Contents

1 Patterns

Patterns come in two different flavours: Patterns in time and patterns in space. This part of the notes will concern synchronization, which is patterns in time. In studying synchronization the subject will be explored through the study of the phase dynamics of the following examples:

1) An external force entrains an nonlinear-oscillator, with and without noise.

2) An oscillator entraining another oscillator, ie. coupled oscillators.

3) A network of oscillators, the example being the kuramoto transition, but that is for the next part of these notes.

Through all these we shall see phenomena's such as limit cycles, local synch- and desynchronization, turbulence and more. The central theme here, a theme which runs through all of the following, is the central "battle" between the strength of the entrainment force, and its' frequency, versus the inner frequency of the oscillator. We have two cases:

a) Weak forcing, this lends itself to a perturbative approach.

b) Strong forcing, here the examples are the circle map, and the standard map.

In both cases we shall study the phase dynamics, where the simplest example will be a limit cycle.

1.1 Phase Dynamics of Limit Cycles

In general, we will be working with a M-dim autonomous system [1].

$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}t} = f(\underline{x}), \tag{1}$$

where $\underline{x}$ has dimension M, and f is independent of time. A limit cycle is then a staple, self-substained oscillation such that $\underline{x}(t+T) = \underline{x}(t)$.

This implies that there is a variable ϕ that describes the motion on the cycle, with $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0$, where ω_0 is a natural frequency of self-sustained oscillation. If the cycle is an attractor, i.e. it eats phase volume then we will have $\sum h_i < 0$, where the h_i's are the Lyapunov exponents. Further more since the spacing is fixed on the cycle we will also have that $\exists h_i = 0$. This makes it phase stable but not asymptotically stable. Thus if we consider a small perturbation on a oscillator we expect the excursion induced by the perturbation to be

[1] Autonomous system: system of ordinary differential equations which does not explicitly depend on the independent variable, *Wikipedia*

small "perpendicular" to the limit cycle, but possibly large on or along the cycle.

Limit cycles shall come in two different flavours: The simple non-attracting (weak) limit cycle, which in the canonical coordinates will look something like:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \alpha p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\alpha q. \tag{2}$$

This is not an attractor in phase space, an example being a limit cycle in a Hamiltonian space. More interestingly is the case of the attracting limit cycle.

For a limit cycle in a M-dim system, we will get a $(M-1)$-dim hypersurface as the attracting cycle, this is the isochrome surface. On isochromes flow takes one period, and flow along isochrome rotates attractor at ω_0.

1.1.1 The Complex Ginzburg Landau

The first example of an attracting limit cycle is the complex Ginzburg Landau equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (1 + i\eta)A - (1 + i\alpha)|A|^2 A. \tag{3}$$

This is an equation for a complex amplitude, A. The term on RHS is a linear growth term, the $i\eta A$ is a linear phase frequency. From this is subtracted a non-linear saturation $-|A|^2 A$ and a non-linear frequency shift $-i\alpha|A|^2 A$. Let $A = Re^{i\theta}$, we then have the equations:

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= R(1 - R^2) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \eta - \alpha R^2 \end{aligned} \tag{4}$$

We see that we have an unstable fixpoint at $R = 0$, and a staple limit cycle at $R = 1$.

The equations in eqs. 4 have the solutions for initial conditions R_0,θ_0:

$$\begin{aligned} R(t) &= [1 + (\frac{1 - {R_0}^2}{R_0})e^{-2t}]^{-1/2} \\ \theta(t) &= [\theta_0 + (\eta - \alpha)t - \frac{\alpha}{2}\ln({R_0}^2 + (1 - {R_0}^2)e^{-2t})\,] \end{aligned} \tag{5}$$

In the limit where $t \to \infty$ and $R \to 1$ we have $\theta = \theta_0 + (\eta - \alpha)t - \alpha \ln R_0$. This lets us define $\phi(R, \theta) = \theta - \alpha \ln R$. We then have:

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \frac{\mathrm{d}\theta}{\mathrm{d}t} - \alpha R^{-1}\frac{\mathrm{d}R}{\mathrm{d}t} \\ &= \eta - \alpha, \end{aligned} \tag{6}$$

by using our result from eqs. 4. We see that ϕ rotates uniformly. This is an isochrome, and rewriting $\phi(R, \theta) = \theta - \alpha \ln R \implies \theta_0 = \theta - \alpha \ln R \implies R \sim \exp[\frac{\theta - \theta_0}{\alpha}]$, we get a spiral. Isochromes is spirals attracted to the limit cycle thus we could describe the phase field using a set of isochromes.

1.1.2 Van-der-Pol equation

The other example is the Van-der-Pol equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\mu\frac{\mathrm{d}x}{\mathrm{d}t}(1-\beta x^2) + {\omega_0}^2 x = 0 \tag{7}$$

Writing $y = \dot{x}$ we have:

$$\dot{x} = y \tag{8}$$
$$\dot{y} = 2\mu y(1-\beta x^2) - {\omega_0}^2 x \tag{9}$$

Here if we set $x = r\,\cos\phi$ and $y = r\,\sin\phi$, we get the two equations:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\mu(r^2\cos^2\phi - 1)r\sin^2\phi, \tag{10}$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -1 - \mu(r^2\cos^2\phi - 1)\cos\phi\sin\phi. \tag{11}$$

These one can treat perturbatively, via method of averaging, to get the resulting figures 1, 2.

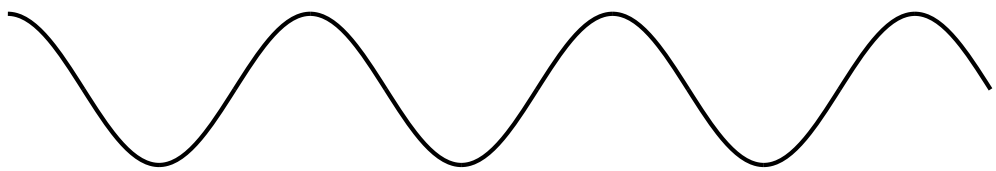

Figure 1: For $\mu \ll 1$ we have a smooth NL oscillation.

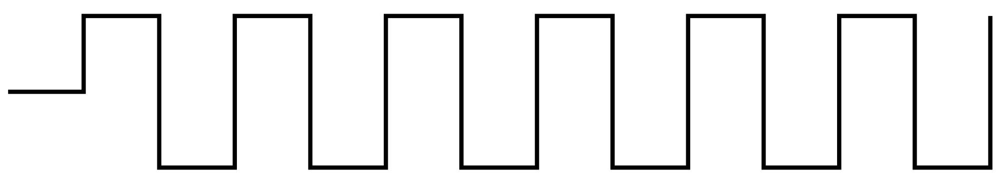

Figure 2: For $\mu \gg 1$ we have a sawtooth oscillation.

1.1.3 Small perturbation from cycle

Now in order to say something general we want to look at $\phi(\underline{x})$ a phase in some neighbourhood of an attracting limit cycle. We have:

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \sum_k \frac{\partial\phi}{\partial x_k}\frac{\mathrm{d}x_k}{\mathrm{d}t} \\ &= \sum_k \frac{\partial\phi}{\partial x_k} f_k(x) \\ &= \omega_0\end{aligned} \tag{12}$$

If we let: $\frac{\mathrm{d}x}{\mathrm{d}t} = f_k(\underline{x}) + \epsilon p(\underline{x}, t)$ then:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sum_k \frac{\partial\phi}{\partial x_k}(f_k(\underline{x}) + \epsilon p_k(\underline{x}, t)) \tag{13}$$

To the lowest order we have $\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_k \frac{\partial \phi}{\partial x_k} f_k(\underline{x})$. Letting $\lim_{t\to\infty} x(t) = x_0$ where x_0 is on the limit cycle we then have x as a function of ϕ with $\phi \Rightarrow \phi_0 + \omega_0 t$. We write to the first order:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 + \epsilon Q(\phi, t), \tag{14}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega_0 + \epsilon Q(\phi_0 + \omega_0 t, t), \tag{15}$$

where Q acts like an external force with its' own frequency ω, and in eq. 15 we have substituted the unperturbed angle on the limit cycle in. If we can write Q, as:

$$\begin{aligned} Q &= \sum_{l,k} a_{l,k} e^{ik\phi} e^{il\omega t} \\ &= \sum_{l,k} a_{l,k} e^{ik\phi_0} e^{i(k\omega_0 + l\omega)t}. \end{aligned} \tag{16}$$

The most important terms here will be the ones giving steady Q, which induce phase singularities. We see that if $k\omega_0 + l\omega \simeq 0 \mod 2\pi$ we have resonances.

The question we can ask now would be: How does the phase evolve? Which frequency wins the fight? Will we get quasi periodicity, or phase locking?

The simple case is for $\omega \sim \omega_0$ but $\omega \neq \omega_0$. We have:

$$Q \equiv q(\phi - \omega t), \tag{17}$$

since the $k = -l$ becomes dominant. If we then define $\psi = \phi - \omega t$ we get:

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\mathrm{d}\phi}{\mathrm{d}t} - \omega \\ &= -\nu + \epsilon q(\psi), \end{aligned} \tag{18}$$

with the mismatch $\nu \equiv \omega - \omega_0$, and $\epsilon q(\psi)$ being the strength of the interaction. This is the simplified phase equation. It's predicated on the existence of a dominant frequency. Now we can directly see that the fight is between ν and $\epsilon q(\psi)$.

Synchronization will show up as phase locking when $\psi = \psi_s$ s.t. $\frac{\mathrm{d}\psi_s}{\mathrm{d}t} = 0 \implies \nu = \epsilon q(\psi_s)$. To see if this fenomena will be stable, one looks at $\psi = \psi_s + \delta\psi \implies \frac{\mathrm{d}\psi_s}{\mathrm{d}t} = q'(\psi_s)\mathrm{d}t$. For:

$$q'(\psi_s) < 0 \implies \text{stable} \tag{19}$$

$$q'(\psi_s) > 0 \implies \text{unstable} \tag{20}$$

The range of the synchronization is set by ϵ and q. In general you will have a synch region $\epsilon q_{min} < \nu < \epsilon q_{max}$ where you will have fix points in stable and unstable pairs, see fig. 3. With the onset of synchronization being a bifurcation.

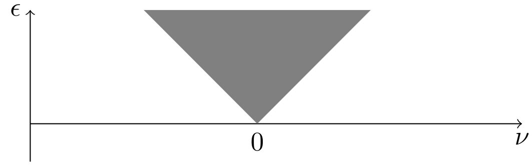

Figure 3: Figure showing the synchronization region. The boundaries are indeed straight lines as shown.

Now for ν outside of the synchronization region we can define a time:

$$t = \int^{\psi} \frac{\mathrm{d}\psi'}{[\epsilon q(\psi') - \nu]}. \tag{21}$$

In this region we expect to see quasi-periodicity with the the beat period:

$$T_\psi = \left| \int_0^{2\pi} \frac{\mathrm{d}\psi}{\epsilon q(\psi) - \nu} \right|, \tag{22}$$

which lets one define the beat frequency:

$$\Omega_\psi = \frac{2\pi}{T_\psi} \tag{23}$$

which gives as the actually observed frequency, the time average:

$$\langle \dot{\phi} \rangle = \omega + \Omega_\psi. \tag{24}$$

If we want to look at what happens close to a bifurcation, we have for example at $\nu_{max} \sim \epsilon q_{max}$:

$$\begin{aligned} \epsilon q(\psi) - \nu &= \epsilon q(\psi) - \nu_{max} - (\nu - \nu_{max}) \\ &= \epsilon q(\psi_{max}) + \epsilon q'(\psi_{max}) \cdot (\psi - \psi_{max}) + \frac{1}{2}\epsilon q''(\psi_{max}) \cdot (\psi - \psi_{max})^2 - \nu_{max} - (\nu - \nu_{max}) \\ &= \frac{1}{2}\epsilon q''(\psi_{max}) \cdot (\psi - \psi_{max})^2 - (\nu - \nu_{max}), \end{aligned} \tag{25}$$

Then we have:

$$T_\psi = \int_0^{2\pi} \frac{\mathrm{d}\psi}{\frac{1}{2}\epsilon q''(\psi_{max}) \cdot (\psi - \psi_{max})^2 - (\nu - \nu_{max})} \tag{26}$$

$$\simeq [\epsilon q''(\psi_{max})(\nu - \nu_m)]^{1/2}. \tag{27}$$

The integral is dominated by the $\psi \simeq \psi_{max}$ giving the above result. The beat frequency is then $\Omega_\psi \sim \sqrt{\epsilon(\nu - \nu_{max})}$. The beat frequency is plotted, in fig. 4 to show how it behaves close to synchronization, the flat part being the synch region.

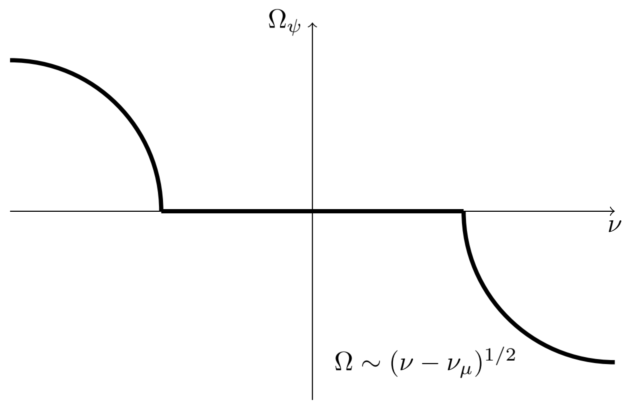

Figure 4: The beat frequency as a function of ν around the synchronization region (flat part).

The frequency slows near the bifurcation point, spending a long time near ψ_m. This will be the frequency of the phase jumps or phase slips to other states of synchronization. This behaviour is show in fig. 5. Near synchronization there will be long periods of nearly constant ψ with brief phase slips in between of period Ω_ψ. As you go to synchrony the time between the slips increases.

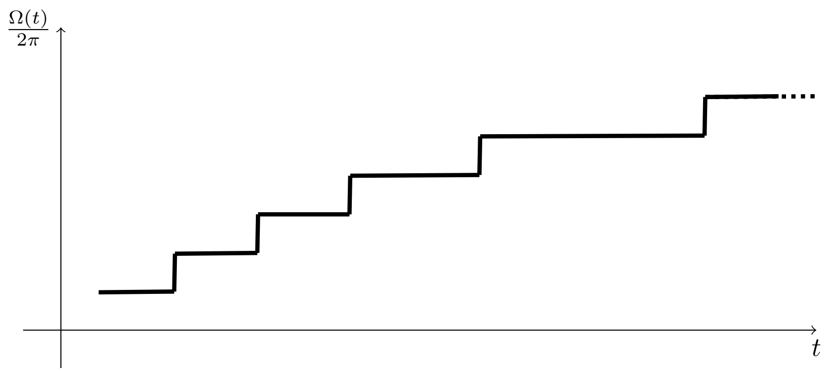

Figure 5: Plot of $\psi = \phi - \omega t$ over time. It shows periods of nearly constant ψ with brief phase slips in between.

1.2 Two Coupled Oscillators

In this section our starting point will be the work we did above for the oscillator with an entrainment force. Here we replace the external entrainment force for a coupling force between two oscillators. Specifically each oscillator has angle variables ϕ_1, ϕ_2, and $\phi_i = \omega_i t$, i.e. they have a stable limit cycle each. Then without loss of generality we can write the functions:

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = \omega_1 + \epsilon Q_1(\phi_1, \phi_2)$$
$$\frac{\mathrm{d}\phi_2}{\mathrm{d}t} = \omega_2 + \epsilon Q_2(\phi_1, \phi_2), \tag{28}$$

where the Q's are periodic in 2π, and the cycles of ϕ_1, ϕ_2 define a 2D torus. As we did for the single oscillator example the entrainment force will be written as:

$$Q_1(\phi_1, \phi_2) = \sum_{k,l} a_1^{k,l} e^{ik\phi_1} e^{il\phi_2} \tag{29}$$
$$Q_2(\phi_1, \phi_2) = \sum_{k,l} a_2^{k,l} e^{ik\phi_1} e^{il\phi_2} \tag{30}$$

In this we can express the ϕ_i with $\omega_i t$, to get:

$$Q_j(\phi_1, \phi_2) = \sum_{k,l} a_j^{k,l} e^{i(\phi_0 + (k\omega_1 + lk\omega_2))}. \tag{31}$$

The dominant terms will be those with $k\omega_1 + l\omega_2 \simeq 0$. If we further take $\frac{\omega_1}{\omega_2} = \frac{m}{n}$, we get resonances at $k = nj$ and $l = -mj$. This let us write the equations governing the phase dynamics as:

$$\frac{\mathrm{d}\phi_j}{\mathrm{d}t} = \omega_j + \epsilon q_j(n\phi_1 - m\phi_2), \tag{32}$$

with:

$$q_1(n\phi_1 - m\phi_2) = \sum_k a_1^{nk,-mk} e^{ik(n\phi_1 - m\phi_2)}$$
$$q_2(n\phi_1 - m\phi_2) = \sum_k a_2^{mk,-nk} e^{ik(m\phi_2 - n\phi_1)}, \tag{33}$$

which naturally leads one to define:

$$\psi \equiv n\phi_1 - m\phi_2. \tag{34}$$

The new equation is then:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\nu + \epsilon q(\psi), \tag{35}$$

where $\nu \equiv m\omega_2 - n\omega_1$ and $q(\psi) \equiv nq_1(\psi) - mq_2(\psi)$. We see that this now has reduced to the one oscillator problem.

1.3 The Nonlinear Oscillator

This section shall treat the general nonlinear oscillator, and on the way show how and when it reduces to the complex Ginzburg Landau (CGL) equation. Furthermore a more general case of coupled oscillators will be explain using the equations of amplitude we shall derive. The CGL is relevant in a lot of different instances, and has also in this course seen a lot traction. If we start with a nonlinear oscillator:

$$\ddot{x} + {\omega_0}^2 x = f(x, \dot{x}) + \epsilon p(t), \tag{36}$$

here $f(x, \dot{x})$ is the nonlinear function and $p(t)$ is a forcing term with it's own frequency ω. We seek a solution of the form:

$$x(t) = \frac{1}{2}(A(t)e^{i\omega t} + c.c.), \tag{37}$$

where $A(t)$ is an amplitude, which is not necessarily slowly varying. We rewrite eq 36 as:

$$\ddot{x} + \omega^2 x = (\omega^2 - {\omega_0}^2)x + f(x, \dot{x}) + \epsilon p(t), \tag{38}$$

in order to write:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x + (\omega^2 - {\omega_0}^2)x + f(x, y) + \epsilon p(t), \end{aligned} \tag{39}$$

which **if**:

$$y(t) = \frac{1}{2}(i\omega A(t)e^{i\omega t} + c.c.) \tag{40}$$

$$\Longrightarrow$$

$$0 = \frac{1}{2}(\dot{A}(t)e^{i\omega t} + c.c.) \tag{41}$$

This lets one rewrite the amplitude equation:

$$\dot{y}(t) = (1/2)(-\omega^2 A(t)e^{i\omega t} + c.c) + (i\omega/2)(\dot{A}(t)e^{i\omega t} - c.c) \tag{42}$$

$$= -\omega^2 x + (\omega^2 - {\omega_0}^2)x + f(x, y) + \epsilon p(t) \tag{43}$$

$$= -\omega^2(\frac{1}{2}(A(t)e^{i\omega t} + c.c.)) + (\omega^2 - {\omega_0}^2)x + f(x, y) + \epsilon p(t) \tag{44}$$

$$\Longrightarrow$$

$$\dot{A}(t) = \frac{e^{-i\omega t}}{i\omega}[(\omega^2 - {\omega_0}^2)x + f(x, y) + \epsilon p(t)]. \tag{45}$$

Which we get by equating eqs. 42 and 44 and using the result from 41. This amplitude equation consist, on the RHS, of a mismatch term $(\omega^2 - {\omega_0}^2)x$, the nonlinear term $f(x, y)$, and the forcing $\epsilon p(t)$. We are interested in the largest, and slowest varying terms on the RHS, so the analysis is done by eliminating the fast oscillating terms via averaging on a scale where we can neglect terms on scale of ω (akin to method of averaging). The three terms will be dealt with in order, each time substituting the expressions for x and y (eqs. 37 and 40).

First, $\dot{A}_1(t) = \frac{e^{-i\omega t}}{i\omega}(\omega^2 - {\omega_0}^2)x$. Averaging we get:

$$\begin{aligned} \dot{A}_1(t) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\tau \frac{e^{-i\omega\tau}}{i\omega}(\omega^2 - {\omega_0}^2)(\frac{1}{2}(A(t)e^{i\omega\tau} + c.c.)) \\ &= \frac{(\omega^2 - {\omega_0}^2)}{2i\omega}A(t) \end{aligned} \tag{46}$$

Secondly, $\dot{A}_2(t) = \frac{e^{-i\omega t}}{i\omega} f(x,y)$. Using that $f(x,y) = \sum_{n,m} c_{n,m}(A(t)e^{i\omega t})^n(\overline{A}(t)e^{-i\omega t})^m$ we get:

$$\dot{A}_2(t) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\tau \frac{e^{-i\omega\tau}}{i\omega}\sum_{n,m} c_{n,m}(A(t)e^{i\omega\tau})^n(\overline{A}(t)e^{-i\omega\tau})^m, \tag{47}$$

the relevant terms will be the ones with $n+m-1=0 \implies m=n-1$ which leaves us with the terms with the factors of the form $A^n\overline{A}^{n-1} = A(A\overline{A})^i$. This leads us to write $A_2(t) = g(|A(t)|^2)A(t)$. Generally $g(|A|^2)$ set by the problem, but the simplest choice would be:

$$\dot{A}_2(t) = \mu A(t) - (\gamma + i\kappa)|A(t)|^2 A(t), \tag{48}$$

here μ is a linear growth term, $(\gamma + i\kappa)$ being the lowest nonlinear term. If we have $\mu, \gamma > 0$ we have a supercritical bifurcation whether as $\mu, \gamma < 0$ lead sub-critical bifurcation we need h.o. to saturate.

Finally $A_3(t) = \frac{e^{-i\omega t}}{i\omega}\epsilon p(t)$ will if we write $p(t) = \sum_n (p_n e^{in\omega t} + c.c.)$:

$$\begin{aligned}\dot{A}_3(t) &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}\tau \frac{e^{-i\omega\tau}}{i\omega}\epsilon p(t) \\ &= -i\epsilon E.\end{aligned} \tag{49}$$

Gathering all the terms we get:

$$\dot{A}(t) = -i\frac{(\omega^2 - {\omega_0}^2)}{2\omega}A(t) + \mu A(t) - (\gamma + i\kappa)|A(t)|^2 A(t) - i\epsilon E, \tag{50}$$

where we can see that we recover the structure of the CGL. We have a term for the mismatch, a term for growth and then the two nonlinear terms which are respectively a saturation term and a frequency shift. The last term is the driving of the oscillator. The derivation is generic to form of nonlinear oscillator. In absence of forcing we recover the Landau-Stuart:

$$\dot{A} = (1 + i\eta)A - (1 + i\alpha)|A|^2 A. \tag{51}$$

For the model to be valid one need $|\omega - \omega_0| \ll \omega_0$ and $\mu \ll \omega_0$ in order to make sure, respectively, that the NL terms are small and the instability around the fixed point $A = 0$ is small. Furthermore $|A|^2 \leq \mu/\gamma$ but $|A|^2 \sim \mu/\gamma$ in order to ensure that the terms representing growth and nonlinear saturation are comparable. This requires small growth, and in practice the model is thus only valid near marginality.

1.4 The Nonlinear Coupled Oscillators

If we consider two weak coupled oscillators, where we in eq. 36 replace the forcing term to obtain the following:

$$\begin{aligned}\ddot{x}_1 + {\omega_1}^2 x_1 &= f_1(x_1, \dot{x}_1) + D_1(x_2 - x_1) + B_1(\dot{x}_2 - \dot{x}_1) \\ \ddot{x}_2 + {\omega_2}^2 x_2 &= f_2(x_2, \dot{x}_2) + D_2(x_1 - x_2) + B_2(\dot{x}_1 - \dot{x}_2),\end{aligned} \tag{52}$$

where we have a linear coupling in the differences between the two phases. The aim of this section is then to give a link between the structure of the coupling and the macro-phenomena that might show. If we use the variables defined in eqs 37, 40, we can, as before (eq. 50), get the amplitude equations, via averaging:

$$\begin{aligned}\dot{A}_1 &= -i\Delta_1 A_1 + \mu_1 A_1 - (\gamma_1 + i\kappa_1)|A_1|^2 A_1 + (\beta_1 + i\delta_1)(A_2 - A_1) \\ \dot{A}_2 &= -i\Delta_2 A_2 + \mu_2 A_2 - (\gamma_2 + i\kappa_2)|A_2|^2 A_2 + (\beta_2 + i\delta_2)(A_1 - A_2),\end{aligned} \tag{53}$$

where we have introduced $\Delta_j = \omega_j - \omega$ to represent the mismatch. The coupling is here the respective terms $\pm(\beta_j + i\delta_j)(A_2(t) - A_1(t))$, where the β's are dissipative terms from the B_j coupling and the δ's are reactive, and from D_j. Now writing $A_j = R_j e^{i\phi_j}$ and $\psi = \phi_2 - \phi_1$ we can write:

$$\dot{R}_1 = \mu_1 R_1(1 - \gamma_1 {R_1}^2) + \beta_1(R_2\cos\psi - R_1) - \delta_1 R_2 \sin\psi \tag{54}$$
$$\dot{R}_2 = \mu_2 R_2(1 - \gamma_2 {R_2}^2) + \beta_2(R_1\cos\psi - R_2) + \delta_2 R_1 \sin\psi \tag{55}$$
$$\dot{\psi} = -\nu + \mu_1\alpha_1 {R_1}^2 - \mu_2\alpha_2 {R_2}^2 + (\delta_2\frac{R_1}{R_2} - \delta_1\frac{R_2}{R_1})\cos\psi + \delta_1 - \delta_2 - (\beta_1\frac{R_2}{R_1} + \beta_2\frac{R_1}{R_2})\sin\psi, \tag{56}$$

with $\nu = \omega_2 - \omega_1$. In order to progress further, the following assumptions is made:

$$\mu_1 = \mu_2 = \mu \tag{57}$$
$$t \to t/\mu \tag{58}$$
$$A \to \frac{A}{(\gamma/\mu)^{1/2}} \tag{59}$$
$$\beta, \delta \ \textit{normalized to}\ \mu \tag{60}$$
$$\alpha \ \textit{normalized to}\ \gamma/\mu. \tag{61}$$

This reduces the system of equations to:

$$\dot{R}_1 = R_1(1 - {R_1}^2) + \beta(R_2\cos\psi - R_1) - \delta R_2 \sin\psi \tag{62}$$
$$\dot{R}_2 = R_2(1 - {R_2}^2) + \beta(R_1\cos\psi - R_2) + \delta R_1 \sin\psi \tag{63}$$
$$\dot{\psi} = -\nu + \alpha({R_1}^2 - {R_2}^2) + \delta(\frac{R_1}{R_2} - \frac{R_2}{R_1})\cos\psi - \beta(\frac{R_2}{R_1} + \frac{R_1}{R_2})\sin\psi. \tag{64}$$

Now we can give our parameters the following interpretations: α is a nonlinear frequency shift, β represents a dissipative coupling and δ a reactive coupling. This system exhibit (at least) two different phenomena: oscillation death/quenching and attractive/repulsive interaction.

Oscillation death. For large β, ν R_1, R_2 becomes stable, effectively killing the oscillations. To see this set $\delta \equiv 0$ and let $\omega = \frac{(\omega_1+\omega_2)}{2} \implies \Delta_1 = -\Delta_2 = \Delta$. Neglecting the nonlinear terms we can write the amplitude equations as:

$$\begin{aligned}\dot{A}_1 &= (-i\Delta + \mu)A_1 + \beta(A_2 - A_1)\\ \dot{A}_2 &= (i\Delta + \mu)A_2 + \beta(A_1 - A_2).\end{aligned} \tag{65}$$

Now if we perturb about $A_1 = A_2 = 0$ with:

$$\begin{bmatrix}A_1(t)\\A_2(t)\end{bmatrix} = \begin{bmatrix}A_1(0)\\A_2(0)\end{bmatrix} e^{\lambda t}, \tag{66}$$

solving gives:

$$\lambda = \mu - \beta \pm \sqrt{\beta^2 - \Delta^2}. \tag{67}$$

In order to get the quenching we need $\lambda < 0$ which implies:

$$\mu < \beta \tag{68}$$
$$\beta < (\mu^2 + \Delta^2)/2\mu \implies \beta < \Delta^2/2\mu + ... \tag{69}$$

The first condition is from the fact that the diffusive coupling introduces dissipation in each oscillator. The second condition makes the detuning so large as to make forcing from the other oscillator unable to excite.

Attractive/repulsive interaction. Assuming β, δ small, by making a small excursion can writing:

$$R_j = 1 + r_j, \qquad r_j \ll 1, \tag{70}$$

get the equations:

$$\dot{r}_1 = -2r_1 + \beta(\cos\psi - 1) - \delta\sin\psi$$
$$\dot{r}_2 = -2r_2 + \beta(\cos\psi - 1) + \delta\sin\psi. \tag{71}$$

When we have strong damping $\dot{r}_j = 0$ this reduces to:

$$r_1 = \frac{\beta}{2}(\cos\psi - 1) - \frac{\delta}{2}\sin\psi$$
$$r_2 = \frac{\beta}{2}(\cos\psi - 1) + \frac{\delta}{2}\sin\psi, \tag{72}$$

which finally by inserting into the phase equation,$\psi = \phi_2 - \phi_1$ gives us.

$$\dot{\psi} = -\nu - 2(\beta + \alpha\delta)\sin\psi. \tag{73}$$

If $\nu = 0$ then for $(\beta + \alpha\delta) > 0$ implies that $\psi = 0$ is a stable fix point, thus synchronizing the two oscillators. On the other hand for $(\beta + \alpha\delta) < 0$ implies that $\psi = \pi$ is a stable fix point, again also a synchronized point. The first case is will then attract the oscillators to each other whereas the second repulse.

To summarize, the β terms from the B's is a form of dissipative coupling which will drive the to oscillators towards synchronization, or attraction. The δ's, from the D's is a reactive coupling, which has no effect on isochronous oscillators ($\nu = 0$), but will be attractive or repulsive depending on sign otherwise.

1.5 Synchronization with Noise

In this section, we will look at the oscillator with noise. So far everything presented has been deterministic, but this is due to change. This requires us to change our approach in order to accommodate the element of randomness that is introduced. The starting point will be eq. 18, representing the phase dynamics, but with an added term representing noise. This gives the Langevin equation for a particle in the potential V:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\nu + \epsilon q(\psi) + \varepsilon(t) \tag{74}$$
$$= -\frac{\mathrm{d}V(\psi)}{\mathrm{d}\psi} + \varepsilon(t) \tag{75}$$

with:

$$V(\psi) = \nu\psi - \epsilon\int^{\psi} q(x)\mathrm{d}x \tag{76}$$

The staple fixpoints of the equation above is for:

$$\frac{\mathrm{d}V}{\mathrm{d}\psi} = 0, \qquad \frac{\mathrm{d}^2V}{\mathrm{d}\psi^2} > 0 \tag{77}$$

This depends on the structure of $V(\psi)$. If the potential is structure as in fig. 6, it is possible to have noise induced phase slips from one ψ_{s1} to another ψ_{s2}. If $V(\psi)$ is smooth with no local minima, the "best" one can hope for is quasi-periodic motion. This has an obvious parallel to Kramer's problem, and the question is, what does it take to overcome the energy barrier, and with what probability? Furthermore we would like to know an average phase rotation frequency:

$$\Omega_\psi = \langle\dot{\psi}\rangle = \int \mathrm{d}\psi P(\psi)\left(-\frac{\mathrm{d}V}{\mathrm{d}\psi}\right) \tag{78}$$

where $P(\psi)$ is a pdf. Here it is important to distinguish between white and coloured (bounded) noise. For white noise we have $\langle\varepsilon(t_1)\varepsilon(t_2)\rangle = {\varepsilon_0}^2\delta(t_2 - t_1)$, and the phasekicks introduced by the noise will be Gaussian. This makes large kicks possible whereas in the case of coloured noise the kicks will be restricted. In the noisy

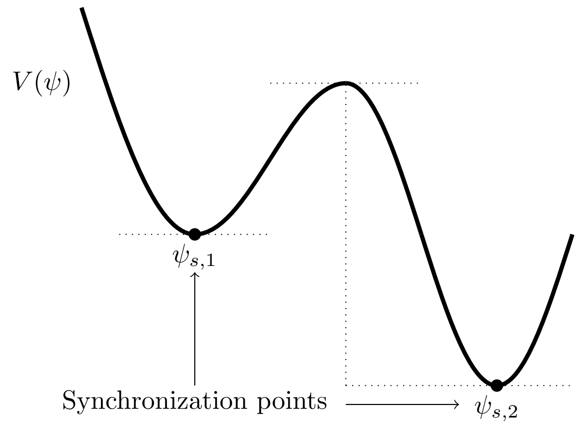

Figure 6: A potential $V(\psi)$ where there is local minima which corresponds to states of synchronization.

environment the condition for synchronization, namely $\Omega_\psi = 0$ needs to be change to Ω_ψ small but finite. Also Ω_ψ needs to be treated statistically.

In order to calculate the probability of one phase slip one needs the probability flux which can be gotten from the Fokker-Planck theory with white noise. We have:

$$\frac{\partial P(\psi)}{\partial t} = -\frac{\partial}{\partial \psi}\left\{\left\langle\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle P - \frac{\partial}{\partial\psi} DP\right\}, \tag{79}$$

with $D = \frac{\langle\delta\psi\delta\psi\rangle}{2\Delta\epsilon}$ and $\delta\psi = \psi - \langle\psi\rangle$. Furthermore, with:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\psi = \varepsilon(t) \tag{80}$$

$$D = \langle\varepsilon^2\rangle = D_0 \tag{81}$$

$$\langle\frac{\mathrm{d}\psi}{\mathrm{d}t}\rangle = -\nu + \epsilon q \tag{82}$$

we can write:

$$\frac{\partial P(\psi)}{\partial t} = -\frac{\partial}{\partial\psi}\left\{(-\nu + \epsilon q(\psi))P - D_0\frac{\partial P}{\partial\psi}\right\}. \tag{83}$$

Equivalently we can introduce Γ_ψ the probability flux:

$$\frac{\partial P}{\partial\psi} + \frac{\partial}{\partial\psi}\Gamma_\psi = 0, \tag{84}$$

with:

$$\langle\Gamma_\psi\rangle = \frac{1}{2\pi}\int_0^{2\pi}\Gamma_\psi \mathrm{d}\psi \tag{85}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{\mathrm{d}V}{\mathrm{d}\psi}P - D_0\frac{\partial\rho}{\partial\psi}\right]\mathrm{d}\psi \tag{86}$$

$$= \frac{1}{2\pi}\langle\Omega_\psi\rangle. \tag{87}$$

Now to solve we use stationarity and periodicity of P, $p(\psi + 2\pi) = p(\psi)$, to write:

$$\frac{1}{\rho}\frac{\partial P}{\partial\psi} = -\frac{1}{\rho_0}\frac{\mathrm{d}V}{\mathrm{d}\psi} \quad\Longrightarrow\quad \log P = -\frac{V}{\epsilon_0^2} + \text{constant} \tag{88}$$

$$\Longrightarrow\quad P = c\int_q^{\psi+2\pi}\mathrm{d}\psi\frac{\exp[V(\dot\psi - V(\psi)]}{D_0} \tag{89}$$

For the Adler equation $q(\psi) = \sin\psi$, we can Fourier analyse P:

$$P = \sum_{n=-\infty}^{\infty} P_n e^{in\psi}, \tag{90}$$

and using that we are looking for stationary solutions, i.e. independent of time we can write:

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial \psi}\Gamma_\psi = 0 \implies \Gamma_\psi\bigg|_{ss} = \Gamma\delta_{n,0}, \tag{91}$$

which leads to:

$$\Gamma\delta_{n,0} = -(inD_0 + \nu)P_n + \frac{\epsilon}{2i}(P_{n-1} - P_{n+1}). \tag{92}$$

$$n = 0 \implies \Gamma = -\nu P_0 + \frac{\epsilon}{2i}(P_{-1} - P_{+1}) \tag{93}$$

$$n \neq 0 \implies \Gamma = 0 = -(in{\varepsilon_0}^2 + \nu)P_n + \frac{\epsilon}{2i}(P_{n-1} - P_{n+1}), \tag{94}$$

normalization then requires $P_0 = \frac{1}{2\pi}$ and we also have $P_{real} \implies P_{-n} = P_n*$. We have:

$$\Gamma = -\frac{\nu}{2\pi} - \epsilon\Im P_1 \tag{95}$$

$$\implies$$

$$\Omega_\psi = 2\pi\langle\Gamma_\psi\rangle = -\nu - 2\pi\epsilon\Im P_1. \tag{96}$$

This is the statistical slip frequency. To get P_1 we notice:

$$0 = -(in{\varepsilon_0}^2 + \nu)P_n + \frac{\epsilon}{2i}(P_{n-1} - P_{n+1}) \tag{97}$$

$$\implies$$

$$\frac{P_n}{P_{n-1}} = \frac{1}{(in{\varepsilon_0}^2 + \nu)\frac{2i}{\epsilon} + \frac{P_{n+1}}{P_n}} \tag{98}$$

$$\implies$$

$$P_1 = \frac{1/(2\pi)}{(i{\varepsilon_0}^2 + \nu)\frac{2i}{\epsilon} + \frac{1}{\nu + 2i\epsilon^2}\frac{2i}{\epsilon} + \frac{P_3}{P_2}}. \tag{99}$$

So we get a continued fraction which we can calculate to large n. And the flux $\langle\Gamma_\psi\rangle \to P_1$. As an interesting aside we can also see that the lyapunov exponent can be found by writing:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\nu + \epsilon\sin\psi + \varepsilon, \tag{100}$$

so

$$\frac{\mathrm{d}\delta\psi}{\mathrm{d}t} = \epsilon\cos\psi\delta\psi, \tag{101}$$

which let us write:

$$h = \left\langle \frac{1}{\delta\psi}\frac{\mathrm{d}\delta\psi}{\mathrm{d}t} \right\rangle = \epsilon\langle\cos\phi\rangle = 2\pi\Re P_1. \tag{102}$$

Now ${\epsilon_0}^2 = 0 \implies h = 0$ unless synchronized and ${\epsilon_0}^2 \neq 0 \implies h < 0$. Thus the effect of noise is to increase the plateau of Ω_ψ, see fig. 7, thus softening the synchronization. Ways to extend this work would be to look at synchronization by quasi-harmonic stochastic force or to look at mutual synchronization of noisy oscillators.

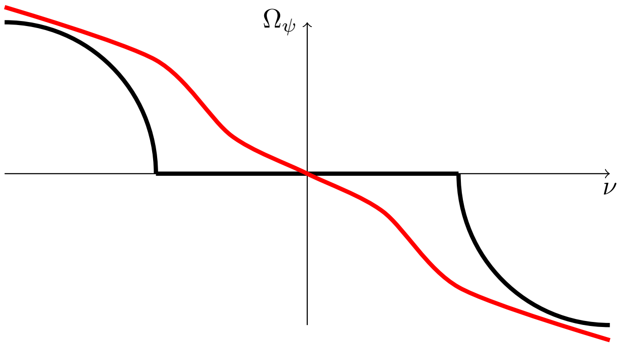

Figure 7: The beat frequency as a function of ν around the synchronization region (flat part). We see that compared to the case without noise that the noise softens the synchronization.

1.6 References

The notes I have tried to cover are the following:

- Lecture 6a: Basics of Synch 1
- Lecture 6b: Basics of Synch 2
- Lecture 6c: Synch-Technical Aside
- Lecture 7: Locking and Quasi-periodicity for Coupled Oscillators, Oscillator with Noise.

Further reading: Review of patterns near threshold-Cross,Hohenberg