V.) Synchronization in Oscillatory Media

($\rightarrow$ Continuua)

Now consider:

- extended system, each site performing self-sustained oscillation

and

- coupling of sites

a.) Oscillator Lattices

Consider:

1) 1 D chain, nearest neighbor coupling

2) N elements, frequencies ω_k $k=1, \dots N$

then generalizing equation for 2 coupled oscillators, find for phase dynamics:
$\mapsto$ osc. ~~and~~ frequency.

$$\frac{d\phi_k}{dt} = \omega_k + \epsilon\, \mathcal{Z}(\phi_{k-1} - \phi_k) + \epsilon\, \mathcal{Z}(\phi_{k+1} - \phi_k)$$

$$k = 1, \dots N.$$

B.C.'s: $\phi_0 = \phi_1$

$\phi_{N+1} = \phi_N$

- $k \rightarrow$ index
- nearest neighbor coupling
- linear $\rightarrow$ Bloch.

obviously: (akin to r vs ϵ)

→ $\epsilon = 0$, independent oscillators → ω_n

→ ϵ large ⟺ $\epsilon \gg |\omega_k|$

⇒ expect synchronization of lattice as coupling stronger than individual frequency differences.

→ in between extremes:

as coupling is nearest neighbor,

expect → synchronized clusters / domains, with $\ll N$ elements

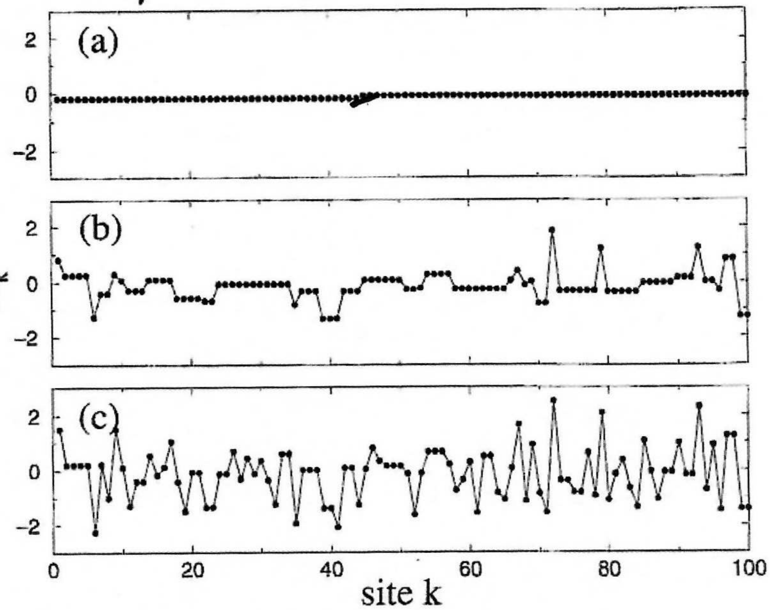
→ several synchronized ω's ⟺ 1 per cluster

(a la' spin Glass) → local domains

[localized domains of magnetization order – quenched, random]

$q = \sin x$

reminiscent of spin glass



→ $\epsilon = 4$ 2 clusters (large)

→ $\epsilon = 1$ several domains

→ $\epsilon = .2$ small clusters + non-synchronized

Figure 11.2. Clusters in a lattice (11.1) of 100 phase oscillators with random natural frequencies (normally distributed with unit variance) and the coupling function $q(x) = \sin x$. (a) A two-cluster state at $\varepsilon = 4$. (b) Many relatively large clusters at $\varepsilon = 1$. (c) A few small clusters plus many nonsynchronized oscillators at $\varepsilon = 0.2$.

n.b. → shear on ω?

For continuum limit:

$$\frac{d\phi_k}{dt} = \omega_k + \epsilon q(\phi_{k-1} - \phi_k) + \epsilon q(\phi_{k+1} - \phi_k)$$

$$\simeq \omega_k + \epsilon q' \left[\phi_{k-1} - 2\phi_k + \phi_{k+1}\right]$$

$$+ \frac{\epsilon}{2} q'' \left[(\phi_{k-1} - \phi_k)^2 + (\phi_{k+1} - \phi_k)^2\right]$$

Taylor expansion

$\therefore$ $\epsilon \to \tilde{\epsilon}/(\Delta x)^2$

$\phi_{k+1} - \phi_k \sim O(\Delta x)$

$\Rightarrow$

$$\frac{\partial \phi(x,t)}{\partial t} = \omega(x) + \alpha \nabla^2 \phi(x,t) + \beta \left(\nabla \phi(x,t)\right)^2 + \text{h.o.t}$$

$\Rightarrow$ <u>Phase 'Diffusion'</u> / Phase Evolution Equation

$$\boxed{\frac{\partial}{\partial t} \phi(x,t) = \omega(x) + \alpha \nabla^2 \phi(x,t) + \beta \left(\nabla \phi(x,t)\right)^2}$$

$$\frac{\partial}{\partial t} \nabla \phi = -\partial \omega / \partial \underline{x} \rightarrow \text{small.}$$

Akin KPZ/Burgers Equation: $(\omega \rightarrow 0)$

$$\left.\partial_t v + v \partial_x v - \nu \partial_x^2 v = 0 \right\} \quad \begin{array}{l}\text{Burgers Eqn.}\\ \text{1D, zero-}P\\ \text{hydro}\end{array}$$

$v = \partial_x \phi \Rightarrow$

$$\left.\partial_t \phi + \frac{(\partial_x \phi)^2}{2} - \nu \partial_x^2 \phi = 0 \right\} \quad \underline{\text{KPZ}}\ \text{equation}$$

N.B. $\Big\{$ First hint of relation/connection to turbulence $\rightarrow$ shocks.

Some Simple Cases:

a) Waves, Modes.

$\rightarrow$ What if $\omega(x)$ non-trivial ?

Like Burgers, simplify by Hopf-Cole substitution

$$\frac{\partial \phi}{\partial t} = \omega(x) + \alpha \nabla^2 \phi + \beta (\nabla \phi)^2$$

$$\phi = \frac{\alpha}{\beta} \ln u$$

$$\frac{\alpha}{\beta} \frac{1}{u} \frac{\partial u}{\partial t} = \omega(x) + \left\{ \alpha \frac{\alpha}{\beta} \left(\frac{1}{u} \partial_x^2 u - \cancel{\frac{1}{u^2} (\partial_x u)^2} \right) + \cancel{\beta} \frac{\alpha^2}{\beta^{\cancel{2}}} \cancel{\frac{1}{u^2} (\partial_x u)^2} \right\}$$

$$= \omega(x) + \frac{\alpha^2}{\beta} \frac{1}{u} \partial_x^2 u$$

$$\Rightarrow \frac{\partial u}{\partial t} = + \frac{\beta}{\alpha} \omega(x) u + \alpha \partial_x^2 u$$

diffn.

$$\frac{\partial u}{\partial t} = \frac{\beta}{\alpha}\,\omega(x)\,u + \alpha \nabla^2 u$$

$$u = u(x)\,e^{\lambda t}$$

$$\Rightarrow \quad \lambda u = \frac{\beta}{\alpha}\,\omega(x)\,u + \alpha \nabla^2 u$$

Sturm-Liouville Problem!

$\rightarrow$ If oscillators identical ?

$$\omega(x) = \text{const.}$$

1D

$$\boxed{\phi(x,t) = kx + (\omega + \beta k^2)t + \phi_0}$$

phase

plane wave solution.

note: — oscillations synchronous
— phase shift between pts. non-zero
— dispersive, sign $\beta \Rightarrow$ type.
— sensitive to b.c.'s

i.e. if $\nabla\phi\Big|_{\text{bndry}} = 0 \quad \Rightarrow K = 0$, only

This establishes – synchrony
– homogeneous phase profile

→ Periodic B.C.'s ?
→ Noise ? – Noisy KPZ/Burgers ?

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– homogeneous phase profile

→ Periodic B.C.'s ?

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→ Recall:

- interested in systems of spatially coupled nonlinear oscillators
- issues of synchronization / coherence primarily concerned with phase

⇒ Phase Diffusion Equation:

$$\boxed{\frac{\partial \phi}{\partial t} = \underbrace{\omega(x)}_{\text{frequency distribution}} + \underbrace{\alpha \nabla^2 \phi + \beta (\nabla \phi)^2}_{\text{phase coupling}} + \underbrace{\varepsilon}_{\text{noise}}}$$

To obtain:

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt}$$

$$\frac{d\underline{x}}{dt} = \underbrace{\underline{F}(\underline{x})}_{\text{basic oscillation}} + \underbrace{\epsilon \underline{p}}_{\text{perturbation}}$$

basic oscillation → solves autonomous system.

Now, no loss of generality to take $\epsilon \underline{p} = D \nabla^2$

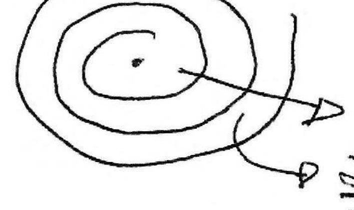
so $$\frac{d\phi}{dt} = \underline{F}(x)\cdot\frac{\partial\phi}{\partial\underline{X}} + \underline{D\nabla^2}\cdot\frac{\partial\phi}{\partial\underline{X}}$$

and, in lowest order perturbation theory:

$$\frac{d\phi}{dt} = \underline{F}(x)\cdot\left.\frac{\partial\phi}{\partial\underline{X}}\right|_{\underline{X}_0(\phi)} + \underline{D\nabla^2}\cdot\left.\frac{\partial\phi}{\partial\underline{X}}\right|_{\underline{X}_0(\phi)}$$

i.e. evaluated at l.c. orbit $\underline{X}_0$, parametrized by ϕ

now $$\underline{Z}(\phi) \equiv \left.\frac{\partial\phi}{\partial\underline{X}}\right|_{\underline{X}_0(\phi)}$$ → phase sensitivity function (gradient ⇔ u.p.o.)



$\left.\frac{\partial\phi}{\partial\underline{X}}\right|_{X_0(\phi)}$

$$\Rightarrow \quad \frac{d\phi}{dt} = \underbrace{\underline{F}(x)\cdot\underline{Z}(\phi)}_{\omega} + \underline{D\nabla^2}\cdot\underline{Z}(\phi)$$

$$= \omega + \underline{D\nabla^2}\cdot\left.\underline{Z}(\phi)\right|_{\underline{X}_0(\phi)}$$

$$= \omega + \underline{D}\,\underline{\nabla}\cdot\underline{\nabla}\,\left.\underline{Z}(\phi)\right|_{\underline{X}_0(\phi)}$$

$$= \omega + D\,\underline{\nabla}\cdot\left(\underline{Z}(\phi)\cdot\frac{d\underline{X}_0}{d\phi}\,\underline{\nabla}\phi\right)$$

$$\frac{d\phi}{dt} = \omega + D\left(\underline{Z}(\phi)\cdot\frac{d\underline{x}_0(\phi)}{d\phi}\right)\nabla^2\phi + D\left(\underline{Z}(\phi)\cdot\frac{d^2\underline{x}_0}{d\phi^2}\right)(\underline{\nabla}\phi)^2$$

so finally have phase diffusion equation: (PDE)

not a pun......

$$\boxed{\frac{d\phi}{dt} = \omega + \alpha\nabla^2\phi + \beta(\underline{\nabla}\phi)^2}$$

here $\alpha = \alpha(\phi) = \underline{Z}(\phi)\cdot\frac{d\underline{x}_0(\phi)}{d\phi}$ → periodic in T (basic u.p.o.)

$\beta = \beta(\phi) = \underline{Z}(\phi)\cdot\frac{d^2\underline{x}_0}{d\phi^2}$ → periodic in T (basic u.p.o.)

consistent with p.t., can average: (slow evolution relative T)

$$\bar{\alpha} = \frac{1}{T}\int_0^T \alpha(\phi(t))\,dt$$

⇒ PDE, with const. coefficients.

$$\bar{\beta} = \frac{1}{T}\int_0^T \beta(\phi(t))\,dt$$

All this now begs the question:

How does one actually compute α, β, for a model ??

→ Computing α, β ?! → key is $\underline{Z}(\phi)$!

- recall, PDE is piece of CGL-esque amplitude equation, i.e. $W = Ae^{i\phi}$ → attracting cycle
  → ϕ dynamics fast
  → can perturb and slave A variation to ϕ perturbation.
- CGL, in turn, derived from reductive perturbation theory (i.e. glorified Poincare-Lindstedt p.t.) which is:

$$\begin{cases} \text{- perturbation about limit cycle} \\ \text{- restricted close to threshold} \\ \text{- solvability equation for secularity removal.} \end{cases}$$

⇒ will look at perturbation about cycle!

Thus, not surprisingly can exploit Floquet theory to construct α, β !

Recall: Floquet theory from parametric instabilities!

↳ solutions of pde/ode with T-periodic potential

Now, consider: $\underline{X}(t) = \underbrace{\underline{X}_0(t)}_{\text{u.p. cycle}} + \underbrace{\underline{u}(t)}_{\text{perturbation}}$

$$\therefore \quad \frac{d\underline{u}}{dt} = \frac{\partial \underline{F}}{\partial \underline{\underline{x}}} \cdot \underline{u}$$

(i.e. small perturbation about cycle)

$$= \underline{\underline{L}}(t) \cdot \underline{u}(t)$$

→ T-periodic system
(i.e. $\underline{\underline{L}}$ has period T)

$$\underline{\underline{L}}(t) = \partial F_i / \partial x_j \Big|_{x_0(t)}$$

In general, Floquet theory tells us :

$$\boxed{\underline{u}(t) = \underbrace{\underline{\underline{S}}(t)}_{\substack{\text{T-periodic} \\ \text{Matrix, } S(0)=1}} \cdot e^{\overbrace{\Lambda}^{\substack{\text{constant} \\ \text{matrix}}} t} \cdot \underline{u}(0)}$$

→ structure of solution

(from linearization)

$\underline{so}$

$$\frac{d}{dt}\left(\underline{\underline{S}}(t) \cdot e^{\Lambda t} \cdot \underline{u}(0)\right) = \frac{d\underline{\underline{S}}(t)}{dt} \cdot e^{\Lambda t} \underline{u}(0) + \underline{\underline{S}}(t) \cdot \underline{\underline{\Lambda}} e^{\Lambda t} \cdot \underline{u}(0)$$

$$= \underline{\underline{L}} \cdot \underline{\underline{S}}(t) \cdot e^{\Lambda t} \cdot \underline{u}(0)$$

$$\Rightarrow \quad \boxed{\frac{d\underline{\underline{S}}(t)}{dt} + \underline{\underline{S}}(t) \cdot \underline{\underline{\Lambda}} - \underline{\underline{L}} \cdot \underline{\underline{S}} = 0}$$

handy identity, equivalent to Floquet solution

$\to$ key to α, β

Now, to construct $\underline{Z}(\phi)$, etc:

define right and left $\left\{\begin{array}{l}\text{eigenvalues}\\ \text{eigenvectors}\end{array}\right.$ ($\underline{\underline{\Lambda}}$ not Hermitian, etc.)

$$\underline{\underline{\Lambda}}\,\underline{u}_\ell = \lambda_\ell \underline{u}_\ell$$

$$\underline{u}_\ell^* \underline{\underline{\Lambda}} = \lambda_\ell \underline{u}_\ell^* \qquad \ell = 0, \dots n-1$$

with $\quad u_\ell^* u_m = \delta_{m,\ell} \quad \to$ normalization condition

Now, as basic motion is cyclic oscillation,
$\exists$ one $\lambda = 0$ (other $\lambda < 0$!)

$\Rightarrow$ corresponds to "rotation mode" - i.e. rotation on cycle

i.e. $$\frac{dx}{dt} = \underline{F}(\underline{x})$$

$$\frac{d^2\underline{x}}{dt^2} = \frac{d}{dt}\left(\frac{d\underline{x}}{dt}\right) = \frac{\partial F}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt}$$

$$\therefore \quad \frac{d}{dt}\underline{u} = \underline{\underline{L}} \cdot \underline{u} \qquad \underline{\text{satisfied}}$$

so $\quad \dfrac{d\underline{x}_0}{dt} = \underline{\underline{S}}(t) \cdot e^{\Lambda t} \cdot \left(\dfrac{d\underline{x}_0}{dt}\right)_{t=0}$ → as particular solution

and since zero eigenvalue for:

$$\underline{u}_0 = \left.\frac{d\underline{x}_0}{dt}\right|_{t=0}$$

$$\underline{\underline{S}}(t) \cdot \underline{u}_0 = \frac{d\underline{x}_0(t)}{dt}$$

Now, noting definitions:

$I(\phi) \equiv$ isochronal surface (surface of orbits $\underline{X}(\phi)$)
$T(\phi) \equiv$ tangent surface to $I(\phi)$

$I(\phi) \to n$ dims.
$T(\phi) \to n-1$ dims.

and Λ eigenvalues $\begin{cases} 1 \text{ zero} \\ \text{others negative} \end{cases}$

so for any $u(t)$ on $T(0)$, $\displaystyle u(t) \xrightarrow[t\to\infty]{} 0$

$\underline{\underline{\text{if}}}$ $\underline{u}(0) \cdot \underline{u}_0 = 0$.

Now, then: $\underline{Z}(0) \cdot \underline{u}_\ell = 0$, for $\ell \neq 0$ (i.e. not corresponding to zero eigenvalue)

$\left.\frac{\partial \phi}{\partial x}\right|_{x(\phi)}$ → so $\underline{Z}(\phi) \perp T(\phi)$

∴ can take $\underline{u}_0^* = \underline{Z}(0)$

i.e. eigenvector for left zero eigenvalue determines $\underline{Z}(0)$!

Note: Proportionality? (= vs ~, above?)

$$u_0^* \cdot u_0 = 1$$

$$\Rightarrow \underline{Z}(0) \cdot \left.\frac{dx_0}{dt}\right|_{t=0} = 1$$, for $\omega = 1$ (Kuramoto normalization)

Now, can note:

$$\underline{Z}(t) \cdot \frac{d\underline{x}_0(t)}{dt} = 1$$, on cycle (defines $\omega = 1$).

and for particular solution ($\lambda = 0$): $\underline{S}(t) \cdot \underline{u}_0 = \frac{dx_0}{dt}$

$\Rightarrow$ so

$$1 = \frac{d\underline{x}_0}{dt} \cdot \underline{Z}(t)$$

$$1 = \frac{d\underline{x}_0}{dt} \cdot \underline{u}_0^{+} \underline{\underline{S}}^{-1}(t)$$

$$\Rightarrow \quad \boxed{\underline{Z}(t) = \underline{u}_0^{+} \cdot \underline{\underline{S}}^{-1}(t)} \quad \rightarrow \text{solves for } \underline{Z}!$$

so, can immediately write: (PDE)

$$\frac{d\phi}{dt} = 1 + \epsilon\,\Omega(\phi) + \Omega^{(1)}(\phi)\,\nabla^2\phi + \Omega^{(2)}(\phi)(\nabla\phi)^2$$

$\rightarrow$ perturbation

$$\Omega(\phi) = \left.\frac{\partial\phi}{\partial \underline{x}}\right|_{x_0(\phi)} \cdot \underline{p}(x) \equiv \underline{Z}(\phi) \cdot \underline{\Pi}(\phi)$$

$$\Omega^{(1)}(\phi) = \underline{Z}(\phi) \cdot \underline{\underline{D}} \cdot \frac{d\underline{x}_0(\phi)}{d\phi} \qquad \text{(in general, } \underline{\underline{D}}\text{)}$$

$$\Omega^{(2)}(\phi) = \underline{Z}(\phi) \cdot \underline{\underline{D}} \cdot \frac{d^2\underline{x}_0(\phi)}{d\phi^2}$$

here: $\Omega(\phi) = \underline{u}_0^{*} \cdot \underline{\underline{S}}^{-1}(\phi) \cdot \underline{\Pi}(\phi)$

and $\Omega^{(1)}(\phi) = \underline{u}_0^{+} \cdot \underline{\underline{S}}^{-1}(\phi) \cdot \underline{\underline{D}} \cdot \underbrace{\underline{\underline{S}}(\phi) \cdot \underline{u}_0}_{d\underline{x}_0/d\phi}$

$$\Omega^{(2)}(\phi) = \underline{u}_0^{+} \cdot \underline{\underline{S}}^{-1}(\phi) \cdot \underline{\underline{D}} \cdot \frac{d\underline{\underline{S}}(\phi)}{d\phi} \cdot \underline{u}_0$$

→ "Now isn't that special!?"

— Church Lady (SNL)

→ Application to CGL

- before, derived phase equation for Stuart-Landau systems, i.e. no spatial coupling
- now, add such coupling

∴ prototypical CGL:

$$\frac{\partial W}{\partial t} = \underbrace{(1+iC_0)W}_{\text{growth}\ /\ \text{frequency}} + \underbrace{(1+iC_1)\nabla^2 W}_{\text{diffusive coupling}\ /\ \text{spatial dispersion}} - \underbrace{(1+iC_2)|W|^2 W}_{\text{nonlinear saturation}\ /\ \text{NL frequency shift}}$$

Now, CGL can be written in matrix form:

$$\begin{cases} \lambda \to \text{growth/stretch} \\ \omega \to \text{rotation} \end{cases}$$

$$\begin{cases} W = X + iY \\ \dfrac{\partial}{\partial t}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \lambda(R) + \nabla^2, & -\omega(R) - c_1\nabla^2 \\ \omega(R) + c_1\nabla^2, & \lambda(R) + \nabla^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \\ \lambda = 1 - R^2 \to \text{per Landau-Stuart (usual)} \\ \omega(R) = c_0 - c_2 R^2 \to \text{differential rotation} \end{cases}$$

$$\begin{cases} \partial_t X = (1-R^2)X + \nabla^2 X - (c_0 - c_2 R^2)Y - c_1\nabla^2 Y \\ \partial_t Y = (c_0 - c_2 R^2)X + c_1\nabla^2 X + (1-R^2)Y + \nabla^2 Y \\ R^2 = X^2 + Y^2 \end{cases}$$

so can re-write as:

$$\frac{dX}{dt} = X - c_0 Y - (X - c_2 Y)(X^2 + Y^2)$$

$$\frac{dY}{dt} = Y + c_0 X - (Y + c_2 X)(X^2 + Y^2)$$

and $\underline{\underline{D}} = \begin{pmatrix} 1 & -c_1 \\ c_1 & 1 \end{pmatrix}$ i.e. $\left\{ \dfrac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \lambda, & -\omega \\ \omega, & \lambda \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \underline{\underline{D}}\begin{pmatrix} X \\ Y \end{pmatrix} \right.$

Now, to construct PDE, need $\underline{\underline{\Lambda}}$ and its eigenvalues, eigenvectors.

so $$W(t) = \underbrace{w_0(t)}_{\text{periodic soln.}} \big[1 + \underbrace{\hat{w}(t)}_{\text{perturbation}}\big]$$

$$\frac{dW}{dt} = (1+ic_0)W - (1+ic_2)|W|^2 W$$

so $$\begin{cases} W_0(t) = \exp[i\omega_0 t] \\ \omega_0 = c_0 - c_2 \end{cases} \rightarrow \text{periodic solution (u.p.o.)}$$

$$\frac{d}{dt}\left(w_0(t)[1+\hat{w}(t)]\right) = (1+ic_0)\, w_0(t)[1+\hat{w}(t)]$$
$$- (1+ic_2)\,|1+\hat{w}(t)|^2 W_0(t)(1+\hat{w}(t))$$

Linearizing $\Rightarrow$

$$\boxed{\frac{d\hat{w}}{dt} = (1+ic_2)(\hat{w}+\hat{w}^*)}$$

Taking $w = \varepsilon + i\eta$, can re-write above:

$$\frac{d}{dt}\begin{pmatrix}\varepsilon \\ \eta\end{pmatrix} = -2\begin{pmatrix}1, & 0 \\ c_2, & 0\end{pmatrix}\begin{pmatrix}\varepsilon \\ \eta\end{pmatrix}$$

$\Rightarrow$

$$\underline{\underline{\Delta}} = -2\begin{pmatrix}1, & 0 \\ c_2, & 0\end{pmatrix} \quad \Rightarrow \begin{pmatrix}\varepsilon(t) \\ \eta(t)\end{pmatrix} = e^{\Delta t}\begin{pmatrix}\varepsilon(0) \\ \eta(0)\end{pmatrix}$$

Also need $S(t)$ (ultimately tied to $W_0(t)$).

Now;

$$W(t) = W_0(t)\left[1 + \hat{\omega}(t)\right]$$

if $\begin{cases} x = X - X_0(t) \\ y = Y - Y_0(t) \end{cases}$

$$\Rightarrow \begin{pmatrix}x \\ y\end{pmatrix} = S(t)\begin{pmatrix}\varepsilon \\ \eta\end{pmatrix}$$

$$S(t) = \begin{pmatrix}\cos\omega_0 t, & -\sin\omega_0 t \\ \sin\omega_0 t, & \cos\omega_0 t\end{pmatrix}$$

so have:

$$\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} = S(t)\, e^{\Delta t}\begin{pmatrix}x(0) \\ y(0)\end{pmatrix}$$

$\hookrightarrow$ requisite form

Now, need diagonalize Λ:

$$\underline{\underline{\Lambda}} = -2\begin{pmatrix} 1 & 0 \\ c_2 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} \underline{u}_0 &= \omega_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \underline{u}_0^{+} &= \omega_0^{-1}\left(-c_2, 1\right) \end{aligned}\right\} \lambda_0 = 0$$

$$\left.\begin{aligned} \underline{u}_1 &= \begin{pmatrix} 1 \\ c_2 \end{pmatrix} \\ \underline{u}_1^{+} &= (1, 0) \end{aligned}\right\} \lambda_1 = -2$$

and can now calculate $\Omega^{(1)}, \Omega^{(2)}$:

$$\Omega^{(1)}(\phi) = \underline{u}_0^{+} \cdot \underline{\underline{S}}^{-1}(\phi) \cdot \underline{\underline{D}} \cdot \underline{\underline{S}}(\phi) \cdot \underline{u}_0$$

$$= 1 + c_1 c_2$$

$$\equiv \alpha$$

Similarly

$$\Omega^{(2)}(\phi) = \underline{u}_0^* \cdot \underline{\underline{S}}^{-1}(\phi) \cdot \underline{\underline{D}} \cdot \frac{d\underline{\underline{S}}}{d\phi} \cdot \underline{u}_0$$

$$\Omega^{(2)} = \omega_0 (C_2 - C_1) \equiv \beta$$

Note: For $1 + C_1 C_2 < 0 \Rightarrow$ negative phase diffusion!

$\Rightarrow$ "phase turbulence".

And now for something completely different ----

— Monty Python

→ Phase Roughening and De Coherence

Roughening → variability of phase profile.

↳ {Coherence / DeCoherence of Oscillator system

have PDE, with α, β = const.

$$\frac{d\phi}{dt} = \omega(\underline{x}) + \alpha(\underline{\nabla}\phi)^2 + \beta\nabla^2\phi$$

(n.b. reversed notation)

if: ω = const. ⇒ KPZ / Burgers Equation

$$\frac{\partial\phi}{\partial t} = \omega + \alpha(\underline{\nabla}\phi)^2 + \beta\nabla^2\phi$$

$$\frac{\partial}{\partial t}\underline{\nabla}\phi = \cancel{\nabla\omega} + \beta\nabla^2(\underline{\nabla}\phi) + 2\alpha\,\underline{\nabla}\phi\cdot\underline{\nabla}(\underline{\nabla}\phi)$$

$$\Rightarrow \left.\begin{aligned} &\frac{\partial\underline{V}}{\partial t} + \lambda\,\underline{V}\cdot\nabla\underline{V} = \nu\nabla^2\underline{V} \\ &\underline{V} = \underline{\nabla}\phi \end{aligned}\right\}$$

N-dim Burgers Eqn. (zero-pressure hydro.)

⇒ { – Burgers solvable by Hopf-Cole subst. $\left(\phi = \frac{\alpha}{\beta}\ln u\right)$

– with noise → Burgers turbulence → Fluids.

– $\omega = \omega(x)$ ⇒ convert to imaginary time Schrodinger equation via Hopf-Cole substitution

in → $\phi = kx + (\omega + \beta k^2)t + \phi_0$ → b.c. sensitivity.

→ Now, consider synchronization with noise:

$$\frac{\partial \phi}{\partial t} = \omega + \alpha (\nabla\phi)^2 + \beta \nabla^2\phi + \underbrace{\varepsilon(x,t)}_{\text{noise}}$$

Key questions: ① how "rough" does phase profile get

$\Rightarrow$ $\boxed{\text{"roughness"} \leftrightarrow \text{measure of deviation from mean value}}$

nb: obviously: "smooth" phase profile ↔ synchronization
"rough" ↔ decoherence
i.e. phase variability large, even for const. ω.

② understand roughness as function of scale → effective coherence domain size !?

Thus: $\boxed{\text{Need pdf of phase fluctuations!}}$ → F-P Eqn.

Note in weak fluctuation limit, can linearize PDA:

$$\frac{\partial \phi}{\partial t} = \omega + \cancel{\alpha (\nabla\phi)^2} + \beta \nabla^2\phi + \varepsilon(x,t)$$

$\langle \phi \rangle$ and linear term ⇒ Dopplershift so, in Fourier space:

$$\frac{\partial \phi_k}{\partial t} = \omega \, \delta_{k,0} - \beta k^2 \phi_k + \varepsilon_k(t)$$

Now, for simple case of white noise:

$$\langle \varepsilon_k(t) \, \varepsilon_{k'}(t') \rangle = 2\sigma^2 \delta_{k,k'} \, \delta(t-t')$$

so, can write F.P.E. from Langevin equation:

$$P = P(\phi_k) \rightarrow \text{pdf of phases} \quad (k \neq 0)$$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi_k} \Big\{ \underbrace{(-k^2 \beta \phi_k)}_{\left\langle \frac{d\phi_k}{dt} \right\rangle} P - \underbrace{\frac{\partial}{\partial \phi_k} D}_{\frac{\langle \delta\phi \delta\phi \rangle}{2\Delta t}} P \Big\}$$

$$\frac{dv}{dt} = -\nu v + \varepsilon$$

$$D = \sigma^2$$

so isomorphic to Brownian motion $\Rightarrow$

$$P_{equ}(\phi_k) = c \exp\left[-\phi_k^2 / (D/\beta k^2) \right]$$

$\therefore$ ϕ_k has: $\begin{cases} \underline{\text{Gaussian}} \text{ Pdf} \quad (\text{linearized}) \\ \text{with variance: } \mathrm{Var}(\phi_k) \simeq D/\beta k^2 \end{cases}$

$\therefore \rightarrow \mathrm{var}(\phi_k) \Rightarrow$ divergent at large scale

$\rightarrow$ for spectrum:

$$|\phi_{k_\omega}|^2 = 2\tau^2 / \omega^2 + (\beta k^2)^2$$

$$|\phi_k|^2 = \int \frac{d\omega}{(\beta k^2)} \frac{2\tau^2}{\left(\left(\frac{\omega}{\beta k^2}\right)^2+1\right)} \frac{\beta k^2}{(\beta k^2)^2}$$

$$\cong 1/\beta k^2.$$

(retaining ω_0 no effect) $\langle\phi\rangle$

$\therefore$ spectrum of phase fluctuations divergent at large scale, too

Thus, roughness "intensity" can be determined by integration:

$$|\tilde{\phi}|^2 = \int_{k_{min}}^{k_{max}} 1/\beta k^2 \; k^{d-1} \, dk$$

(for arbitrary dimension)

$$= \begin{cases} 1/k_{min} \sim L_{max} & - \quad d=1 \\ \ln L & - \quad d=2 \\ \text{infared convergent} & - \quad d \geq 3 \end{cases}$$

$\Rightarrow$ "phase roughness is a strong function of dimensionality of system" is the lesson ....

What is the Point - What Does "Roughening" Mean ?

- smooth phase profile $\Rightarrow$ coherence
  "rough" " " $\Rightarrow$ decoherence !
  $\hookrightarrow$ large variance

in spite of:

- all frequencies can, in principal, be the same at all points

$\Rightarrow$

- oscillators can be synchronized but not coherent

- profile random walks

- synchronized and coherent on small scale
- de-coherent on large scale.

→ The Fans now ask – "What about the nonlinear case"?

i.e. $$\frac{\partial \phi_k}{\partial t} = -\beta k^2 \phi_k + \alpha(\underline{\nabla}\phi\cdot\underline{\nabla}\phi)_k + \tilde{\varepsilon}_k(t)$$

seek extract "effective β" from $(\underline{\nabla}\phi\cdot\underline{\nabla}\phi)_{\underline{k}}$

i.e.

$$(\underline{\nabla}\phi\cdot\underline{\nabla}\phi)_{\underline{k}} = \alpha\sum_{\underline{k}',\omega'} -i\hat{\phi}_{-\underline{k}'}\,\underline{k}'\cdot i(\underline{k}+\underline{k}')\,\hat{\phi}^{(2)}_{\underline{k}+\underline{k}'} \equiv -\gamma_{\underline{k}}\,\hat{\phi}_{\underline{k}}$$

⇒

$$\frac{\partial \hat{\phi}_{k+k'}}{\partial t} + \beta k^2\hat{\phi}_{k+k'} + \gamma_{k+k'}\,\hat{\phi}_{k+k'} = -\alpha\,\underline{k}'\cdot\underline{k}\,\hat{\phi}_{\underline{k}'}\,\hat{\phi}_{\underline{k}}$$

$$\hat{\phi}_{\substack{k+k'\\ \omega+\omega'}} = \left(-i\omega'' + \beta k''^2 + \gamma_{\underline{k}+\underline{k}'}\right)^{-1}\left(-\alpha\,\underline{k}'\cdot\underline{k}\,\hat{\phi}_{\substack{\underline{k}'\\ \omega'}}\hat{\phi}_{\substack{\underline{k}\\ \omega}}\right)$$

$$\Rightarrow +\gamma_{\underline{k}} = +\alpha^2\sum_{\underline{k}',\omega'}(\underline{k}'\cdot\underline{k}+\underline{k}')(\underline{k}'\cdot\underline{k})\left(-i\omega''+\beta k''^2+\gamma_{\substack{k+k'\\ \omega+\omega'}}\right)^{-1}|\phi_{\substack{\underline{k}'\\ \omega'}}|^2$$

$$\cong \alpha^2\sum_{\substack{\underline{k}'\\ \omega'}}(\underline{k}\cdot\underline{k}')^2|\phi_{\substack{\underline{k}'\\ \omega'}}|^2\left(-i\omega''+\beta k''^2+\gamma_{k+k'}\right)^{-1}$$

(parity)

$$\gamma_{\underline{k}} = \alpha^2 \sum_{\substack{\underline{k}' \\ \omega'}} (\underline{k}\cdot\underline{k}')^2 |\phi_{\substack{\underline{k}' \\ \omega'}}|^2 \left\{ \frac{\beta k''^2 + \gamma_{k+k'}}{\omega'^2 + (\beta k''^2 + \gamma_{k+k'})^2} \right\}$$

1D

$$\begin{cases} \gamma_{\underline{k}} = \alpha^2 k^2 \sum_{\substack{\underline{k}' \\ \omega'}} k'^2 |\phi_{\substack{k' \\ \omega}}|^2 \left\{ \frac{\beta k''^2 + \gamma_{k+k'}}{\omega'^2 + (\beta k''^2 + \gamma_{k+k'})^2} \right\} \\ \quad \equiv \beta_k \, k^2 \\ \beta_k = \alpha^2 \sum_{\underline{k}', \omega'} k'^2 |\phi_{\substack{k' \\ \omega'}}|^2 \left\{ \frac{(\beta + \beta_k) k''^2}{\omega''^2 + ((\beta + \beta_k) k''^2)^2} \right\} \end{cases}$$

so $$|\hat{\phi}_{\substack{\underline{k} \\ \omega}}|^2 = \left\{ \frac{2\sigma^2}{\omega^2 + ((\beta + \beta_k) k^2)^2} \right\}$$

and need compute $\beta_k \Big|_{k \to 0}$ by recursion.

is effectively a "renormalized" $\begin{cases} \text{dispersion} \to \text{phase} \\ \text{viscosity} \to \text{Burgers} \end{cases}$.

TBC