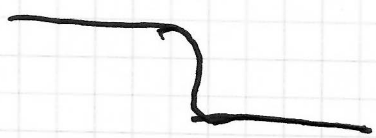
Fronts II: Review Fisher (Unistable)
Bi-stable

Recall:

Turing → $\begin{cases} \text{NL reaction-diffusion system} \\ D=0 \to \text{dynamical system.} \end{cases}$

$\begin{cases} \text{Turing instability results from} \\ Pr \neq 1 \text{ diffusion + perturbation} \\ \text{about fixed point.} \end{cases}$

∴ – if multiple fixed points, notion of a "front" linking differing domains  is naturally of interest.

– In MFE, fronts appear in avalanche modelling, transport barriers, etc.

Fronts come in two (at least) basic categories:

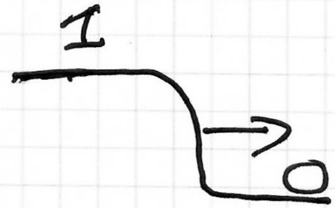
– uni-stable / 2^{nd} order / Fisher
– bi-stable / 1^{st} order / Fitzhugh-Nagumo

For Fisher:

$$\partial_t u = D\,\partial_x^2 u + f(u)$$

$$f(u) = \gamma(u - \alpha u^2)$$

- uni-stable $\begin{cases} u = 0 \text{ unstable} \\ u = 1 \text{ stable} \end{cases}$



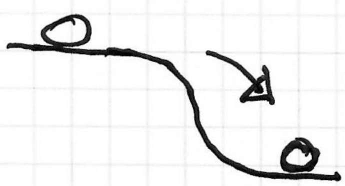
- Front prop. → diffn + γ
  → will show $c = 2\sqrt{\gamma D}$

- $u = \underline{f}(x - ct)$ Key question → what is speed?
  ↳ travelling wave solution

$$-c\underline{f}' = D\underline{f}'' + f(\underline{f})$$

$$-\gamma\dot{x} = m\ddot{x} - \partial V/\partial x$$

c s/t $\underline{}$
γ

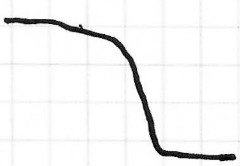


$\left\{\begin{array}{l}\text{ball rolls to} \\ \text{bottom and stops}\end{array}\right.$

i.e. $\left\{\begin{array}{l} c \\ \gamma \end{array}\right.$ to dissipate work done on

- For analysis: Leading edge

<u>i.e.</u>



- ~~treat~~ Front as exponential in moving frame

- accurate for front, not saturated state u.

so $u = A e^{-k(x-ct)}$

⇒ consistency of front solution is
$$c = 2(\gamma D)^{1/2}.$$

speed set the front is marginally stable in co-moving frame

i.e. γ vs $c\partial_x$

— front is stable → can extract speed from stability analysis

— front: asymptotics support leading edge.

Bi-stable Front

→ What of front which connects two stable fixed points?

→ In practice:

— front, in effect, is a switch between 2 states.

- in practice, such systems will have 2 stable, 1 unstable roots

classic example: FN system (Toy version of H-H model of neuron signals)

$$\frac{\partial v}{\partial t} = \underset{\text{bistable}}{\underset{\uparrow}{f(v)}} - w + I_a + D\frac{\partial^2 v}{\partial x^2}$$

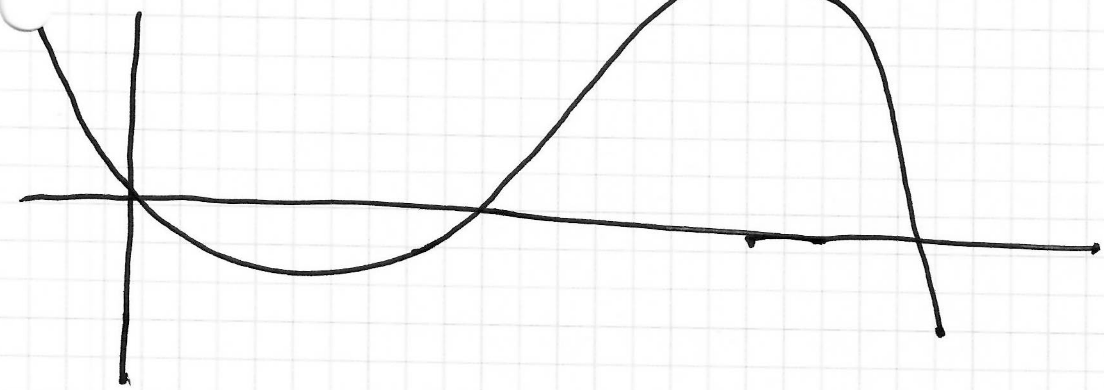
fast (i.e. sodium)

$$\frac{\partial w}{\partial t} = bv - \gamma w$$

slow (i.e. calcium)

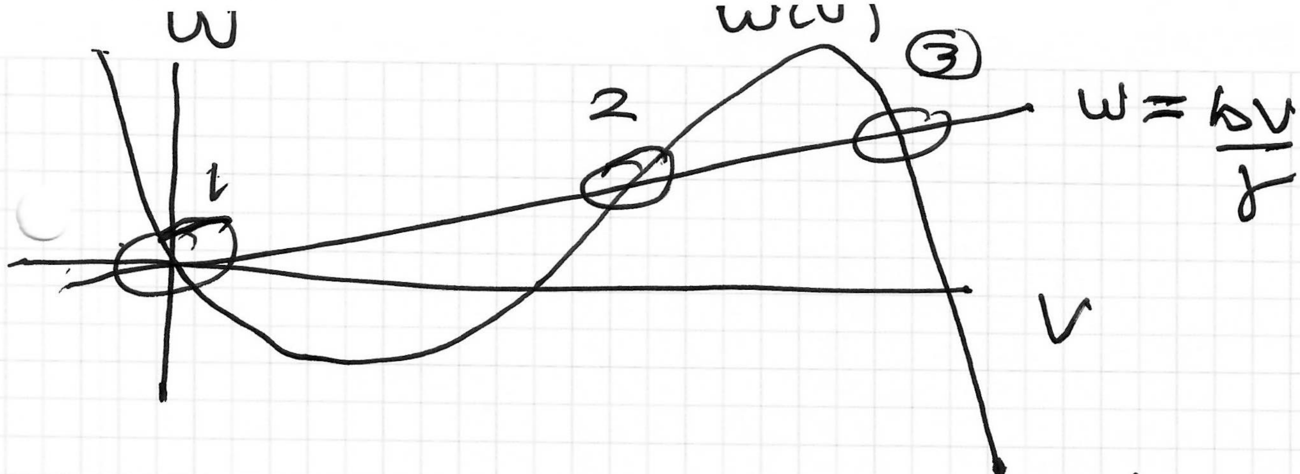
f(v):

f(v)



so (ion I)

$$\left.\begin{aligned} w &= f(v) = W(v) \\ w &= bv/\gamma \end{aligned}\right\} \to \text{fixed pts, states of system}$$



evident: ①, ③ stable: $f' < 0$

② unstable: $f' > 0$

$\therefore$ ⇒ front can link/allow transition between 2 stable fixed pts.

→ unstable root 'powers' front motion (akin Fisher)

Simple/Basic Bistable Problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u) + D\,\partial_x^2 u \\[2ex] f(u) = A(u-u_1)(u_2-u)(u-u_3) \end{cases}$$

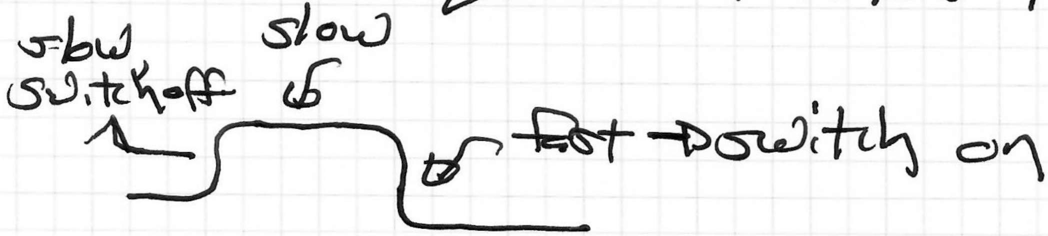
- counter-part of Fisher
- but bistable, i.e. $f'(u_{1,3}) < 0$ → stable

  $f'(u_2) > 0$ → unstable

Aside:

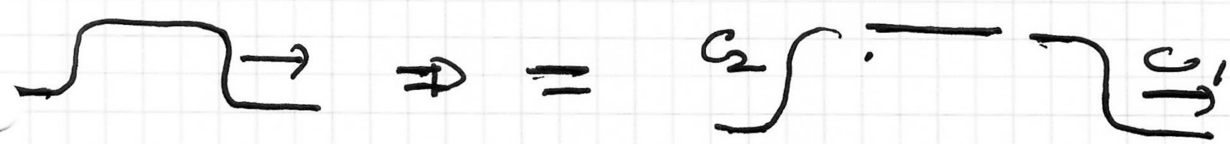
- aim of FN system is to describe pulses in excitable media

- Pulses require multiple time scale



Idea of FN:

switchoff flat top switch on



To maintain identity/coherence of pulse, need: $c_1 = c_2$

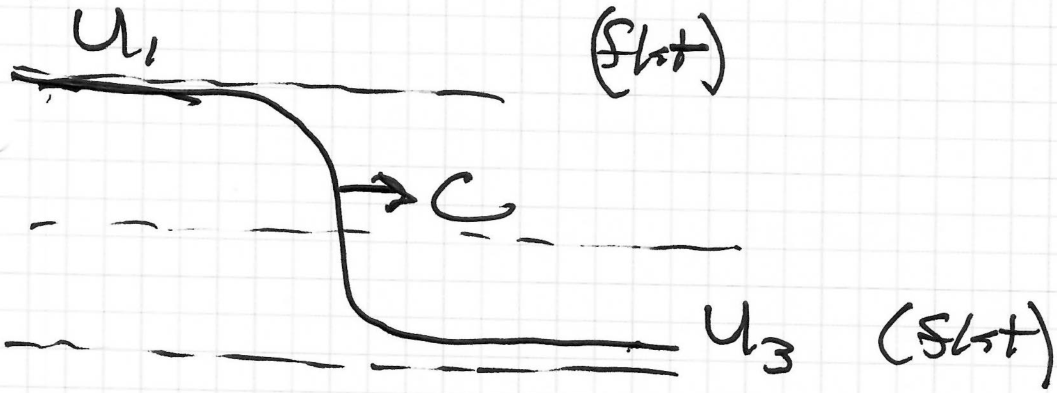
- key element: switch-on/off front.

- switch-on starts from $v = 0$

$$\Rightarrow \boxed{\partial_t u = D\partial_x^2 u + f(u)}$$

simple system above.

As usual:



As usual, look for speed c via traveling wave:

$$u = u(x-ct)$$

$$-cu' = Du'' + f(u)$$

$$u'\left[-cu' = Du'' + f(u)\right]$$

so

$$-c\int_{x(u_3)}^{x(u_1)} u'^2\,dx = D\int_{x(u_3)}^{x(u_1)} u'u''\,dx + \int_{x(u_3)}^{x(u_1)} u' f(u)\,dx.$$

$$\quad ① \qquad\qquad ② \qquad\qquad ③$$

$$① = \int_{x(u_3)}^{x(u_1)} u'u''\,dx = \int_{x(u_3)}^{x(u_1)} \frac{d}{dx}\left(u'^2/2\right)$$

$$= 0,$$ as $u' = 0$ on both sides front.

② $$\int_{x(u_3)}^{x(u_1)} dx\, u' f(u) = \int_{x(u_3)}^{x(u_1)} du\, f(u)$$

$$= \int_{u_3}^{u_1} du\, f(u)$$

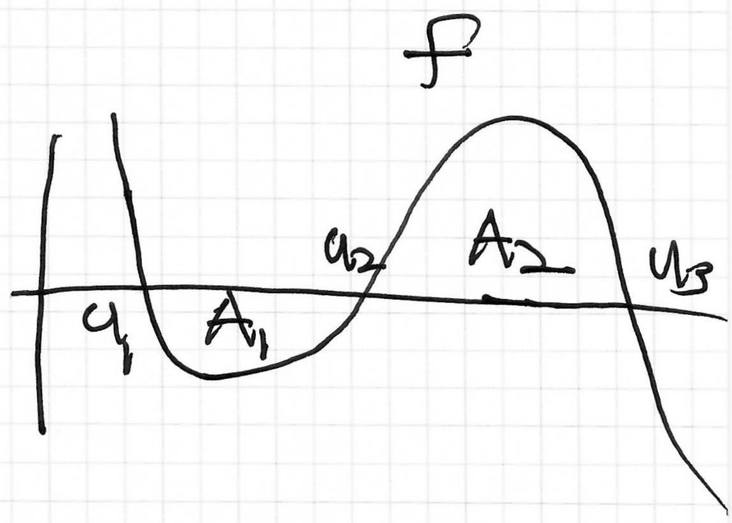
③ $$= -c \underbrace{\int_{-\infty}^{+\infty} u'^2 dx}_{\text{non zero}}$$

$$\boxed{c = -\int_{-u_3}^{u_1} f(u)\, du \Big/ \int_{-\infty}^{+\infty} u'^2 dx}$$

→ front speed.

$$c \sim -\int_{u_3}^{u_1} f(u)\, du$$

$$= \int_{u_1}^{u_3} du\, f(u)$$



i.e. c linked to area under f → curve with 3 zero crossings.

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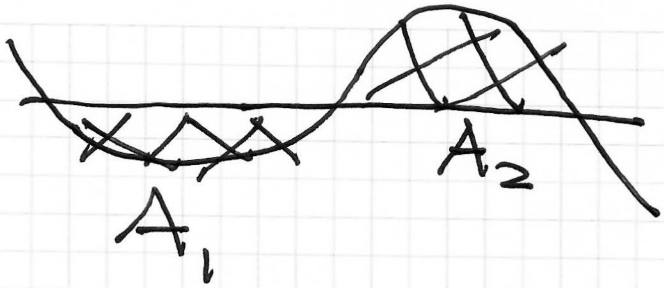
So



$A_2 > A_1 \rightarrow c > 0$ — front advances toward $x > 0$

$A_1 < A_2 \rightarrow c < 0$ — front advances toward $x < 0$

$A_1 = A_2 \rightarrow c = 0$ — front stationary — co-existance

→ Area rule is akin to Maxwell construction in thermodynamics.

Equal areas $\Leftrightarrow$ co-existance of phases.

→ Bistable dynamics is much richer than simple Pac-Man dynamics of Fisher.

$$L(u) = D\left(a(u-u_1)(u-u_3)\right)$$
$$+\left[c\,a\,(u-u_1)(u-u_3) + A\,(u-u_1)(u_2-u)(u-u_3)\right]$$

$$= Da\left(u'\,(u-u_3) + (u-u_1)\,u'\right)$$
$$+ [\quad]$$

$$= Da\left[a(u-u_1)(u-u_3)(u-u_3) + a(u-u_1)^2(u-u_3)\right]$$
$$+ [\quad] = 0$$

re-grouping:

$$L(u) = (u-u_1)(u-u_3)\left[(2Da^2 - A)\,u - \left(Da^2(u_1+u_3) - ca - Au_3\right)\right]$$

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so $L(u) = 0$

$\Rightarrow$ $2Da^2 = A$

$Da^2(u_1 + u_3) - Au_2 - ca = 0$

so $a = (A/2D)^{1/2}$

and

$$\boxed{c = (AD/2)^{1/2}(u_1 - 2u_2 + u_3)}$$

$A \Leftrightarrow \gamma$ $\Rightarrow$ $c \sim (D\gamma)^{1/2}$, as before

and $c \cong \sqrt{D\gamma}\; c(u_1, u_2, u_3)$

param in reaction.

$u_2 = (u_1 + u_3)/2$ $\Rightarrow$ stationary front.

Thus, speed set by parameters in reaction function. Weighted balance between these is the key.

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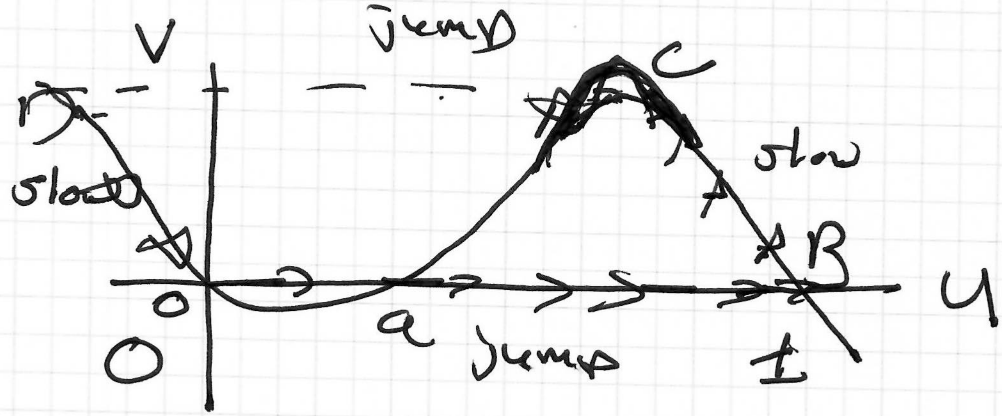
Toward Pulses/Waves

Considers:



0 B C D O

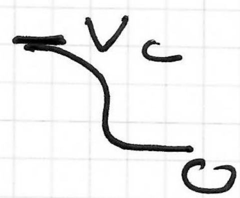
$$\partial_t u = f(u) - v - D\partial_x^2 u$$

$$\partial_t v = bu - \gamma v \qquad 0<a<1$$

for slow eqn:

$$v_t = \epsilon(Lu - Mv)$$

treat v as fixed for fast bifurcations



if $v \to 0$ i.e.

$$\partial_t u = D u_{xx} + f(u)$$

$$f(u) = u(a-u)(u-1)$$

→ leading edge set at $v = 0$

it V finite:

$$u_t = D u_{xx} + f(u) - V_c$$

changes effective reaction function

so

$$c_- = \left(\frac{D}{2}\right)^{1/2}(u_c - 2u_p + u_0)$$

where u_c, u_p, u_0 roots of

$$f(u) = V_c$$

→ speed is function of amplitude of V

then pulse condition:

$$\boxed{c_+ = c_-(V_c)}$$

guarantees that pulse will not disperse → i.e. forward and backward transitions propagate together at same speed. This sets a critical amplitude V_c for pulse to be excited.

slow evolution links the up, back transitions.

" "

- The FN model is basic simple model of pulse in excitable media.
- so constructed from basic element of bistable front.

If time → spirals, etc.

In MFE – "Reaction is in the diffusion."

– More akin Cahn-Hilliard