→ Synchronization of Oscillator Ensembles

— Kuramoto Transition

- seek to examine self-organization of oscillator ensembles, via transition to synchronous behavior

- can develop obvious analogy with phase transitions (a la' ferromagnetism), i.e.

order	synchronization
disorder	asynchrony
mean magnetization $\underline{M}$	mean field Q (phase forcing)
thermal fluctuations	noise
$\underline{S}_i \cdot \underline{S}_j$ coupling	$\left\{\begin{array}{l}\text{oscillator coupling} \\ \text{frequency distribution}\end{array}\right\}$ ~~offset~~ one another

- for theoretical description, consider:

⇒ Kuramoto transition

- N mutually coupled oscillators, with different natural frequencies.

∴ transition must involve some competition between coupling and frequency spread i.e. why? → in single oscillator phase locking $\Leftrightarrow \omega_{\text{mis-match}}$ vs. ϵQ

- for un-restricted mutual coupling, seek coupling of form:

$$\epsilon \sum_{j=1}^{N} F(\phi_i - \phi_j) \Big/ N \equiv Q_{eff}$$

c.c. **mean-field coupling.**

why $1/N$? ⇒ so $\sum \sim N/N \sim 1$ (if maximally coherent)

so coupling finite in thermodynamic limit.

$$\frac{d\varphi}{dt} = -\nu + \epsilon\xi$$

N.B.: Otherwise, synchrony guaranteed as Q overwhelms mismatch as $N \to \infty$.

For minimal problem (Kuramoto transition):

$$\boxed{\frac{d\phi_k}{dt} = \omega_k + \frac{\epsilon}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_k)} \qquad (*)$$

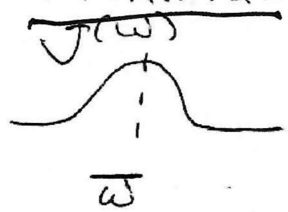
Adler Form.

- basic dynamical equation
- noise to be added later.

- Also, need distribution of frequencies (i.e. how does ω_k vary ?) : → varies $\sqrt{MN}$

$g(\omega)$ → distribution / spectral density

→ Assume symmetric about maximum $\bar{\omega}$, i.e.



$\Rightarrow g(\bar{\omega}+x) = g(\bar{\omega}-x)$

(A)

For Kuramoto transition:

why MF?

- as all oscillators couple, and coupling is identical, expect mean field approach will be accurate, as $N \to \infty$

- for M.F. theory:

Solution:
→ entrainment condition (by mean)
→ mean eqn

① → need: $\boxed{\text{order parameter}}$

② → need: to represent coupling of each oscillator to mean field

③ → need: represent mean field assembly to oscillators (i.e. self-consistency)

SO

① Order parameter:

$$Z = Ke^{i\theta} = X + iY$$

$$= \frac{1}{N}\sum_{k=1}^{N} e^{i\phi_k}$$

(relates mean to individuals)

Note: $\frac{1}{N}\sum_{j=1}^{N} \sin(\phi_j - \phi_k)$

$$= \left(\frac{1}{N}\sum_{j=1}^{N} \sin\phi_j \cos\phi_k - \cos\phi_j \sin\phi_k\right)$$

$$= \left(\frac{1}{N}\sum_{j=1}^{N} \sin\phi_j\right)\cos\phi_k - \left(\frac{1}{N}\sum_{j=1}^{N}\cos\phi_j\right)\sin\phi_k$$

$$= \underline{K\sin\theta}\cos\phi_k - \underline{K\cos\theta}\sin\phi_k$$

$$= \underline{K\sin(\theta - \phi_k)}$$

② Thus, each oscillator couples to mean field by:

$$\boxed{\frac{d\phi_k}{dt} = \omega_k + \epsilon K\sin(\theta - \phi_k)} \quad (**)$$

EOM.

→ entrainment by mean field.

(3) For relation of mean field to oscillators:

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}$$

$\Rightarrow$

$$\boxed{K = \int_{-\pi}^{\pi} e^{i\psi}\, \underbrace{n(\psi)}_{\text{phase distribution}}\, d\psi}$$

(1) $\Rightarrow$ Now, can proceed:

$$\frac{d\phi_k}{dt} = \omega_k + \epsilon K \sin(\theta - \phi_k)$$

Define:

$\theta \equiv \bar{\omega} t$ $\qquad\qquad$ $\psi_k = \phi_k - \bar{\omega} t$ $\qquad$ usual

$K =$ const.

$$\Rightarrow \quad \frac{d\psi_k}{dt} = \omega_k - \bar{\omega} - \epsilon K \sin\psi_k$$

$\rightarrow$ a la' basic synchronization problem!

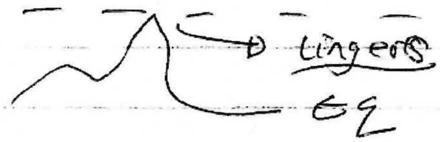
so $$\boxed{\frac{d\psi_k}{dt} = \omega_k - \bar{\omega} - \epsilon K \sin \psi_k}$$

$\therefore$ - oscillator 'entrained' by mean field if:

$$\psi_k = \sin^{-1}\left[\frac{\omega_k - \bar{\omega}}{\epsilon K}\right] \qquad \left|\frac{\omega_k - \bar{\omega}}{\epsilon K}\right| < 1$$

$$= \sin^{-1}\left[\frac{\omega_{MM}}{\epsilon K}\right]$$

- otherwise, phase rotates. Important to recall 'phase 'lingers' near peak Q if close to entrainment $\Rightarrow$ slips.



2)

$\Rightarrow$ Now, must re-construct mean field:

$$K = \int_{-\pi}^{\pi} e^{i\psi}\, n(\psi)\, d\psi$$

$\rightarrow$ {distribution of synchronized osc. phases

$$n(\psi) = n_{synch}(\psi) + n_{asynch}(\psi)$$ $\rightarrow$ {asynch phase distribution

$$= n_S(\psi) + n_{AS}(\psi)$$

For $n_S(\psi)$:

$$n_S(\psi) = g(\psi) = g(\omega)\left|\frac{d\omega}{d\psi}\right|$$

$$d\omega/d\psi = \epsilon k \cos\psi$$
$$\omega = \bar{\omega} + \epsilon k \sin\psi$$

$$n_S(\psi) = g(\omega)\,\epsilon k \cos\psi$$
$$= g(\bar{\omega} + \epsilon k \sin\psi)\,\epsilon k \cos\psi$$

For $n_{AS}(\psi)$:

$n_{AS}(\psi) \sim$ relative ammount of time oscillator spends at each value of ψ (n.b.: a/a' information dimension)

$$\sim t_\psi / T_\psi$$

T_ψ ↓ total period

as for measure

$$\sim T_\psi^{-1}\,|\dot{\psi}|^{-1}$$

$$|\dot{\psi}| = |\omega - \bar{\omega} - \epsilon k \sin\psi|$$

and recall:

$$T_\psi = \int_0^{2\pi} d\psi \Big/ |\omega - \bar{\omega} - \epsilon k \sin\psi|$$

$$\cong 1 \Big/ \left[(\omega-\bar{\omega})^2 - \epsilon^2 k^2\right]^{1/2}$$ → i.e. divergent approaching threshold

So $P(\psi, \omega) = \dfrac{1}{2\pi} \dfrac{\left((\omega-\bar{\omega})^2 - \epsilon^2 k^2\right)^{1/2}}{|\omega - \bar{\omega} - \epsilon k \sin\psi|}$

probability of observing at ψ with ω

So

$$\Lambda_{AS} = \int_{|\omega-\bar{\omega}|>\epsilon k} g(\omega)\, P(\psi, \omega)\, d\omega$$

i.e. excludes "synched" region.

$$= \int_{\bar{\omega}+\epsilon k}^{\infty} \frac{g(\omega)\left((\omega-\bar{\omega})^2 - \epsilon^2 k^2\right)^{1/2} d\omega}{2\pi\,(\omega - \bar{\omega} - \epsilon k \sin\psi)}$$

$$+ \int_{-\infty}^{\bar{\omega}-\epsilon k} \frac{g(\omega)\, d\omega \left((\omega-\bar{\omega})^2 - \epsilon^2 k^2\right)^{1/2}}{2\pi\,(-\omega + \bar{\omega} + \epsilon k \sin\psi)}$$

if re-define: $\omega - \bar{\omega} = x$

and $g(\bar{\omega}+x) = g(\bar{\omega}-x)$

$\Rightarrow$

$$n_{AS}(\psi) = \int_{\epsilon k}^{\infty} \frac{g(\bar{\omega}+x)\, x\, (x^2-\epsilon^2 k^2)^{1/2}\, dx}{\pi[x^2-\epsilon^2 k^2 \sin^2\psi]}$$

symm

Now, can finally write:

$$K = \int_{-\pi}^{\pi} d\psi\, e^{i\psi}\, [n_S(\psi) + n_{AS}(\psi)]$$

but note: n_{AS} has period π in angle ψ

$\Rightarrow$ No contribution to integral!

(N.B. reasonable, $N \to \infty \to$ preferred angles?)

$$\boxed{K = \int_{-\pi}^{\pi} d\psi\, e^{i\psi}\, n_S(\psi)}$$

$\to$ sensible.

$\Rightarrow$

$$K = \int_{-\pi}^{\pi} d\psi\, e^{i\psi}\, \epsilon K g(\bar{\omega} + \epsilon k \sin\psi) \cos\psi$$

so, for real and imaginary part:

$$\to K = K\epsilon \int_{-\pi/2}^{\pi/2} \cos^2\psi \; g(\overline{\omega} + \epsilon K \sin\psi)\, d\psi \qquad (1)$$

$\to$ for mean field strength

$$\to 0 = \epsilon K \int_{-\pi/2}^{\pi/2} \cos\psi \sin\psi \; g(\overline{\omega} + \epsilon K \sin\psi)\, d\psi \qquad (2)$$

$\to$ for mean field frequency

note can re-write (2) as:

$$0 = \int_{-\epsilon K}^{\epsilon K} dx \, \frac{x}{\epsilon K} \, g(\overline{\omega} + x)$$

as $g(\overline{\omega}+x) = g(\overline{\omega}-x)$, (2) is automatically satisfied for choice of g.

i.e. CHOICE OF $\overline{\omega}$ at peak, s/t $g(\omega)$ symmetric, is consistent!

Thus, remains to determine mean field strength by:

$$\boxed{K = \epsilon K \int_{-\pi/2}^{\pi/2} \cos^2\psi \; g(\overline{\omega} + \epsilon K \sin\psi)\, d\psi}$$

self-consistency condition

Thus:

→ for $g(\omega)$ (frequency distribution) symmetric about peak $\bar{\omega}$, have mean field with

→ frequency $\omega = \bar{\omega}$

→ K set by: condition

$$(\ast 1) \quad \boxed{1 = \epsilon \int_{-\pi/2}^{\pi/2} \cos^2\psi \, g(\bar{\omega} + \epsilon K \sin\psi) \, d\psi}$$

($N \to \infty$, homogeneous coupling)

→ can obtain explicit result via solution of (∗1) above for only a few cases.

i.e.

– For Lorentzian:

$$g(\omega) = \gamma \Big/ \pi \big[(\underbrace{\omega - \bar{\omega}}_{\text{peak}})^2 + \underbrace{\gamma^2}_{\text{width}} \big]$$

{ sets range of frequencies to synchronize.

expect competition between:

- coupling strength $\Rightarrow \epsilon$
- range of frequencies $\Rightarrow \gamma$

$\therefore$ not surprisingly, solution $\Rightarrow$

$$\begin{cases} K = \left(1 - \dfrac{2\gamma}{\epsilon}\right)^{1/2} \\ \epsilon_{crit} = 2\gamma \quad ; \quad \underline{\underline{K \sim (\epsilon - \epsilon_c)^{1/2}}} \end{cases}$$

$\Rightarrow$ NB: typical of unimodal distributions.

i.e. For small K,

$$g(\bar{\omega} + \epsilon K \sin\psi) \approx g(\bar{\omega}) + g' \epsilon K \sin\psi + g''(\epsilon K \sin\psi)^2 + \ldots$$

$$K = \underline{\epsilon} K \int_{-\pi/2}^{\pi/2} d\psi \, \cos^2\psi \left[g(\bar{\omega}) + \cancel{g' \epsilon K \sin\psi} + g''(\epsilon K \sin\psi)^2 \right]$$

symm

so obtain:

$$\epsilon_c = 2 / \pi g(\bar{\omega})$$

and $$K^2 \cong \frac{8 g(\bar{\omega})}{|g''| \epsilon^3} (\epsilon - \epsilon_c) \qquad (g'' < 0)$$

$K \sim (\epsilon - \epsilon_c)^{1/2}$ generic

i.e.

$$1 = \epsilon \frac{\pi}{2} g(\bar{\omega}) + \frac{\epsilon^3}{8} g''(\bar{\omega}) K^2$$

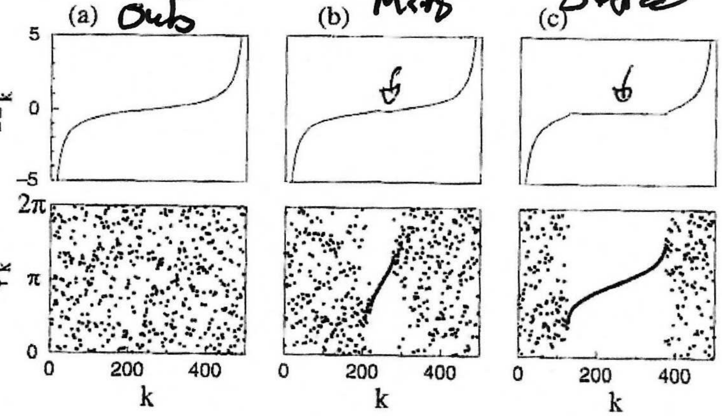
$$\therefore \frac{\epsilon^3}{8} |g''(\bar{\omega})| K^2 = \epsilon \frac{\pi}{2} g(\bar{\omega}) - 1$$

$\xi \rightarrow$ mean field

(a) Sub (b) Crit (c) Super

Ω_k

phase



note widening flat spot

const slope $\Rightarrow$ const freq.

Figure 12.1. Dynamics of a population of 500 phase oscillators governed by Eq. (12.1). The distribution of natural frequencies is the Lorentzian one (12.13) with $\gamma = 0.5$ and $\bar{\omega} = 0$; the critical value of coupling is $\varepsilon_c = 1$. (a) Subcritical coupling $\varepsilon = 0.7$. The oscillators are not synchronized, the mean field fluctuates (due to finite-size effects) around zero. (b) Nearly critical situation $\varepsilon = 1.01$. A very small part of the population near the central frequency is synchronized. The observed frequencies $\Omega_k = \langle \dot{\phi}_k \rangle$ are the same for these entrained oscillators. (c) $\varepsilon = 1.2$, a large part of the population is synchronized, the mean field is large. The amplitude of the mean field is $K \approx 0.1$ for (b) and $K \approx 0.41$ for (c), in good agreement with the formula (12.14).

→ More on Kuramoto Model, and its extensions

Recall: – system of N oscillators, $N \to \infty$
– distribution of frequencies $g(\omega)$
s.t. $g(\omega)$ symmetric about a maximum $\bar{\omega}$

– mean field: $Z = X + iY \equiv K e^{i\theta}$

$$= \frac{1}{N} \sum_{k=1}^{N} e^{i\phi_k}$$

so

$$\frac{d\phi_k}{dt} = \omega_k + \epsilon K \sin(\theta - \phi_k)$$

$$K = \int_{-\pi}^{\pi} e^{i\psi} \left[n_s(\psi) + n_{as}(\psi) \right] d\psi$$

⇒ integral equation for K → mean field strength.

– for $g(\omega) = \gamma / (\omega - \bar{\omega})^2 + \gamma^2$, K small
→ critical c.c. for onset of synch.

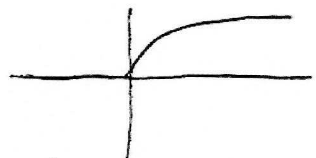
$$\begin{cases} \epsilon_{crit} = 2/\pi g(\bar{\omega}) \\ K^2 \approx \dfrac{8 g(\bar{\omega})}{\epsilon^2 |g''|} (\epsilon - \epsilon_c) \end{cases}$$

mean field.

Some comments and questions:

- is the Kuramoto transition a "true" bifurcation?

$$\mu = (K - K_c)/K$$



i.e. does $K=0$ state go unstable when $\mu > 0$?

Answer: Tough to show rigorously, as infinite number of phase configurations $\{\psi_i : i = 1 \ldots N\}$ belong to macro state of particular K.

- how many / what fraction of oscillators participate in a synchronized cluster?

Recall, only synchronized population contributes, and synchronization $\Rightarrow$

$$\psi_\alpha = \overline{\omega} t + \sin^{-1}\left(\frac{\omega_\alpha - \overline{\omega}}{\epsilon K}\right)$$

so, fraction of synchronized oscillators

$$r \equiv \frac{N_s}{N} = \int_{\overline{\omega} - \epsilon K}^{\overline{\omega} + \epsilon K} g(\omega)\, d\omega$$

So, near threshold:

$$r \cong 2\epsilon k\, g(\bar{\omega})$$

$$= 2\epsilon \frac{2}{\pi g(\bar{\omega})} g(\bar{\omega}) = \frac{4}{\pi}\epsilon + O(\epsilon^3)$$

- similarly, what is distribution of "effective frequencies" $G(\tilde{\omega})$

i.e. → important to distinguish:

$g(\omega)$ → "input" distribution of frequencies

$G(\tilde{\omega})$ → what an experimentalist would actually measure

(N.B.: Synchronization ⇒ flat spot in Ω_k)

Now, $G(\tilde{\omega}) = g(\omega) \dfrac{d\omega}{d\tilde{\omega}}$ (i.e. density of states equal).

- as before: $G(\tilde{\omega}) = G_{synch}(\tilde{\omega}) + G_{asynch}(\tilde{\omega})$

→ sharply peaked on $\bar{\omega}$

Now, $$G_S(\tilde{\omega}) = \underbrace{\rho}_{\text{fraction in cluster}}\, \delta(\tilde{\omega} - \bar{\omega})$$

Now, $$\tilde{\omega}_k = \bar{\omega} + (\omega_k - \bar{\omega})\left(1 - \left(\frac{k\epsilon}{\omega_k - \bar{\omega}}\right)^2\right)^{1/2}$$

$\Rightarrow$

$$G_{AS}(\tilde{\omega}) = g\left(\bar{\omega} + \left[(\tilde{\omega} - \bar{\omega})^2 + (\epsilon k)^2\right]^{1/2}\right) * \frac{|\tilde{\omega} - \bar{\omega}|}{\left((\tilde{\omega} - \bar{\omega})^2 + (\epsilon k)^2\right)^{1/2}}$$

Then: $k = 0$ (no clustering / collective mode phenomena)

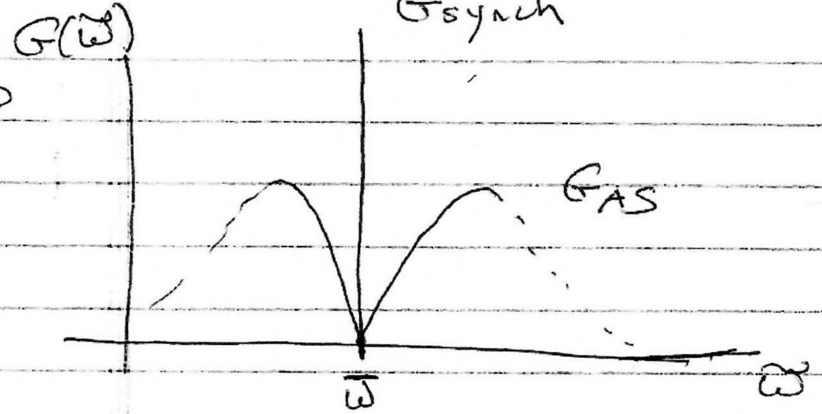
$\Rightarrow$ $G_{AS}(\tilde{\omega}) = g(\tilde{\omega})$

$k \neq 0$ (clustering occurs)

$\Rightarrow$ $G_{AS}(\bar{\omega}) = 0$

and $$\lim_{x \to \bar{\omega}\pm} \frac{d}{dx} G_{AS} = \pm\, g(\bar{\omega} + k\epsilon) / k\epsilon$$

$\Rightarrow$



i.e. → feature noted by Wiener (1965), (per DK) in spectrum of α rhythm of brain waves.

→ physically:

- frequencies near central peak $\bar{\omega}$ "pulled" to peak by Kuramoto transition

- naturally depletes neighboring populations (i.e. $\underset{\tilde{\omega}\to\bar{\omega}}{G_{AS}} \to 0$). of non-synched oscillators.