68.

i.) Pattern Formation Paradigms I : Epidemic Front Propagation – Fisher Equation

i.) Motivation

→ basic paradigm in nonlinear dynamics: $\left\{ \begin{array}{l} \text{logistic} \\ \text{problem} \end{array} \right.$

map : $X_{n+1} = a X_n (1 - X_n)$ — growth, saturation

continuous system : $\dfrac{dx}{dt} = \gamma_0 x - \alpha x^2$

Fixed pts : $\left\{ \begin{array}{l} x_0 = 0 \\ x_0 = \gamma_0/\alpha \end{array} \right.$

stability: $\left\{ \begin{array}{l} -i\omega = \gamma_0 - 2\alpha x_0 \\ \gamma = \gamma_0 - 2\alpha x_0 \end{array} \right.$

∴ Transition : $x_0 = 0$ (unstable) to $x_0 = \gamma_0/\alpha$ (stable) $\left\{ \begin{array}{l} \text{i.e. growth to} \\ \text{saturation of} \\ \text{population} \end{array} \right.$

→ in spatio-temporal generalization, allow diffusive dispersal of population ρ:

$$\boxed{\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \gamma \rho - \alpha \rho^2} \quad \left\{ \begin{array}{l} \text{Fisher} \\ \text{Equation} \end{array} \right.$$

note: TDGL : $\dfrac{\partial \eta}{\partial t} - D\dfrac{\partial^2 \eta}{\partial x^2} = a(T)\eta - b\eta^3$

→ seek solutions of form:

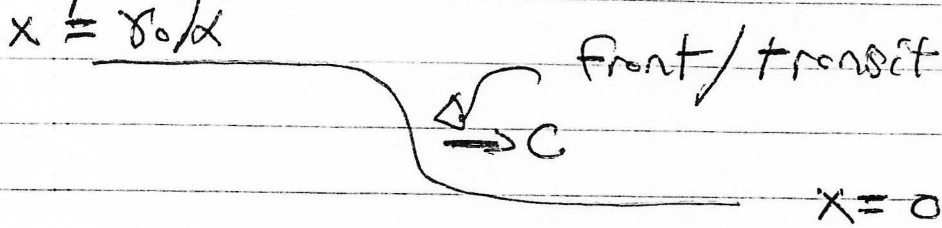
$$\rho = \rho(x - ct)$$

i.e. propagating solution of nonlinear equation

expect:

- propagation driven by $X_0: 0 \to \gamma_0/\alpha$ transition instability

- solution to have form of front → domain wall separating regions of two phases



front / transition layer / domain wall
i.e. ① similar phase transition
② combustion etc.

- seek:

① → structure of solution
② → propagation speed c → { what sets? physics of ε_{min}.
③ → stability of front

ii.) Formulating problem:
if Fisher eqn:

$$\frac{\partial \rho}{\partial t} = k\rho(1-\rho) + D\frac{\partial^2 \rho}{\partial x^2}$$

$t^* = kt$

$x^* = x\,(k/D)^{1/2}$ and omitting $*$ $\Rightarrow$

$$\frac{\partial P}{\partial t} = P(1-P) + \frac{\partial^2 P}{\partial x^2}$$

$P = P(x - ct)$ $\Rightarrow$

$$\left.\begin{array}{l} P'' + cP' + P(1-P) = 0 \\ P(-\infty) = 1, \quad P(\infty) = 0 \end{array}\right\}$$

Now, can analyze via # of strategies:

① dynamical system

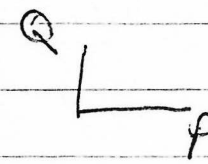
$$\left\{\begin{array}{l} Q = P' \\ Q' = -cQ - P(1-P) \end{array}\right.$$

$\Rightarrow$

$$P' = Q$$
$$Q' = -cQ - P(1-P)$$

and

$$\left.\frac{dQ}{dP} = \frac{-cQ - P(1-P)}{Q}\right\}$$ phase plane trajectories



Observe similarity:

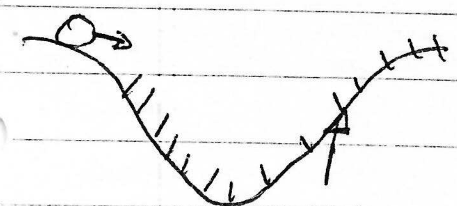
- Fisher Eqn. (generalized) and 1D mechanics

$$-D\frac{\partial^2 \rho}{\partial x^2} - c\frac{\partial \rho}{\partial x} = -\frac{\delta U(\rho)}{\delta \rho}$$ → c to stabilize transition in moving frame

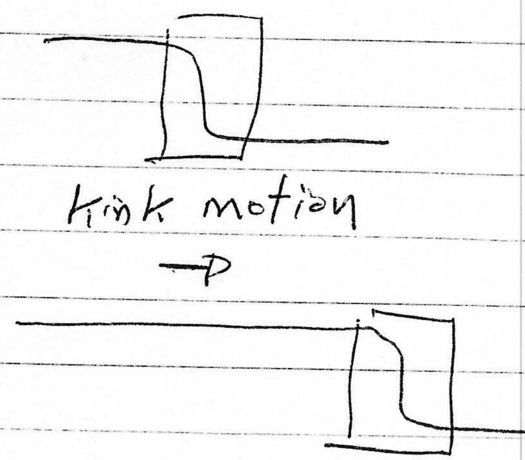
↕ inertia ↕ friction ↕ force

$$m\ddot{x} + \gamma\dot{x} = -\frac{\delta U(x)}{\delta x}$$ → γ drag to balance force

i.e. ball motion



arrival here depends on i.c.'s due friction



∴ can expect:

→ sensitivity of trajectory to initial condition
↕ (i.e. push at $t=0$ to arrive at X_0?)

→ condition for propagation (over/under damping)

Now, trajectories have two critical points:

$$\rho = 0, \quad Q = 0$$
$$\rho = 1, \quad Q = 0$$

Can linearize about these:

$$-\gamma\tilde{\rho} = \tilde{Q}$$

$$-\gamma\tilde{Q} = -c\tilde{Q} - \tilde{\rho} + 2\rho_0\tilde{\rho}$$

For $(0,0)$:

$$-\gamma\tilde{\rho} = \tilde{Q}$$
$$-\gamma\tilde{Q} = -c\tilde{Q} - \tilde{\rho}$$

$$0 = \begin{vmatrix} -\gamma, & -1 \\ 1, & -\gamma - c \end{vmatrix} \Rightarrow \begin{array}{l} \gamma(\gamma + c) + 1 = 0 \\ \gamma^2 + c\gamma + 1 = 0 \end{array}$$

$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 - 4\right)^{1/2}$$

$c \geq c_{min} = 2$ for non-negative definite ρ (avoid oscillation)

For $(0,1)$:

$$0 = \begin{vmatrix} -\gamma, & -1 \\ -1, & -\gamma - c \end{vmatrix} \Rightarrow \gamma(\gamma + c) - 1 = 0$$

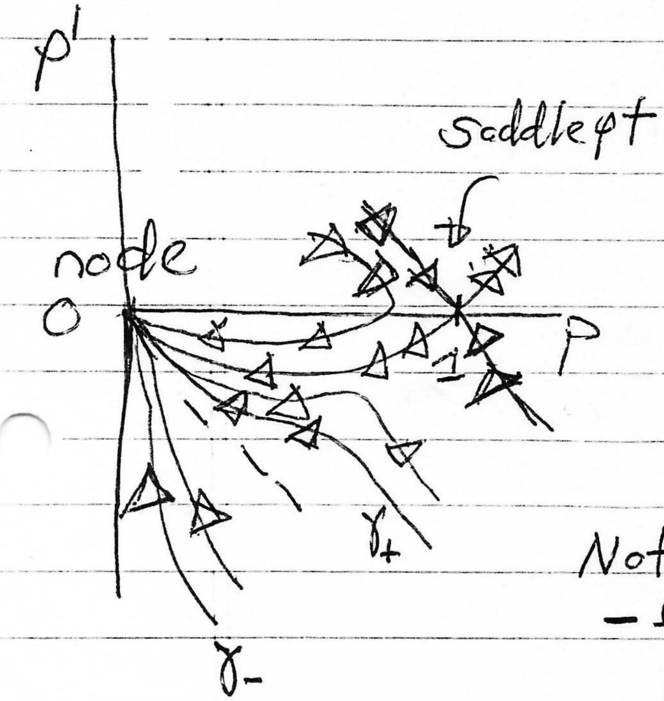
$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 + 4\right)^{1/2}$$

Thus, $(0,0)$: stable node for $c^2 > 4$

stable $\left\{\begin{array}{l}\text{focus} \\ \text{spiral}\end{array}\right.$ for $c^2 < 4$

$(0,1)$: saddle point

$\Rightarrow$ phase plane trajectories:



Clearly, $\exists$ a phase space trajectory from $(1,0) \rightarrow (0,0)$ which

i.) falls in $\rho > 0$

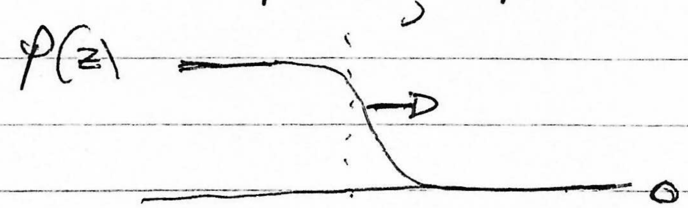
ii.) $\rho' < 0$ (front)

for all wave speeds $c > 2$

$\Rightarrow$ front solution

Note:

- formally, travelling wave solutions exist for $c < c_{min} = 2$, but these are unphysical as ρ oscillates ($\rho < 0$?)
- $c > c_{min}$, solution has $\rho > 0$, $\rho' < 0$ $\Rightarrow$ front

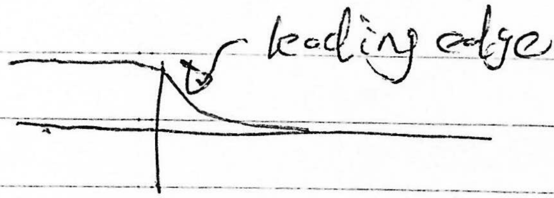


(physical solution)

$\therefore$ analysis establishes minimum speed for propagating front solution $c_{min} = 2(kD)^{1/2}$

leading edge analysis

- consider edge of evolving wave propagating from $-\infty$ to $+\infty$

- 

$$\rho(x,0) \sim Ae^{-\alpha x} \qquad x \to \infty$$

- linearizing Fisher Eqn (about unstable fixed point):

$$\frac{\partial}{\partial t}\rho = \rho + \frac{\partial^2 \rho}{\partial x^2}$$

$$\rho = Ae^{-\alpha(x-ct)}$$ (propagating leading edge)

$$\alpha c = 1 + \alpha^2 \Rightarrow \alpha^2 - \alpha c + 1 = 0$$

$$\alpha = \frac{c}{2} \pm \frac{1}{2}(c^2 - 4)^{1/2}$$

∴ consistency with leading edge hypothesis structure forces $c > c_{min} = 2 \to 2(\kappa D)^{1/2}$

Key Point:

- in fixed frame, instability occurs at each point, as ρ transitions $0 \to 1$
- c_{min} specifies minimal speed such that marginal

Observe:

$$\rightarrow \begin{cases} c_{min} = 2(kD)^{1/2} \\ \Delta x = (D/k)^{1/2} \\ \downarrow \end{cases}$$

kink width

can sharpen kink via $D \downarrow$ or $k \uparrow$ (increase rate local instability)

→ observe that with diffusion: Δx, c_{min} emerge from marginal stability analysis

$$\gamma = k - k'^2 D$$

$$\gamma = 0 \quad \Rightarrow k' \sim 1/\Delta x \sim \left(\frac{k}{D}\right)^{1/2}$$

→ $c_{min} \sim (kD)^{1/2}$ but diffusion → D/L^2

$$\frac{1}{\tau} \sim \frac{c}{L} \sim \left(\frac{kD}{L^2}\right)^{1/2}$$

local transition instability → k

$$1/\tau_{trans} \sim \left(\frac{D}{L^2} k\right)^{1/2}$$

→ geometric mean of $\begin{cases}\text{diffusion} \\ \text{transition}\end{cases}$ time scale

i.e. propagation is synergism of local transition instability with diffusive coupling (spatially) stability maintained $\left(\partial/\partial t \, (\sim \gamma) \rightarrow -c \frac{\partial}{\partial x} \right)$

- leading edge analysis illustrates wave speed dependence on conditions at $x = \pm\infty$.

$\Rightarrow$

Note: KPP proved that if

a.) $P(x,0)$ has compact support

b.) $P(x,0) = P_0(x) > 0$

$$P_0(x) = \begin{cases} 1, & x \leq x_1 \\ 0, & x \geq x_2 \end{cases} \qquad x_1 < x_2$$

c.) $P_0(x)$ continuous $\quad x_1 < x < x_2$

(i.e. kink structure), then:

$P(x,t)$ evolves to $P(x - c_{min} t)$.

key issue: minimum speed is one selected

i.e. Counter-intuitive point is that pattern/front in Fisher Equation which is selected is one with minimum speed (marginal stability!?)

iii.) Front Stability

→ clearly, physically interesting solution should be stable

→ while wave-front unstable to far-field perturbations, KPP thm. suggests insensitivity to i.c. perturbations with compact support /

∴ natural to investigate stability.

For stability $\quad p = p(\underbrace{x-ct}_{\text{front prop. time dependence}}, \underbrace{t}_{\to \text{ instability}})$

$$\frac{\partial p}{\partial t} = p(1-p) + c\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x^2}$$

$$p = p_0(\overbrace{x-ct}^{z}) + \epsilon\tilde{p}(z,t) \qquad (z \equiv x-ct)$$

$$\therefore \quad \frac{\partial \tilde{p}}{\partial t} = \tilde{p} - 2p_0(z)\tilde{p} + c\frac{\partial \tilde{p}}{\partial z} + \frac{\partial^2 \tilde{p}}{\partial z^2}$$

$$\Rightarrow \quad \frac{\partial \tilde{p}}{\partial t} = (1-2p_0(z))\tilde{p} + c\frac{\partial \tilde{p}}{\partial z} + \frac{\partial^2 \tilde{p}}{\partial z^2}$$

Now $\quad \tilde{p} = \tilde{p}(z)e^{-\gamma t}$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \tilde{p}}{\partial z^2} + c\frac{\partial \tilde{p}}{\partial z} + (1+\gamma-2p_0(z))\tilde{p} = 0}$$

eigenmode equation

$\gamma > 0 \Rightarrow$ stable

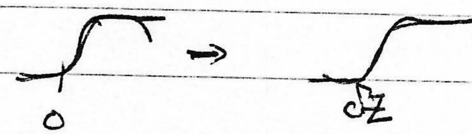
for $\gamma = 0$ have: $\tilde{p}'' + c\tilde{p}' + (1-2p_0(z))\tilde{p} = 0$

observe

$$0 = \frac{\partial^2 \rho}{\partial z^2} + c\,\frac{\partial \rho}{\partial z} + \rho(1-\rho)$$

$\rho = \rho_0(z)$ is solution. Now, consider infinitesimal shift of solution:

$\rho_0(z+\delta z)$



$\Rightarrow$

$$0 = \frac{\partial^2}{\partial z^2}\left(\cancel{\rho_0(z)} + \delta z\,\frac{d\rho_0(z)}{dz}\right) + c\,\frac{\partial}{\partial z}\left(\cancel{\rho_0(z)} + \delta z\,\frac{d\rho_0}{dz}\right)$$

$$+ \cancel{\rho_0(1-\rho_0)} + \left(\frac{d\rho_0}{dz} - 2\rho_0(z)\,\frac{d\rho_0}{dz}\right) + O(\delta z^2)$$

$$= (\rho_0')'' + c(\rho_0')' + \left(1 - 2\rho_0(z)\right)\rho_0' \quad \text{eigenmode at } \gamma = 0.$$

$\therefore$ $\gamma = 0$ is "translation mode" $\Rightarrow$ related to translational invariance of system / momentum conservation of kink.

$\Rightarrow$ for stability, need:

$$\gamma > 0 \Rightarrow \lim_{t\to\infty} \tilde{\rho} = 0$$

$$\gamma = 0 \Rightarrow \lim_{t\to\infty} \tilde{\rho} = \frac{d\rho_0}{dz} \quad \text{(front translation)}$$

Now, substituting $\tilde{p} \to \delta p\, e^{-cz/2}$ $\Rightarrow$

$$\delta p'' + \left(\gamma - \left(2P_0(z) + \frac{c^2}{4} - 1\right)\right)\delta p = 0$$

$$\delta p(\pm L) = 0$$

$\Rightarrow$

$$\gamma = \frac{1}{\left(\int \delta p^2 dz\right)} \left[\int dz \left\{\left(\frac{c^2}{4} - 1 + 2P_0(z)\right)\delta p^2 + \frac{1}{2}(\delta p')^2\right\}\right]$$

some L

$$\gamma \geq 0 \iff c^2 \geq c_{min}^2 = 4 \checkmark$$

i.e. c_{min} emerges from stability analysis for front.

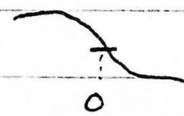
(iv.) Asymptotic Analysis of Nonlinear Problem

$\rightarrow$ would be re-assuring to demonstrate viability of (leading edge) analysis — i.e. obtain analytic form for nonlinear front

$\rightarrow$ proceed via singular perturbation theory approach

No loss of generality to assume:

$\Rightarrow$ $z=0$ is where $P = 1/2$



Then, in front: $\quad y = \frac{z}{c} = \epsilon^{1/2} z \qquad \epsilon = \frac{1}{c^2}$

$$P(z) = g(y)$$

$\therefore$ Fisher eqn. becomes:

$$\epsilon \frac{d^2 g}{dy^2} + \frac{dg}{dy} + g(1-g) = 0$$

$g(-\infty) = 1$
$g(\infty) = 0$
$g(0) = 1/2$
$0 < \epsilon \leq 1/c_{min}^2 = 1/4$

$$g(y, \epsilon) = g_0(y) + \epsilon g_1(y) + \dots$$

$$\Rightarrow \quad \frac{dg_0}{dy} = -g_0(1-g_0)$$

$$\frac{dg_1}{dy} + (1-2g_0)g_1 = \frac{d^2 g_0}{dy^2}$$

$$\begin{cases} g_0(\mp\infty) = \begin{smallmatrix}1\\0\end{smallmatrix}, \quad g_0(0) = 1/2 \\ g_i(\pm\infty) = 0 \\ g_i(0) = 0 \end{cases}$$

$\therefore$

$$\frac{dg_0}{g_0(1-g_0)} = -dy \quad \Rightarrow \int \left(\frac{1}{g_0} + \frac{1}{1-g_0}\right) dg_0 = -y + c$$

$$+\ln g_0 - \ln(1-g_0) = -y + c$$

$$\ln\left(g_0/(1-g_0)\right) = -y + c$$

$$\frac{g_0}{1-g_0} = c\,e^{-y} \qquad g_0 = ce^{-y}(1-g_0)$$

$$g_0(1+ce^{-y}) = ce^{-y}$$

$$\therefore\ g_0 = c/(c+e^{y}) \qquad c = 1 \text{ for B.C.'s } g(0) = 1/2$$

$$= 1/1+e^{y}$$

$$= 1/(1+e^{z/c}) \quad \checkmark$$

For g_1:
$$\frac{dg_1}{dy} - \frac{g_0''}{g_0'}\,g_1 = -g_0''$$

crank ⇒
$$g_1 = e^{y}(1+e^{y})^{-2}\ln\left[\frac{4e^{y}}{(1+e^{y})^{2}}\right]$$

etc.

So

$$P(z,\theta) = \frac{1}{(1+e^{z/c})} + \frac{1}{c^2}\frac{e^{z/c}}{(1+e^{z/c})^2}\ln\left[4e^{z/c}(1+e^{z/c})^{-2}\right]$$

$$+ \cdots$$

Curiously → asymptotics least accurate for $c=2$

but

→ $O(1)$ is excellent fit (few %) to exact numerical solution

$\Rightarrow$ observe: if interested in relative steepness, then asymptotics at $z = 0 \Rightarrow$

$$-P'(0) = \frac{1}{4c} + o\left(\frac{1}{c^5}\right)$$

i.e. $\begin{cases} \text{faster fronts are less steep} \\ \text{slow fronts are more steep} \end{cases}$

Next: Dynamics of Fronts in S-curve Reaction-Diffusion Systems.