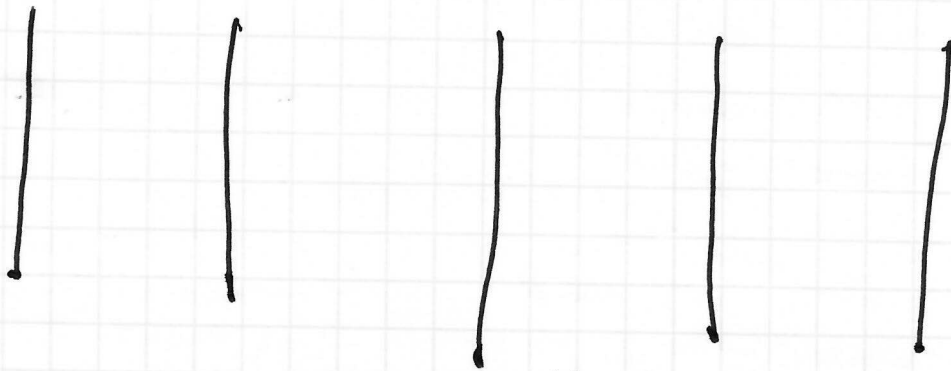
→ bunch.

cluster

→ negative diff.



collapse to roll pairs (i.e. mergers)



final pattern.

Process is one of modulation  $\rightarrow$   
coalescence  $\rightarrow$  condensation

Points:

— modulation brings rolls  
closer, fragments  $\Rightarrow$

— vortex merger (attraction)

— configuration in lowest energy  
(i.e.  $(\nabla A)^2 \downarrow$ )

$\Rightarrow$  instability:

Phenomena referred to as Eckhaus  
Instability  $\rightarrow$  i.e. E.I.

# Calculation:

6.

Have N-W eqn:

$$\tau_0 \partial_t A = rA + \sum_D^2 \left( \partial_x + \frac{1}{2i\kappa_0} \partial_y^2 \right)^2 A - g_{\text{eff}} |A|^2 A$$

$$A(x, y, t) = \tilde{A}(x, y, t) e^{i\kappa x}$$

(recall  $|\kappa| < \sqrt{r}/\epsilon_0$ )  
so  $(\kappa_0 \rightarrow 1)$ ,  $\tau_0 \rightarrow 1$  (general)

$$\frac{d\tilde{A}}{dt} = (r - \kappa^2) \tilde{A} + 2i\kappa (\partial_x - i\partial_y^2) \tilde{A} + (\partial_x - i\partial_y^2)^2 \tilde{A} - |\tilde{A}|^2 \tilde{A}$$

Uniform:  $\tilde{A}_0 = (r - \kappa^2)^{1/2}$

Now,

$$\tilde{A} = \tilde{A} + a \quad (\text{a.b. notation})$$

$\downarrow$   
perturbation about  
uniform phase winding  
state

$$a = u + iv$$

and linearization  $\Rightarrow$

$$\left\{ \begin{aligned} \partial_t u &= \left[ -2(r - \sigma k^2) + \partial_x^2 + \sigma k \partial_y^2 - \sigma y^4 \right] u - (2\sigma k - \partial_y^2) \partial_x u \\ \partial_t v &= (2\sigma k - \partial_y^2) \partial_x u \\ &\quad + (\partial_x^2 + \sigma k \partial_y^2 - \sigma y^4) v \end{aligned} \right.$$

so, everything straight forwardly:

$$u = U e^{\sigma t} \cos(k_x x) \cos(k_y y)$$

$$v = V e^{\sigma t} \sin(k_x x) \cos(k_y y)$$

$\Rightarrow$  dispersion relation:

$$0 = \sigma^2 + 2(r - \sigma k^2) + k_x^2 + k_y^2 \sigma k + \sigma y^4 \sigma + (2(r - \sigma k^2) + k_x^2 + \sigma y^2 \sigma k + \sigma y^4) (k_x^2 + k_y^2 \sigma k + \sigma y^4) - k_x^2 (2\sigma k + \sigma y^2)^2$$

For Eckhaus,  $Z_y = 0$ .

8.

$$s^2 + 2((r - \sigma k^2) + \epsilon_x^2)s + \epsilon_x^2(2(r - 3\sigma k^2) + \epsilon_x^2) = 0$$

$$(s - s_1)(s - s_2) = 0$$

$$s^2 - (s_1 + s_2)s + s_1 s_2 = 0$$

Roots real  $\Rightarrow$  stability of

$$s_1 + s_2 < 0 \quad s_1 < 0, \quad s_2 < 0$$

$$\Rightarrow s_1, s_2 > 0$$

so, instability if:

$$(2(r - 3\sigma k^2) + \epsilon_x^2) < 0$$

$$\epsilon_x^2 < 2(3\sigma k^2 - r)$$

requires:

$$|\sigma k| > \sqrt{r/3}$$



Instability requires:

7

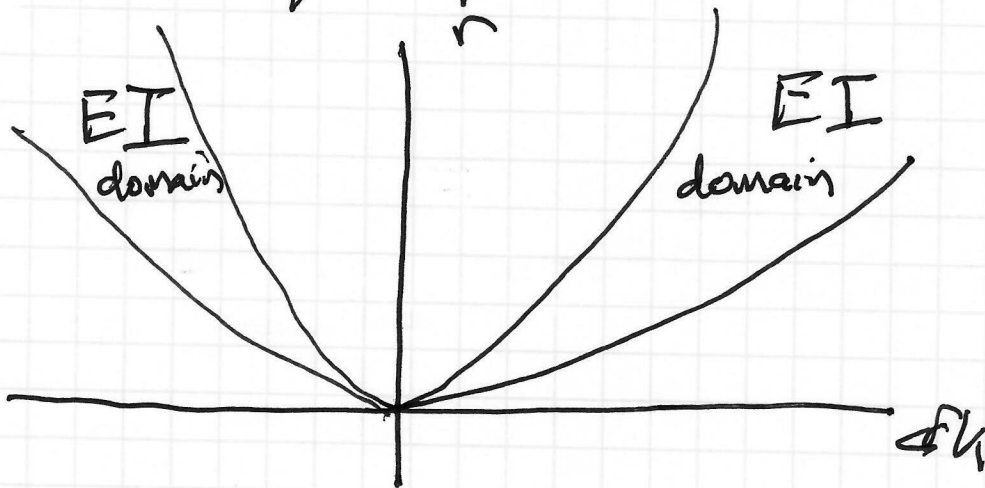
$$|k| < \sqrt{r}/\epsilon_0 \rightarrow \text{phase winding}$$

$$|k| > \sqrt{r/3} \rightarrow \text{instability}$$

$$\Gamma_y = 0 \Rightarrow \text{Eckhaus}$$

$$\epsilon_0 = 1 \quad \sqrt{r/3} < |k| < \sqrt{r}$$

$\rightarrow$  hydrodynamic mode  $\rightarrow$  broken translational invariance



DB:  
can represent  
as negative  
diffn.

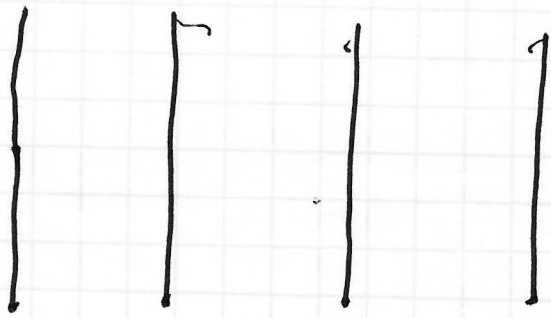
## ② Rotational Invariance

$\rightarrow$  see  
negative diffn  
phase.

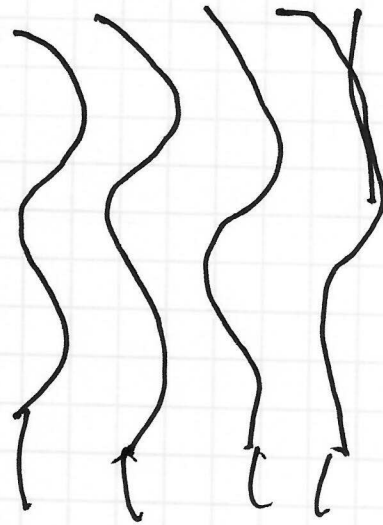
$\rightarrow$  Another symmetry is  
rotational invariance in  $x, y$   
plane for primary instability.

⇒ zig-zag

10.

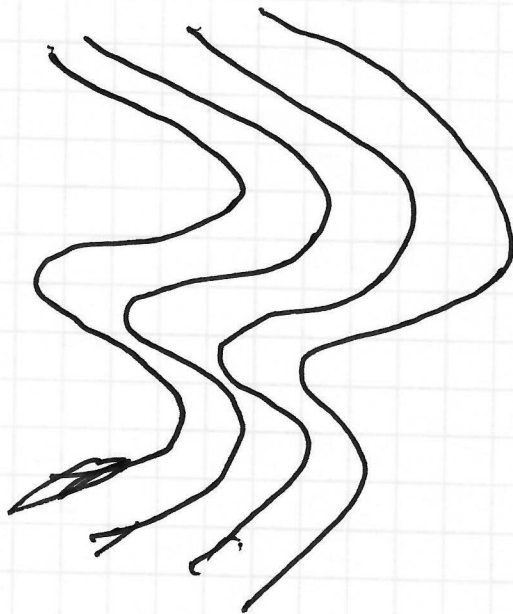


uniform



bending

(roll rotations)  
in variance)



saturation when

$(\partial A / \partial y^2)^2$  high in. → flex.

In analysis, take  $z_x = 0$ ,  
 $z_y$  finite; then:

$$S = -2(n - \sigma k^2) - z_y^2 \sigma k - z_y^4$$

LO.

But  $S_+ = -\epsilon_j^2 (z_j^2 + \delta k)$  11  
 $> 0$

for  $\delta k < 0$

$\delta k + z_j^2 < 0$

s.e

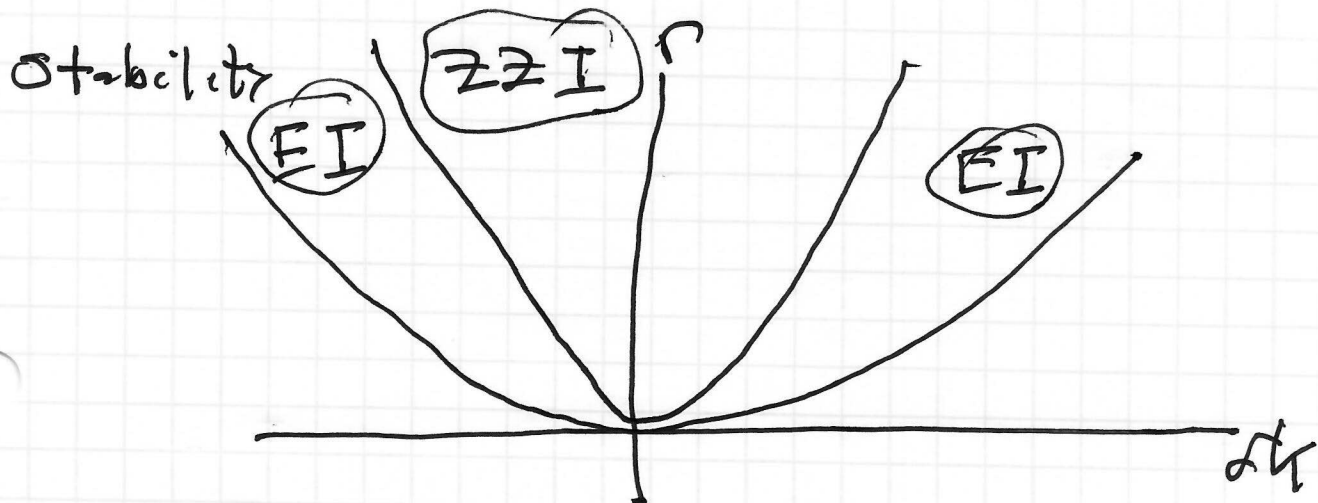
$\delta k < -z_j^2$

$\Rightarrow$  Zig-Zag instability

$\leadsto$  bending.

Eckhaus and Zig-Zag are two typical pattern forming instabilities.

$\Rightarrow$  Types of huge ice-bars.



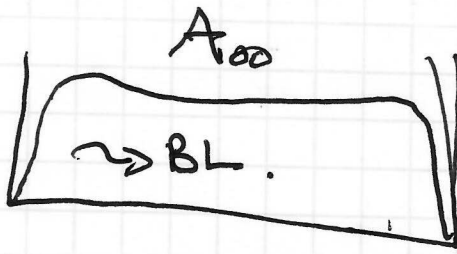
→ other Pattern Problems / Related: 12.

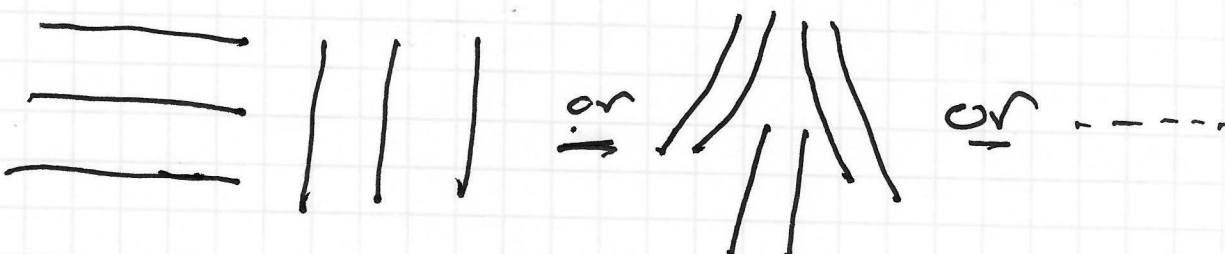
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① → waves ⇒ NLS  
special case of CGL

② → Geometry ⇒ boundary conditions  
on amplitude eqns

③ → Grain boundaries  
dislocations ⇒ pattern  
discontinuity

\*  
④ → Phase Diffusion ⇒ clustering  
(negative diffn.)

⑤  obviously  
quantized  
(h.o. eqn.)

⑥  or ...  
(director field, etc.)

⑦ Waves ⇒ NLS as  
envelope eqn.

C.f. Classic Plasma problem of B.  
Langmuir Turbulance reduces to  
NLS in subsonic case.

But NLS is far more generic.

Also, observe:

dispersion  
↓

CGLE:  $\partial_t A = \nu A + (1 + i\alpha) \partial_x^2 A$

-  $(1 + i\gamma) |A|^2 A$

\*  
NL frequency shift

Now, conservative limit:  $\nu \rightarrow 0$

$|A|, \alpha \gg 1$

$\gamma \nabla A$   
 $\nabla \sim |A|^2$  ↓  
quadratic feedback on mean.

$$*i \partial_t A = \alpha \partial_x^2 A + \gamma |A|^2 A$$

i.e. NLS  $i\hbar \frac{\partial \psi}{\partial t} = H \psi$   $v_0$   
 $= \frac{-\hbar^2}{2m} \nabla^2 \psi + |\psi|^2 \psi$

Point: NLS is generic

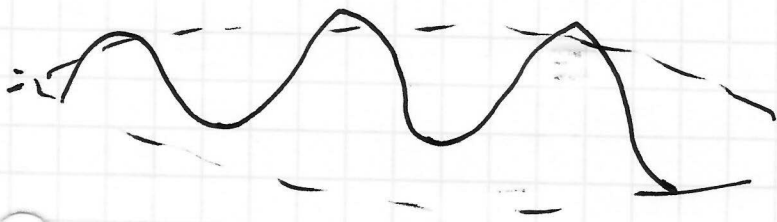
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$\Leftrightarrow$  mean field interaction:

For linear, dispersive wave train:

$$\Phi = \int dk F(k) \exp[ikx - i\omega(k)t]$$

$\omega = \omega(k) \rightarrow$  dispersion relation



wave train

modulation

$k_0 \rightarrow k_0 + \delta k$   
small

$k_0 \rightarrow$  carrier

$$\begin{aligned} \omega &= \omega(k_0 + k - k_0) \\ &= \omega_0 + (k - k_0) \omega_0' + \frac{(k - k_0)^2}{2} \omega_0'' \end{aligned}$$

$$\begin{aligned} \Phi &= \int dk F(k) \exp \left[ ikx - i \left( \omega_0 + (k - k_0) \omega_0' \right. \right. \\ &\quad \left. \left. + \frac{(k - k_0)^2}{2} \omega_0'' \right) t \right] \\ &\stackrel{\text{envelope}}{\downarrow} \\ &\equiv \psi e^{i(k_0 x - \omega_0 t)} \end{aligned}$$

$$\psi = \int dk F(k_0 + k) \exp \left[ ikx - i \left( k \omega_0' + \frac{k^2}{2} \omega_0'' \right) t \right]$$

obviously  $\Rightarrow \psi$  accounts for modulation  
 $\Leftrightarrow$  is envelope

Now,  $\psi$  clearly satisfies:

15.

$$i(\partial_t \psi + \omega_0' \partial_x \psi) + \frac{1}{2} \omega_0'' \partial_x^2 \psi = 0$$

so modulation has dispersion

relation

$$\omega = k \omega_0' + \frac{k^2}{2} \omega_0''$$

i.e.  $\psi = a_0 e^{i(kx - \omega t)}$ .

Also observe: could also entertain

nonlinear frequency shift

(phase invariant)

amplitude

$$\omega = k \omega_0' + \frac{k^2}{2} \omega_0'' - \frac{1}{2} \frac{a^2}{|\psi|^2}$$

(Stokes effect), then  $\psi$

satisfies:

$$i(\partial_t \psi + \omega_0' \partial_x \psi) + \frac{1}{2} \omega_0'' \partial_x^2 \psi + \frac{1}{2} \frac{a^2}{|\psi|^2} \psi = 0.$$

where:  $\psi = a \exp(ikx - i\omega t)$

16.

$\omega = a v$  above.

Now, frame co-moving with  $v_{gr}$

$$\Rightarrow \boxed{i \partial_t \psi + \frac{\omega_D}{2} \partial_x^2 \psi + \gamma |\psi|^2 \psi = 0}$$

NLS.

explains why  
h.o. in  $\partial_x^2$ .

Point:

→ NLS is generic to:

— weakly nonlinear dispersive  
wave train

— nonlinear frequency shift

$$\sim a^2 \sim |\psi|^2$$

→ key point is:

— sign  $\omega_D$

— sign  $\gamma$ .



→ NLS has imaginary coeffs.

17.

⇒ need treat via

$$\psi = A e^{i\phi}$$

so usual crank (linearly):

$$\Omega^2 = \omega''_z A_0^2 k^2 + \frac{\omega''}{4} k^4$$

so instability if:

$$\omega''_z < 0$$

$\omega'' > 0$

⇒ Benjamin - Feir instability

⇒ Modulation grows ..

→ Linear relative of self-focusing.

→ 1D: NLS → soliton solution

→ 3D: collapse → singularity (exact).