# PHYS 221A Lecture Notes

# The Fisher equation and fronts

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## Introduction

Over the last couple of lectures, we have been introduced to reaction-diffusion systems, i.e. systems in which a set of scalar fields that undergo interconversion processes, governed by rates, while spreading over space with the corresponding diffusion coefficients. Moreover, we learned that, for a pair of reacting fields, a difference in such diffusion coefficients produces instabilities that eventually lead to pattern formation. It becomes then interesting to look at the way states propagate in such systems.

A typical reaction-diffusion system is described by equations of the form:

$$\frac{\partial c}{\partial t} = \gamma F(c) + D\nabla^2 c.$$

When $F(c)$ is non -linear, the system depicts layers of fixed points. Said layers are connected through fronts whose motion describes the propagation from unstable fixed points to stable ones (Figure 1).

## The Fisher-KPP equation

At this point, it becomes relevant to ask for the factors determining the structure of the fronts, i.e. their speed, width, and stability. In order to do so we will study the simplest non-linear reaction diffusion equation, the Fisher-KPP (Kolmogorov-Petrovski-Puskinov):

$$\frac{\partial P}{\partial t} = \gamma P - bP^2 + D\frac{\partial^2 P}{\partial x^2} \tag{1}$$

The reaction part of the equation can be understood by taking the predator-prey model:

$$\begin{aligned} \dot{u} &= auv - bu \quad \text{(predator)} \\ \dot{v} &= cv - duv \quad \text{(prey)}, \end{aligned}$$

and enslaving the prey to the predator ($u = \alpha v$). In this way, the set of equations adopts the form of equation (1). Moreover, with the change of variable $P^* = \frac{b}{\gamma}P$, equation (1) becomes:

$$\frac{\partial P}{\partial t} = \gamma P\,(1 - P) + D\frac{\partial^2 P}{\partial x^2}. \tag{2}$$

Where the * has been dropped for clarity. By ignoring the diffusion term, it is possible to identify the logistic differential equation, i.e. the continuous realization of the logistic map:

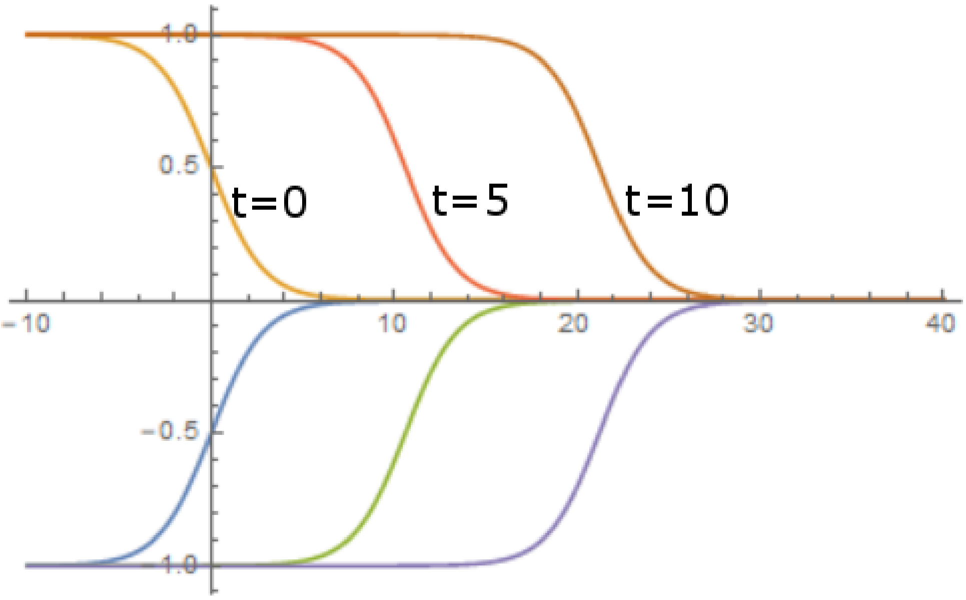


Figure 1: Plots of solutions of the Time Dependent Ginzburg Landau equation at different times showing the front moving from the unstable fixed points $P = \pm 1$ to the stable one $P = 0$

$x_{n+1} = ax_n(1-x_n)$. Both analog models yield to the conclusion that the underlying mechanism is that of the growth of a self-limiting reproductive population. In fact, the equation including diffusion was first suggested by Ronald Fisher in 1937 as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population.

It can be easily proven that $P_0 = 0$ and $P_0 = 1$ are fixed points of equation (2). For $\gamma$ purely imaginary they both are centers of neutral stability, whereas for $\gamma > 0$, $P_0 = 0$ is unstable and $P_0 = 1$ is stable (Fig 2). We will concern ourselves to the second case, corresponding to populations, therefore negative values of $P$ are physically meaningless. Consequently, the wave front traveling solutions should inhabit the $0 \leq P \leq 1$ interval.

# Traveling front analysis

Since we are looking for propagating solutions, we try the form:

$$P = P(x - ct).$$

With which equation (2) becomes:

$$P'' + \tfrac{c}{D}P' + \tfrac{\gamma}{D}P\,(1 - Q) = 0, \tag{3}$$

with boundary conditions

$$P(-\infty) = 0 \quad P(\infty) = 1.$$

As we shall see, there is a variety of ways to analyze this problem.

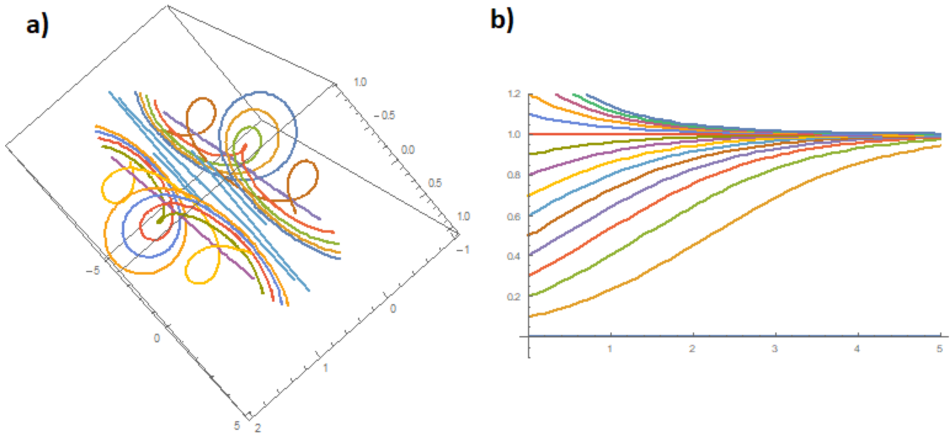


Figure 2: Fixed points of the logistic equation (a) Parametric plot showing the complex plane as function of the initial conditions of the logistic equation with imaginary coefficient. The fixed points are centers thus show marginal stability. (b) Solutions of the logistic equation with real coefficient at several initial conditions. $P = 0$ is an unstable fixed point whereas $P = 1$ is stable

## Dynamical system

Upon a closer look, equation (3) is reminiscent of an equation of motion with damping:

$$\begin{array}{cccc} -DP'' & -cP' & = & -\dfrac{\delta u(P)}{\delta P} \\ \updownarrow & \updownarrow & & \updownarrow \\ \text{inertia} & \text{friction} & & \text{force} \\ m\ddot{x} & +\gamma\dot{x} & = & -\dfrac{\delta u\left(x\right)}{\delta x} \end{array}$$

With this simile in mind it is possible to argue that just as the damping balances the force in the equation of motion, the role of the speed is to stabilize the transition in the moving frame. In the same way that in a damped periodic mechanical system there is a threshold for the damping over which the system will not reach a given point (fig 3), we can expect that there will be a set of requirements to ensure the transition between fixed points. Furthermore, the analogy can be pushed to the extent that the front propagation can be studied in the same way as damped systems are by exploring the trajectories on a phase plane. In this situation we will inquire the phase plane $(P, Q)$ where $Q = P$. Therefore:

$$Q' = -\tfrac{c}{D}Q + \tfrac{\gamma}{D}P\left(P-1\right),$$

and the trajectories are of the form:

$$\frac{dQ}{dP} = \frac{-cQ + \gamma P\left(P-1\right)}{DQ}. \tag{4}$$

This expression has two singular points at $(0, 0)$ and $(1, 0)$, evidently corresponding to the steady states. Linearization around the singular points yields to:

$$\frac{d}{dz}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} -c & -\gamma \\ D & 0 \end{pmatrix}_{(0,0)} \begin{pmatrix} Q \\ P \end{pmatrix}$$
$$\frac{d}{dz}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} -c & \gamma \\ D & 0 \end{pmatrix}_{(1,0)} \begin{pmatrix} Q \\ P \end{pmatrix},$$

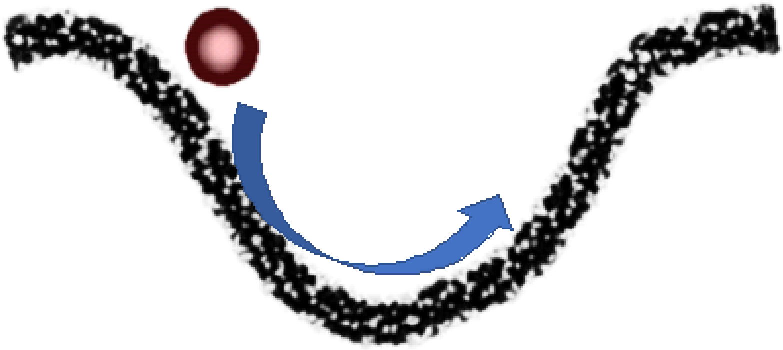

Figure 3: Damped periodic system. If the surface is rough enough the ball will never reach the same height unless pushed.

where $z = x - ct$.

These systems are characterized by the eigenvalues:

$$\lambda_{\pm}(0,0) = \frac{-c \pm \sqrt{c^2 - 4D\gamma}}{2} \qquad \lambda_{\pm}(1,0) = \frac{-c \pm \sqrt{c^2 + 4D\gamma}}{2}. \tag{5}$$

This result indicates that the singular point (0,0) is stable. And depending upon whether $c^2 > 4D$ or $c^2 < 4D$ the point corresponds to a node or a spiral respectively. On the other hand (1,0) is a saddle point (figure 4) It becomes clear that, around the origin, the minimum possible velocity is $c_{\text{min}} = 2\sqrt{D\gamma}$ The former analysis concludes that trajectories from (0,0) to (1,0) are found as long as $P > 0$, $P < 0$ and $c > 2\sqrt{D\gamma}$ (formally there are also trajectories for $P < 0$, but as discussed before, this situation is unphysical).

## Leading edge

An alternative to the phase plane analysis can be realized by assuming that the propagation of the front is governed by its leading front. Near to the unstable fixed point $P \approx 0$, which means that $P^2$ is negligible, and equation (2) adopts the linear form:

$$\frac{\partial P}{\partial t} = \gamma P + D\frac{\partial^2 P}{\partial x^2}. \tag{6}$$

Considering a traveling wave solution:

$$\begin{aligned} P(x,0) = &\quad Ae^{-\alpha x} \\ P(x,t) = &\quad Ae^{-\alpha(x-ct)}, \end{aligned} \tag{7}$$

yields:

$$\alpha c = \gamma + D\alpha^2. \tag{8}$$

Therefore:

$$\begin{aligned} D\alpha^2 - c\alpha + \gamma = 0 \\ \alpha = \tfrac{c \pm \sqrt{c^2 - 4D\gamma}}{2D}, \end{aligned}$$

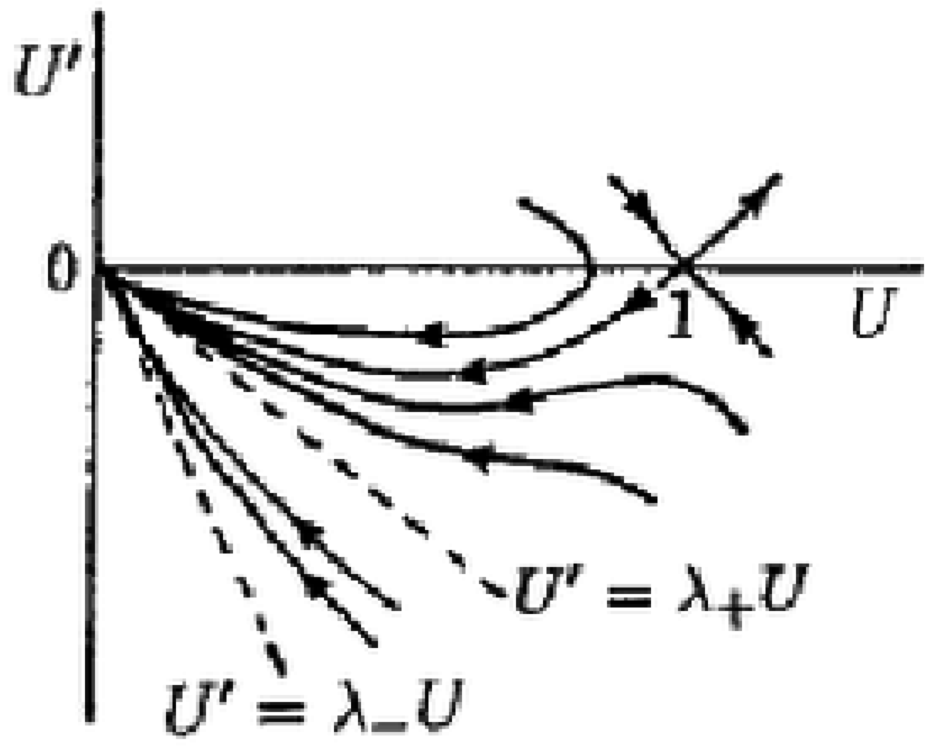


Figure 4: Phase plane trajectories for equation (3)

The leading front structure holds for $c > c_{\min} = 2\sqrt{D\gamma}$, which is consistent with the previous result.

This approach also allows to determine the width of the kink, which is given by:

$$\Delta x(c) = \alpha^{-1} = \frac{2D}{c \pm \sqrt{c^2 - 4D\gamma}},$$

or evaluated at $c_{\min}$:

$$\Delta x(c_{\min}) = \sqrt{D/\gamma}.$$

This result implies that the front can be sharpened by either decreasing the diffusivity or increasing the rate of the local instability.

On the other hand, marginal analysis with the dispersion relation (8) yields:

$$\begin{aligned} c &= \frac{d\,(\alpha c)}{d\alpha} \\ 0 &= D\alpha - \frac{\gamma}{\alpha} \\ \alpha &= \sqrt{\frac{\gamma}{D}} \end{aligned}$$

Then:

$$\Delta x = \sqrt{\frac{D}{\gamma}} \quad c = 2\sqrt{D\gamma}. \tag{9}$$

It is worth noting that the relation between the timescale,$\tau$, and the length scale, $L$, of the transition:

$$\frac{1}{\tau} \sim \frac{c}{L} \sim \sqrt{\gamma \frac{D}{L^2}}$$

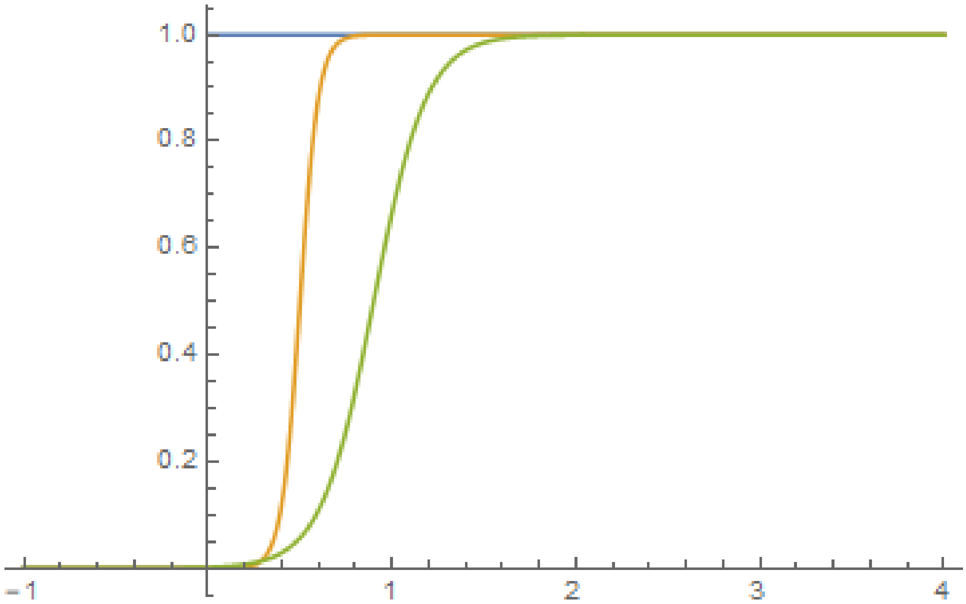


Figure 5: Solution with compact support propagating over time

corresponds to the geometric mean of the time scales involved in both, the diffusion and the local process.

Before moving on to the study of the stability, it is worthy to mention that in 1937 Kolmogorv, Petrovskii and Puskinov showed that a solution $P(x,t)$ subject to initial conditions such that $P(x,0)$ is compactly supported, i.e.

$$P(x,0) = P_0(x) \geq 0; \quad P_0(x) = \begin{cases} 1 & x \leq x_1 \\ 0 & x \geq x_2 \end{cases}, \tag{10}$$

with $x_1 < x < x_2$ and $P_0(x)$ a continuous function, evolves to $P(x - c_{\min}t, t)$ (Fig **??**). This is a surprising fact since the front selected is the one with marginal stability. Nevertheless, according to the previous analysis, for every other set of initial conditions, the evolution of the system depends critically on the behavior of $P(x,0)$ as $x \to \pm\infty$.

## Front stability

According to the discussion in the last paragraph of the former section, the front is highly susceptible to far field effects whenever it does not fulfill (10). This brings up the question for stability under local perturbations.

In order to investigate the stability of the front, we will perform a change of variable such that $P(x,t) \to P(z,t)$ with $z \equiv x - ct$. The Fisher-KPP equation is then written in the form

$$\frac{\partial P}{\partial t} = \gamma P\left(1 - P\right) + c\frac{\partial P}{\partial z} + D\frac{\partial^2 P}{\partial z^2}. \tag{11}$$

The local perturbations $\tilde{P}(z,t)$ can be introducedby writing solitions in the form

$$P(z,t) = P_0(z) + \epsilon\tilde{P}(z,t).$$

Plugging this ansatz in equation (11) yields:

$$\epsilon\frac{\partial \tilde{P}}{\partial t} = \gamma\left(\epsilon\tilde{P} - 2\epsilon P_0\tilde{P} - \epsilon^2\tilde{P}^2\right) + c\epsilon\frac{\partial \tilde{P}}{\partial z} + D\epsilon\frac{\partial^2 \tilde{P}}{\partial z^2}.$$

By keeping only the terms up to first order in $\epsilon$, the last expression becomes

$$\frac{\partial \tilde{P}}{\partial t} = \gamma \tilde{P} \left(1 - 2P_0\right) + c\frac{\partial \tilde{P}}{\partial z} + D\frac{\partial^2 \tilde{P}}{\partial z^2}. \tag{12}$$

The perturbation can be written as

$$\tilde{P}(z,t) = \tilde{p}(z)e^{-\lambda t},$$

with which (12) turns to

$$D\frac{d^2\tilde{p}}{dz^2} + c\frac{d\tilde{p}}{dz} + \left[\gamma\left(1 - 2P_0\right) + \lambda\right]\tilde{p} = 0, \tag{13}$$

this is an eigenmode equation. The stability of the system requires $\lambda > 0$. Keeping in mind that $P_0(z)$ is solution of the original equation, let us consider consider an infinitesimal shift $P_0(z + \delta z)$ and plug it into (2):

$$\begin{aligned}
0 &= D\frac{d^2}{dz^2}\left(P_0 + \delta z P_0'\right) + c\frac{d}{dz}\left(P_0 + \delta z P_0'\right) + \gamma\left(1 - P_0 - \delta z P_0'\right)\left(P_0 + \delta z P_0'\right) + \mathcal{O}(\delta z^2) \\
&= \delta z\left[D\frac{d^2P_0'}{dz^2} + c\frac{dP_0'}{dz} + \gamma\left(1 - 2P_0\right)P_0'\right] + \cancel{D\frac{d^2P_0}{dz^2} + c\frac{dP_0}{dz} + \gamma P_0\left(1 - P_0\right)} + \mathcal{O}(\delta z^2) \\
&= D\frac{d^2P_0'}{dz^2} + c\frac{dP_0'}{dz} + \gamma\left(1 - 2P_0\right)P_0'
\end{aligned} \tag{14}$$

The remaining equation is no other than (13) with $\lambda = 0$ and the solution $\tilde{p}(z) = P_0'(z)$. This means that $\lambda = 0$ is a translational mode.

To figure out the possible eigenvalues, we write

$$\tilde{p}(z) = \tilde{q}(z)e^{-\frac{cz}{2D}}.$$

Evaluating (13) with this ansatz yields

$$D\tilde{q} + \left\{\gamma\left[1 - 2P_0(z)\right] + \lambda - \frac{c^2}{4D}\right\}\tilde{q} = 0.$$

Multiplying this equation by $\tilde{q}'(z)$ and integrating, allows to isolate $\lambda$ as

$$\lambda = \frac{c^2}{4D} - \gamma + \frac{2\gamma\int P_0(z)\tilde{q}(z)\tilde{q}'(z)dz - \int \tilde{q}'d\tilde{q}'}{\int \tilde{q}d\tilde{q}} \tag{15}$$

Demanding the boundary condition that $\tilde{q}(\pm L) = 0$ for some appropiate value of $L$, the requirement of the positive eigenvalues can be fulfilled only if

$$c^2 > 4D\gamma.$$

This result is, again, consistent with what we have gotten so far.

## Asymptotic analysis of non-linear problem

Up to this point, the leading edge approach has been very useful. However there is still a lack of validation for its application in non-linear problems. In order to fill in this gap we will test whether this approach leads to analytic expressions for the non-linear front.

Using the standard singular perturbation technique, we perform a change of variable in the neighborhood of the front. Let $P(z=0)=1/2$, and introduce the transformation

$$\xi = \frac{z}{c} = \epsilon^{1/2} z; \quad P(z) \to g(\xi),$$

with which the normalized Fisher equation becomes

$$\epsilon \frac{d^2 g}{d\xi^2} + \frac{dg}{d\xi} + g\,(1-g) = 0,$$

subject to

$$g(-\infty)=1, \quad g(\infty)=0, \quad g(0)=1/2, \quad 0<\epsilon\le c_{\min}^{-2}$$

This equation can be solved with a regular perturbation series in $\epsilon$:

$$g(\xi;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n g_n(\xi) \tag{16}$$

For the zeroth order term:

$$\begin{gathered} \frac{dg_0}{d\xi} = -g_0\,(1-g_0) \\ g_0(-\infty)=1, \quad g_0(\infty)=0, \quad g_0(0)=1/2 \end{gathered} \tag{17}$$

Which yields

$$\begin{aligned} \frac{dg_0}{g_0\,(1-g_0)} &= -d\xi \\ \int \left(\frac{1}{g_0} + \frac{1}{1-g_0}\right) dg_0 &= -\xi + C \\ \ln\left(\frac{g_0}{1-g_0}\right) &= -\xi + C \\ \therefore \quad g_0 &= \frac{C}{C+e^{\xi}} = \frac{1}{1+e^{z/c}}. \end{aligned} \tag{18}$$

While for the first order term:

$$\begin{gathered} \frac{dg_1}{d\xi} + (1-2g_0)\,g_1 = -\frac{d^2 g_0}{d\xi^2} \\ g_i(\pm\infty) = g_i(0) = 0 \end{gathered} \tag{19}$$

Which yields

$$\begin{aligned} \frac{dg_1}{d\xi} - \frac{g"_0}{g'_0} g_1 &= -g"_0 \\ g_1 &= \frac{e^{z/c}}{\left(1+e^{z/c}\right)^2} \ln\left[\frac{4e^{z/c}}{\left(1+e^{z/c}\right)^2}\right]. \end{aligned} \tag{20}$$

And so on.

An interesting feature of the result is that the asymptotic behavior is the least accurate for the marginal velocity $c=2$. Novertheless the truncation to first order is an excellent fit to the exact numerical solution.

Finally let us analize the relative steepness. The gradient of the solution at $z=0$ is

$$-P'(0) = \frac{1}{4c} + \mathcal{O}\left(\frac{1}{c^5}\right) \tag{21}$$

This result implies that the faster the front the less steep it is.

# References

[1] Murray, James D, *Mathematical Biology. I An Introduction {Interdisciplinary Applied Mathematics V. 17}*, 2001, Springer-Verlag New York Incorporated.

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