Phys 221(A) Notes

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Contents

1 Introduction

1.1 Attractors

Why study Strange Attractors? Certain classes of attractors (called 'strange attactors') have fractal dimension. Compare to usual focus / limit cycle. Main topics in study of attractors: How to describe them? Dimension, and Measure.

2 Box Counting Dimension

Box-Counting (Fractal) Dimension, Fractals and Self-Similarity. To measure the 'size' of a fractal, defnie the box-counting dimension in N-dimensional cartesian space as the number of N-dimensional cubes of size ϵ required to cover the structure, denote this by $N(\epsilon)$. Then the boxing count dimension is

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}}. \tag{1}$$

Some examples:

- Two (different) points, $N(\epsilon) = 2$, $D_0 = \lim_{\epsilon \to 0} \frac{\ln 2}{\ln 1/\epsilon} = 0$
- A line requires $\frac{l}{\epsilon}$ boxes, so $D_0 = \lim_{\epsilon \to 0} \frac{\ln l/\epsilon}{\ln 1/\epsilon} = 1$

(a) Middle Third Cantor Set

2.1 Middle Third Cantor Set

Definition: Starting with [0, 1] continually remove the middle third from each sub-interval.
Step 1: [0, 1] (Requires 1 box of size 1 to cover)
Step 2: [0, 1/3], [2/3, 1] (Requires 2 boxes of size 1/3)
Step 3: [0, 1/9], [2/9, 1/3]; [2/3, 7/9], [8/9, 1] (Requires 4 boxes of size 1/9)

And so on. We have $\epsilon_p = (\frac{1}{3})^p, N(\epsilon_p) = 2^p$. Now, consider $p \to \infty$.

$$\begin{aligned} D_0 &= \lim_{p\to\infty} \tfrac{\ln N(\epsilon_p)}{\ln 1/\epsilon_p} \\ &= \lim_{p\to\infty} \tfrac{\ln 2^p}{\ln((1/3)^{-p})} \\ &= \lim_{p\to\infty} \tfrac{p \ln 2}{p \ln 3} = \tfrac{\ln 2}{\ln 3} \approx 0.63. \end{aligned} \tag{2}$$

Fractal dimension (i.e. effective dimensionality between 0 and 1).

2.2 Koch Curve (Snowflake) / Coastline

In the Koch snowflake, each line segment breaks into four small line segments, each of size $l/3$, where l was the original segment length. This leads to $n_p = 4^p$, $\epsilon_p = (1/3)^p$.

$$\begin{aligned} D_0 &= \lim_{p\to\infty} \tfrac{\ln N(\epsilon_p)}{\ln 1/\epsilon_p} \\ &= \lim_{p\to\infty} \tfrac{\ln 4^p}{\ln((1/3)^{-p})} \\ &= \lim_{p\to\infty} \tfrac{p \ln 4}{p \ln 3} = \tfrac{\ln 4}{\ln 3} \approx 1.2618. \end{aligned} \tag{3}$$

Fractal dimension is 'in between' line and plane (1 and 2). (Note also that Koch curve has exactly twice the dimension of the Cantor set).

2.3 Cantor Square ('Cake-Cutting Fractal')

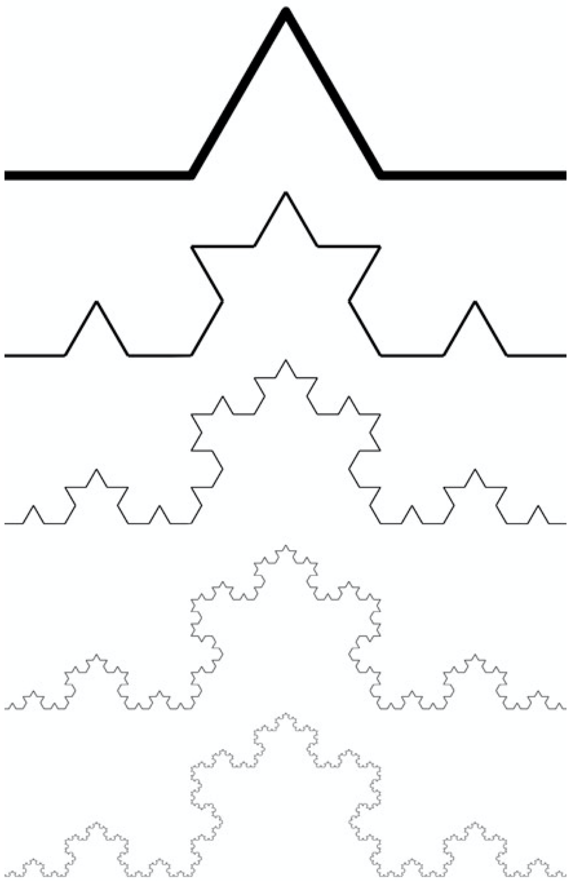
$$D_0 = \lim_{p\to\infty} \frac{\ln N(\epsilon_p)}{\ln 1/\epsilon_p} = \lim_{p\to\infty} \frac{\ln 4^p}{\ln((1/3)^{-p})} \approx 1.2618. \tag{4}$$

Note box-counting dimension of cake-cutting fractal identical to dimension of Koch curve. Somewhat counter-intuitive as (a) koch curve 1D structure, which is 'fattened' while (b) cake-cutting fractal 2D structure which is 'thinned'. More intuitively, the Cake-Cutting fractal has exactly twice the dimension of the middle third cantor set as it can be seen as the direct product of two cantor sets (x and y dimensions).

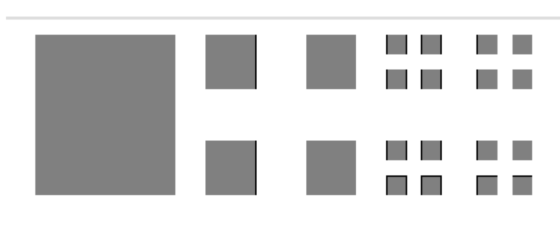
2.4 Self Similarity, Scale Similarity

Fractals tend to exhibit invariance under magnifaction (i.e. koch curve), and to have power law scaling (which is symptomatic of scale invariance).

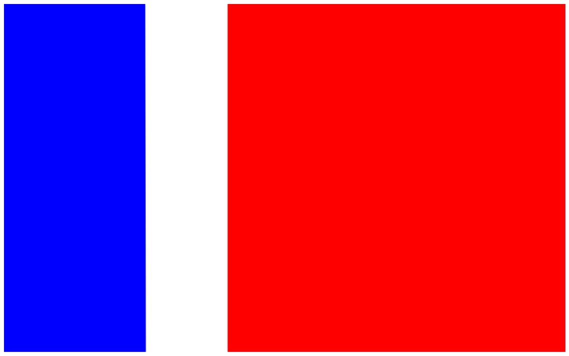
Consider Koch curve, $N(\epsilon_p) = n(p) = 4^p$, which is the number of cubes of scale l_p to cover the curve. At a particular p, $Area_p = l_p^2 = (\frac{1}{3})^{2p} \implies \ln l_p = -p \ln 3$. So, $n(p) = e^{p \ln 4} = e^{(-\frac{\ln l_p}{\ln 3} \ln 4)} \approx 1/l_p^D$. This example of a 'scaling relation' can be thought of as occupation density or number on scale l_p.



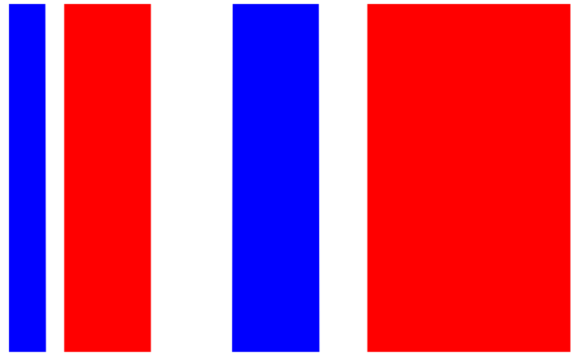
(a) Koch Curve



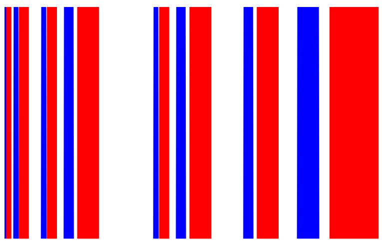
(a) Cake-Cutting Fractal



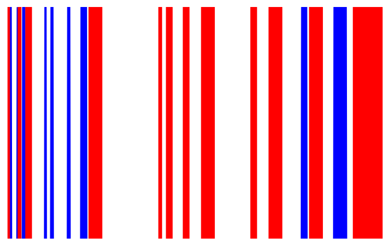
(a) n=1



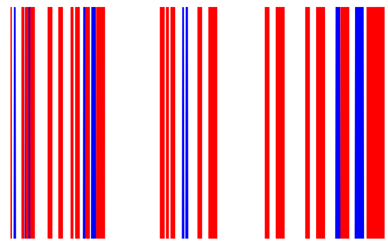
(b) n=2



(a) n=3



(b) n=4



(c) n=5

Figure 5: Pictures of animals

3 Case Study: Baker's Map

The Baker's map is a model of compressible ($\lambda_a + \lambda_b < 1$) or incompressible ($\lambda_a + \lambda_b = 1$) mixing.

$$x_{n+1} = \begin{cases} \lambda_a x_n, & \text{if } y_n < \alpha \\ (1 - \lambda_b) + \lambda_b x_n, & \text{if } y_n > \alpha \end{cases} \tag{5}$$

$$y_{n+1} = \begin{cases} \frac{y_n}{\alpha}, & \text{if } y_n < \alpha \\ \frac{y_n - \alpha}{\beta}, & \text{if } y_n > \alpha \end{cases} \tag{6}$$

Here are repeated iterations of the baker's map with $\lambda_a = 0.25$, $\lambda_b = 0.6$, $\alpha = 0.5$ $\beta = .5$. The red and blue regions are of different mixing materials, while the white shows empty space resulting from the compression of red and blue regions.

Note that after applying the BM twice, the diagram (b) has stripes of length λ_a^2, $\lambda_a\lambda_b$, $\lambda_a\lambda_b$, λ_b^2 (considering only the red and blue).

After n iterations the number of stripes having width $\lambda_a^m \lambda_b^{n-m}$ is $Z(m,n) = \frac{n!}{m!(n-m)!}$.

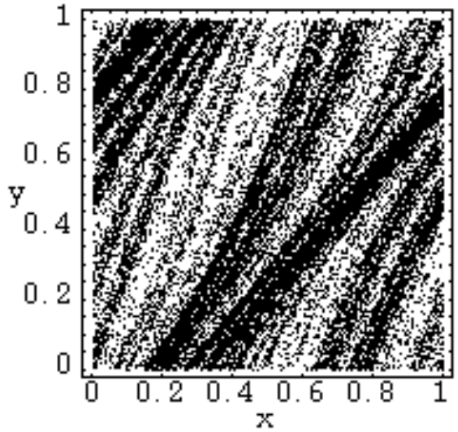
Note also that a horizontal slice of the baker's map gives the Cantor Set, so that we expect the dimension of the BM attractor, $D_0 = 1 + \hat{D}_0$, where $\hat{D}_0$ is the dimension of the Cantor Set.

However to calculate the box counting dimension, D_0 from the map itself we have to leverage the self-similarity of the map.

In particular, the map can be viewed as generating holes gaps in such a way that all stripes where $x < \lambda_a$ arise from the $x < \lambda_a$ of the previous iterate. Consider figure (b), and select the left two stripes, if one were to zoom in on those two, they'd look exactly like figure (a).The same can be said for the right two stripes. The respective stretching factors are $\frac{1}{\lambda_a}$ and $\frac{1}{\lambda_b}$.

We can quantify this observation by letting $\hat{N}(\epsilon) =$ of ϵ-length intervals required to cover a horizontal slice. Then $\hat{N}(\epsilon) = \hat{N}_a(\epsilon) + \hat{N}_b(\epsilon)$, where $\hat{N}_a$ is the contribution from $[0, \lambda_a]$, and similarly for $\hat{N}_b$.

However the self-similarity of the $[0, \lambda_a]$ interval implies that $\hat{N}_a(\epsilon) = \hat{N}(\frac{\epsilon}{\lambda_a})$, $\hat{N}_b(\epsilon) = \hat{N}(\frac{\epsilon}{\lambda_b})$



(a) Sinai Map

Recall, $D_0 = \lim_{\epsilon\to 0} \frac{\ln(\hat{N}(\epsilon))}{\ln(\frac{1}{\epsilon})} \implies \hat{N}(\epsilon) \approx k\epsilon^{-D_0}$. Subsituting into the previous equation, we get $k\epsilon^{-D_0} = k(\frac{\epsilon}{\lambda_a})^{-\hat{D}_0} + k(\frac{\epsilon}{\lambda_b})^{-\hat{D}_0} \implies 1 = \lambda_a^{\hat{D}_0} + \lambda_b^{\hat{D}_0}$. And also, $\lambda_a + \lambda_b < 1 \implies 0 < \hat{D}_0 < 1$. So attractor of Baker's Map has $1 < D_0 < 2$, and for $\lambda_a = \lambda_b = \frac{1}{3}$, we get $\hat{D}_0 = \frac{\ln 2}{\ln 3}$ = dimension of Cantor Set.

4 Attractor Measure

The box counting dimension D_0 is a purely geometric description of the attractor. It gives the scale of the number of ϵ-cubes required to cover the attractor.

However D_0 does not contain information about how often each ϵ-cube is visited, or how long the system stays in any given cube. From the Baker's map figures, we can see that strange attractors exhibit intermittancy, or bursts of dynamics between regions of inactivity (the white space).

To include timing information we need to define a dynamical concept of dimensionality, or measure.

To illustrate this further, we consider the Sinai Map $[0,1] \to [0,1]$ (Ott pg. 81).

$$x_{n+1} = x_n + y_n + \Delta \cos 2\pi y_n \mod 1 \tag{7}$$

$$y_{n+1} = x_n + 2y_n \mod 1 \tag{8}$$

For $\Delta << 1$, the attractor can be shown to cover the entire unit square, so $D_0 = 2$ and $N(\epsilon) \approx \epsilon^{-2}$. But, as can be seen, the trajectories tend to cluster in tight bands rather than explore the space uniformly. (Also see Ott pg. 81, fig 316).

Evidently, the tightly clustered bands are more important for the dynamics of the system than the low density regions. It's natural to relate the dimension of the attractor to the 'dwell time' in small cubes (i.e. replace $N(\epsilon)$ with a weighted sum). This leads to the definition of the natural measure of a cube.

$$\mu_i = \lim_{T\to\infty} \frac{\eta(C_i, \underline{x}_0, T)}{T} \tag{9}$$

Which describes the time an orbit starting from an initial condition, $\underline{x}_0$, spends in cube C_i as $T \to \infty$. (i.e. μ_i is 'fractional dwell time').

Aside: Concept of Measure. Probability measure μ for bounded region R:

- Assigns non-negative number to any set in R
- Is countably additive

For any countably family of disjoint sets S_i in R, then

$$\mu(\cup_i S_i) = \sum_i \mu(S_i),$$

which simply says that the measure of the union of sets is the sum of the measures of the individual sets.

- Assigns $\mu(R) = 1$.
- For $M^{-1}(S)$ = Set of points mapping to S on 1 iterate $M(M^{-1}(S)) = S$, then μ is invariant if $\mu(S) = \mu(M^{-1}(S))$ (defined by fractional dwell time).
- μ is natural if, for interval S and $\underline{x}_0$ in base of attractor of chaotic attractor, $\mu(S, \underline{x}_0)$ is the same for all x_0 in basin except for measure zero sets.

This allows the definition of D_q,

$$D_q = \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln \frac{1}{\epsilon}} \tag{10}$$

$$I(q,\epsilon) = \sum_{i=1}^{N(\epsilon)} \mu_i^q \tag{11}$$

Note:

- $I(q,\epsilon)$ is a weighted sum over all cubes, where the weight is given by the natural measure (fractional dwell time spent in that cube).
- If $q \to 0$, $D_q \to D_0$ (recall $D_0 = \lim_{\epsilon \to 0} \frac{\ln(\hat{N}(\epsilon))}{\ln(\frac{1}{\epsilon})}$)
- If all μ_i are equal, $\mu_i = \frac{1}{N(\epsilon)}$.

$$\begin{aligned} D_q &= \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln \sum_{i=1}^{N(\epsilon)} (\frac{1}{N(\epsilon)})^q}{\ln \epsilon^{-1}} \\ &\approx \frac{1}{1-q} \frac{\ln(\frac{1}{N(\epsilon)})^{q-1}}{\ln \frac{1}{\epsilon}} \\ &\to D_0 \end{aligned}$$

5 Information Dimension

5.1 Back to the Baker's Map

To develop the ideas behind the natural measure μ, dimension spectrum D_q, and how to calculate $I(q,\epsilon)$.

First, observe that an initial distribution uniform in y stays uniform upon iteration. The 'strangeness' of the attractor is due to $\lambda_a + \lambda_b < 1$ while $\alpha + \beta = 1$. This implies that $D_0 = 1 + \hat{D}_0$.

- Natural measure of attractor in $y \leq \alpha$ is α.
- Natural measure of attractor in $y > \alpha$ is β.

but $y \leq \alpha \implies x \leq \lambda_a$, $y > \alpha \implies x \geq 1 - \lambda_b$ (so $\alpha + \beta = 1$). Also consider $\mu[(0 < y < \alpha) \cup (\alpha < y < 1)] = \mu(y < \alpha) + \mu(y > \alpha) = 1$ (this is the first measure requirement). Measure in y is invariant, i.e. $\mu(s) = \mu(M^{-1}(s))$. So, the naturally invariant measure for $x \leq \lambda_a$ is α, while for $1 - \lambda_b < x < 1$ is β.

This implies that $D_q = 1 + \hat{D}_q$ (as the vertical dimension is 1).

To calculate $\hat{D}_q$:

$$\hat{D}_q = \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln \hat{I}(q,\epsilon)}{\ln \epsilon^{-1}} \tag{12}$$

$$\hat{I}(q,\epsilon) = \sum_{i=1}^{\hat{N}(\epsilon)} \hat{\mu}_i^q \tag{13}$$

$$\mu = \begin{cases} \alpha, & \text{if } x \le \lambda_a \\ \beta, & \text{if } x > 1-\lambda_b \end{cases} \implies \hat{I}(q,\epsilon) = \hat{I}_a(q,\epsilon) + \hat{I}_b(q,\epsilon) \tag{14}$$

$\hat{I}_a(q,\epsilon) \leftrightarrow 0 < x < \lambda_a$.
$\hat{I}_b(q,\epsilon) \leftrightarrow 1-\lambda_b < x < 1$.
Recall that self-similarity under magnification we have

$$\begin{cases} \hat{I}_a(q,\epsilon) = \alpha^q \hat{I}(q,\epsilon/\lambda_a), & (\mu=\alpha) \\ \hat{I}_b(q,\epsilon) = \beta^q \hat{I}(q,\epsilon/\lambda_b), & (\mu=\beta) \end{cases} \implies \hat{I}(q,\epsilon) = \alpha^a \hat{I}(q,\epsilon/\lambda_a) + \beta^q \hat{I}(q,\epsilon/\lambda_b). \tag{15}$$

From $D_q = \lim_{\epsilon\to 0} \frac{1}{1-q}\frac{\ln I(q,\epsilon)}{\ln \epsilon^{-1}} \implies \hat{I}(q,\epsilon) = k\epsilon^{(q-1)D_q}$. Substituting this expression into (15) gives a transcendental equation for $\hat{D}_q$,

$$1 = \alpha^q \lambda_a^{(1-q)\hat{D}_q} + \beta^q \lambda_b^{(1-q)\hat{D}_q} \tag{16}$$

For the case that $\lambda_a = \lambda_b = \lambda$, we can calculate $\hat{D}_q$ explicitly.

$$\hat{D}_q = \frac{\ln(\alpha^2+\beta^2)}{(q-1)\ln(\lambda)} = \frac{\ln(e^{q\ln\alpha}+e^{q\ln\beta})}{(q-1)\ln\lambda} \tag{17}$$

$$q=0,\ \ \hat{D}_0 = \frac{\ln 2}{\ln(1/\lambda)} \tag{18}$$

$$q=1,\ \ \hat{D}_1 = \frac{\alpha\ln\alpha+\beta\ln\beta}{\ln\lambda} \tag{19}$$

In general, for $q_j > q_i \implies D_{q_j} \le D_{q_i}$. Now, what does information dimension (D_1), and in general D_q actually mean? A remarkable property of Information Dimension: consider subset of fractal attractor with fraction $0 < \theta < 1$ of natural measure of attractor, then $D_0(\theta) = D_1$. That is, D_1 gives box-counting dimension of smallest set with fraction θ of attractor. Since $\theta < 1$, D_1 represents the dimension of the "core" of the attractor, that is, the region with high natural measure. Equivalently, the dimension of the portion of the attractor which contributes significantly to the dynamics of the system (i.e. excludes regions with infinitesimal μ, or places that are very infrequently visited in phase space).

Here we present a proof by example of $D_0(\theta) = D_1$ for the Baker's Map. To do this, we require

- Characteristic natural measure (i.e. distribution of strips, measure of strips size)
- Distribution of natural measure μ
- Compare D_1, $D_0(\theta)$

Recall that after n-iterations of the Baker's Map there are 2^n strips with width distribution $Z(m,n) = \frac{n!}{m!(n-m)!}$. Also recall

$$\mu = \begin{cases} \alpha, & \text{if } x \le \lambda_a \\ \beta, & \text{if } x > 1-\lambda_b \end{cases}, \text{natural measures of basic strips, rescaling defined by } \ y < \alpha, y > \alpha.$$

Thus, define the natural measure of a strip of width $\lambda_a^m \lambda_b^{n-m}$ as $W(m,n) = Z(m,n)\alpha^m\beta^{n-m}$, where $Z(m,n)$ denotes the number of m, $n-m$ strips, and $\alpha^m\beta^{n-m}$ is μ for an m, $n-m$ strip.

$$W(m,n) = \alpha^m \beta^{n-m} \frac{n!}{m!(n-m)!} \tag{20}$$

Note: $\sum_{m=0}^{n} W(n,m) = \sum_{m=0}^{n} \alpha^m (1-\alpha)^{n-m} \frac{n!}{m!(n-m)!} = (\alpha + 1 - \alpha)^n = 1$. So, $W(n,m)$ fulfills the criteria of a measure. To obtain information about the information entropy, we can leverage the $n \to \infty$ behavior. Recall Stirling's Formula,

$$\ln p! = (p + 1/2)\ln(1+p) - (p+1) + \ln((2\pi)^{1/2}) + O(1/p) \tag{21}$$

Applying Stirling's formula to $Z(m,n)$ and expanding about the point of maximal probability (for binomial coefficients, $m_{max} \approx n/2$, central line of Pascal's triangle) yields

$$Z(m,n) \approx \frac{2^n}{(2\pi)^{1/2}} (\frac{4}{n})^{1/2} \exp(-\frac{1}{2}(\frac{4}{n}(m - n/2)^2)) \tag{22}$$

And similarly,

$$W(m,n) \approx \frac{1}{(2\pi n\alpha\beta)^{1/2}} \exp(-\frac{(m-\alpha n)^2}{2\alpha\beta n}) \tag{23}$$

Where the expansions are valid near their respective maxima, so Z valid for $|m/n - 1/2| << 1$ and W for $|m/n - \alpha| << 1$. Note that Z is clustered about $m/n \approx 1/2$ and W is clustered about $m/n \approx \alpha$, both having width at half-max $1/\sqrt{n}$. In particular, note that $\lim_{n\to\infty} W = \delta(m/n - \alpha)$, $\lim_{n\to\infty} \frac{Z}{2^n} = \delta(m/n - 1/2)$. This confirms that most of the natural measure is concentrated in a small fraction of strips near the maximum, also that the concentration increases with iteration (peaks get tighter).

Recall $\hat{D}_1 = \frac{\alpha \ln \alpha + \beta \ln \beta}{\ln \lambda}$. To check $\hat{D}_0(\theta) = \hat{D}_1$, need to calculate $D_0(\theta)$. Consider the sum of measures less than the fraction θ, i.e. $\sum_{m=0}^{m_\theta - 1} W(m,n) \leq \theta$, where m_θ is greatest integer such that $\hat{N}(\epsilon,\theta) = \sum_{m=0}^{m_\theta} Z(m,n)$. ($\hat{N}(\epsilon,\theta)$ is the smallest number of intervals of length $\epsilon = \lambda_a^n$ to cover θ of x-axis. To relate m_θ to θ, consider:

$$\begin{aligned} \theta &\approx \sum^{m_\theta} W(m,n) \\ &\approx \int_0^{m_\theta} W(m,n)dm \\ &\approx \frac{1}{(2\pi\alpha\beta n)^{1/2}} \int_0^{m_\theta} e^{-\frac{(m-\alpha n)^2}{2\pi n\alpha\beta}} dm \end{aligned} \tag{24}$$

$$m_\theta \approx n(\alpha + (\frac{\alpha\beta}{n})^{1/2} \text{erfc}^{-1}(\theta)) \tag{25}$$

Now, we still need $\hat{N}(\epsilon,\theta)$, however for $\alpha < \beta$, and recalling $\hat{N}(\epsilon) = \hat{N}_a(\epsilon) + \hat{N}_b(\epsilon)$, we can see that $\hat{N}_a(\epsilon) = \hat{N}(\epsilon, \lambda_a)$ has the dominant contribution. In other words, as $n \to \infty$, low powers of α dominant. However, the asymptotic form $Z(m,n)$ is invalid away from $m/n \approx 1/2$. We can get around this by using the approximation:

$$Z(m,n) \approx \frac{\beta^{-n}(\beta/\alpha)^m}{(2\pi\alpha\beta n)^{1/2}} e^{-\frac{(m-\alpha n)^2}{2\pi n\alpha\beta}}, \tag{26}$$

which is valid near $m/n \approx \alpha$. This implies

$$\hat{N}(\epsilon,\theta) \approx \sum_{m=0}^{m_\theta} \frac{\beta^{-n}(\beta/\alpha)^m}{(2\pi\alpha\beta n)^{1/2}} e^{-\frac{(m-\alpha n)^2}{2\pi n\alpha\beta}} \tag{27}$$

Further, the variation in $(\beta/\alpha)^m$ on $[0, m_\theta]$ is stronger than $e^{-\frac{(m-\alpha n)^2}{2\pi n\alpha\beta}}$ as $n \to \infty$. So we have,

$$\hat{N}(\epsilon,\theta) \approx \sum_{m=0}^{m_\theta} (\beta/\alpha)^m \frac{\beta^{-n}}{n^{1/2}} \approx (\beta/\alpha)^{m_\theta} \frac{\beta}{\beta - \alpha} \frac{\beta^{-n}}{\sqrt{n}}, \tag{28}$$

And finally,

$$\hat{N}(\epsilon,\theta) \approx \beta^{-(n-m_\theta)} \alpha^{-m_\theta} n^{-1/2} \tag{29}$$

From above, we have $m_\theta/n \approx \alpha + O(1/\sqrt{n})$, and $\epsilon = \lambda_a^n$. Now consider

$$\begin{aligned} \hat{D}_0(\theta) &= \lim_{n\to\infty} \frac{\ln \hat{N}}{\ln \epsilon^{-1}} \\ &= \lim_{n\to\infty} \frac{m_\theta \ln \alpha + (n - m_\theta)\ln \beta}{n \ln \lambda_a} \\ &= \lim_{n\to\infty} \frac{n\alpha \ln \alpha + n \ln \beta - n\alpha \ln \beta}{n \ln \lambda_a} \\ &= \frac{\alpha \ln \alpha + \beta \ln \beta}{\ln \lambda_a} \\ &= \hat{D}_1 \end{aligned} \tag{30}$$

This equates the box counting dimension of the attractor with fraction θ of natural measure to the information dimension of the attractor.

- Most cubes covering attractor have small fraction of measure (orbit dwell time is tiny fraction).
- Few cubes compose the 'core' of the attractor, though those contain most of the measure.
- Representative of intermittancy in the dynamics!

6 References

Cantor Set
Arnold Cat Map and Sinai as Chaotic Numbers Generators in Evolutionary Algorithms
Oseledet's Theorem
Kaplan-Yorke Conjecture
Kolmogorov Microscales