Lecture 18: Bistable Fronts

PHYS 221A, Spring 2017

Lectures: P. H. Diamond
Notes: Xiang Fan

June 15, 2017

1 Introduction

In the previous lectures, we learned about Turing Patterns. Turing Instability is simple mechanism for generating heterogeneous spatial patterns via reaction-diffusion systems. Turing instability results from $D_A \neq D_B$ diffusion + perturbation about fixed points. If there are multiple fixed points, notion of a "front" linking differing domains is naturally of interest. In MFE, fronts appear in avalanche modeling, transport barriers, etc.

The basic equation for the diffusion-reaction system is:

$$\partial_t u = D\partial_x^2 u + f(u) \tag{1}$$

where $f(u)$ is the reaction. Fronts come in (at least) two basic categories by different choices of $f(u)$:

1. uni-stable/2nd order/Fisher
2. bi-stable/1st order/FitzHugh-Nagumo

2 Uni-stable Front

The choice $f(u) = \gamma(u - \alpha u^2)$ yields Fisher's Equation:

$$\partial_t u = D\partial_x^2 u + \gamma(u - \alpha u^2) \tag{2}$$

There are two fixed points: $u = 0$ and $u = 1$. It's easy to see that $u = 0$ is unstable and $u = 1$ is stable. This is called uni-stable. So there will be a propagating front, see Fig. 1. We will show next that the propagating speed is $c = \sqrt{\gamma D}$.

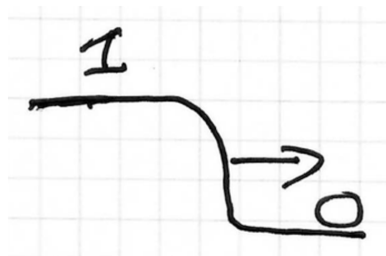

Figure 1: Fixed points and front for Fisher's Equation.

Key question: what is speed? In order to obtain the front speed, we assume the traveling wave solution $u = \mathbf{f}(x-ct)$. Then the equation becomes:

$$c\mathbf{f}' = D\mathbf{f}'' + f(\mathbf{f}) \tag{3}$$

This resembles the equation of motion of a particle with friction and external forcing:

$$\gamma \dot{x} = m\ddot{x} - \delta V/\delta x \tag{4}$$

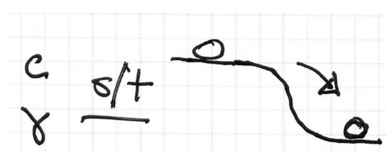

Figure 2: A ball falling off a cliff.

This analogy helps us understand the physics for the front. It's like a ball falls to the bottom of a cliff and stops. The role of c and γ is to dissipate work done on leading edge. We treat front as exponential in the moving frame: $u = A\mathrm{e}^{-k(x-ct)}$. This is accurate for the front, not saturated state for u. Consistency of front solution is

$$c = 2\sqrt{\gamma D} \tag{5}$$

The speed is set so that the front is marginally stable in the co-moving frame, i.e., γ vs $c\partial_x$.

Front is stable means we can extract speed from stability analysis. Front: asymptotics support leading edge.

3 Bi-stable Front

What of front which connects *two stable* fixed points? In this section we use the FitzHugh-Nagumo model as an example to discuss bi-stable front.

In practice, front, in effect, is a switch between 2 states. Such systems will have 2 stable, 1 unstable roots. The classical example is the FitzHugh-Nagumo system (toy version of HodgkinHuxley model of neuron signals):

$$\frac{\partial v}{\partial t} = f(v) - w + I_a + D\frac{\partial^2 v}{\partial x^2} \tag{6}$$

$$\frac{\partial w}{\partial t} = bv - \gamma w \tag{7}$$

where I_a is external stimulus. The evolution of v is fast (i.e., sodium) and that of w is slow (i.e., calcium). An example of $f(v)$ is $f(v) = v(a-v)(v-1)$ (see Fig. 3).

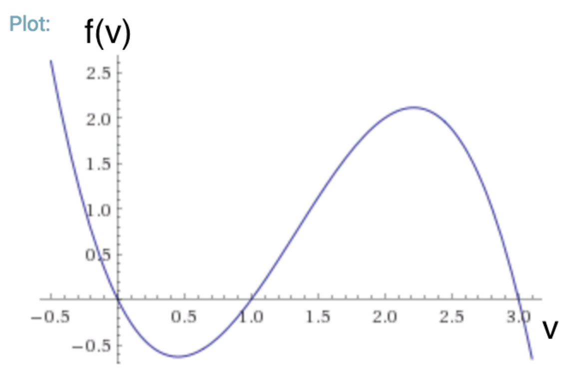

Figure 3: An example of $f(v)$.

Now let's find the fixed points state of the system, and ignore I_a for the moment. Then we get $w = f(v) = w(v)$ and $w = bv/\gamma$. There are three fixed points, as shown in Fig. 4. Obviously, (1) and (3) are stable ($f' < 0$) and (2) is unstable ($f' > 0$). So the front can link/allow transition between two stable fixed points. The unstable root 'powers' front motion (akin Fisher).

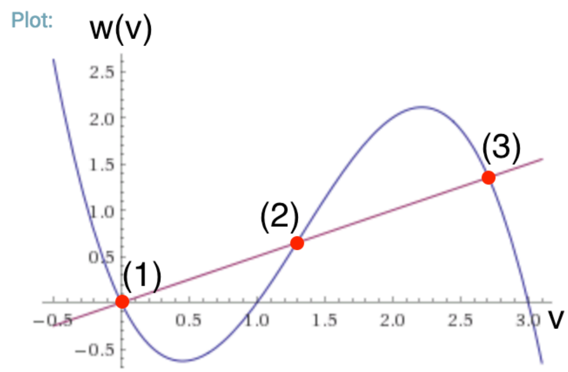

Figure 4: An example of $w(v)$.

The aim of FitzHugh-Nagumo system is to describe *pulses* in excitable media. What's the difference between pulses and fronts? Pulses require multiple time scales. Pulses contain a leading edge, a slow evolution part, and a trailing edge. The same speed is required for both edges $c_+ = c_-$ in order to maintain identity/coherence of the pulse. The key element is the switch-on/off fronts. Switch-on starts from $v = 0$.

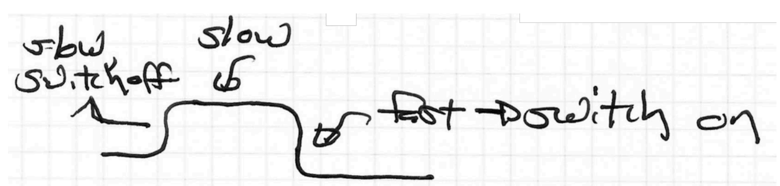

Figure 5: Pulse.

4 Simple/Basic Bi-stable Problem

As discussed above, we are interested in the case where $v = 0$. This leads to a simpler problem:

$$\boxed{\partial_t u = f(u) + D\partial_x^2 u} \tag{8}$$

where $f(u) = A(u - u_1)(u_2 - u)(u - u_3)$. This is the counter-part of Fisher, it is bi-stable, i.e., $f'(u_{1,3}) < 0$ (stable) and $f'(u_2) > 0$ (unstable).

As usual, we look for the speed c via traveling wave: $u = u(x - ct)$:

$$cu' = Du'' + f(u) \tag{9}$$

Multiply u' to the above equation, and then integral:

$$c \int_{x(u_3)}^{x(u_1)} u'^2 \,\mathrm{d}x = D \int_{x(u_3)}^{x(u_1)} u'u'' \,\mathrm{d}x + \int_{x(u_3)}^{x(u_1)} u'' \,\mathrm{d}u \tag{10}$$

After some algebra and the assumption that $u' = 0$ on both sides of the front, we obtain the speed:

$$\boxed{c = -\int_{-u_3}^{u_1} f(u)\,\mathrm{d}u \Big/ \int_{-\infty}^{+\infty} u'^2 \,\mathrm{d}x} \tag{11}$$

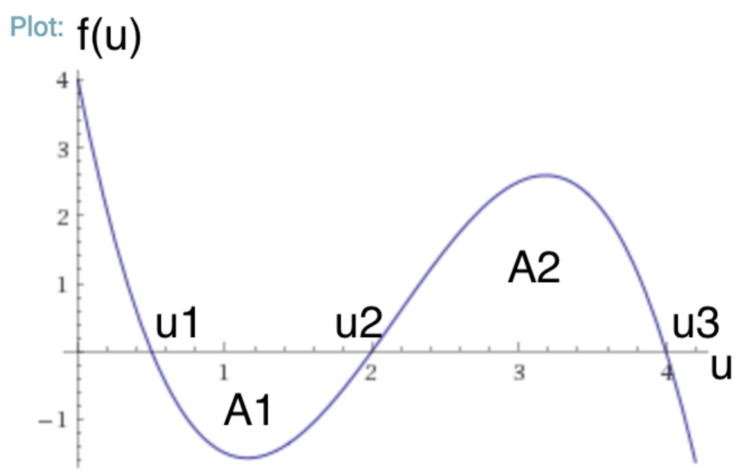

Figure 6: $f(u)$ and its area.

Note that the speed c is linked to the area under $f(u)$, the curve with 3 zero crossings, see Fig. 6. So:

- if $A_2 > A_1$ then $c > 0$: front advances towards $x > 0$.
- if $A_2 < A_1$ then $c < 0$: front advances towards $x < 0$.
- if $A_2 = A_1$ then $c = 0$: front stationary, co-existance.

The area rule is akin to Maxwell construction in thermodynamics. Equal areas means co-existance of phases.

Now we continue to calculate c with concrete f: $f(u) = A(u - u_1)(u_2 - u)(u - u_3)$.

$$\partial_t u = A(u - u_1)(u_2 - u)(u - u_3) + D\partial_x^2 u \tag{12}$$
$$u(+\infty) = u_1 \tag{13}$$
$$u(-\infty) = u_3 \tag{14}$$

Try fit: $u' = a(u - u_1)(u - u_3)$ where a is a coefficient to be determined. Plug in u':

$$\begin{aligned}
0 = L(u) &= Du'' + cu' + A(u - u_1)(u_2 - u)(u - u_3) && (15)\\
&= D[a(u - u_1)(u - u_3)]' + [ca(u - u_1)(u - u_3) + A(u - u_1)(u_2 - u)(u - u_3)] && (16)\\
&= Da[u'(u - u_3) + (u - u_1)u'] + [...] && (17)\\
&= Da[a(u - u_1)(u - u_3)^2 + a(u - u_1)^2(u - u_3)] + [...] && (18)\\
&= (u - u_1)(u - u_3)[(2Da^2 - A)u - (Da^2(u_1 + u_3) - ca - Au_2)] && (19)
\end{aligned}$$

The above equation $L(u) = 0$ must be true for all u, so we have:

$$2Da^2 = A \tag{20}$$
$$Da^2(u_1 + u_3) - ca - Au_2 = 0 \tag{21}$$

so $a = (A/2D)^{1/2}$, and

$$\boxed{c = (AD/2)^{1/2}(u_1 - 2u_2 + u_3)} \tag{22}$$

Note that A and γ are analogous, so $c \sim (D\gamma)^{1/2}$, as before. And $c \sim \sqrt{D\gamma}c(u_1, u_2, u_3)$. If $u_2 = (u_1 + u_3)/2$, it's stationary front. Thus, the speed is set by parameters in reaction function. Weighted balance between there is the key.

5 Toward Pulses/Waves

Let's go back to FitzHugh-Nagumo. Consider:

$$\partial_t u = f(u) - v + D\partial_x^2 u \tag{23}$$
$$\partial_t v = bu - \gamma v \tag{24}$$

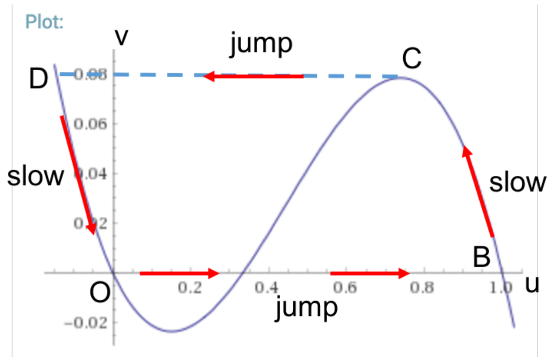

Figure 7: FitzHugh-Nagumo. OBCDO.

u is fast, and v is slow. For the slow equation:

$$v_+ = \epsilon(Lu - Mv) \tag{25}$$

Treat v as fixed for fast bifurcations. Leading edge is set at $v = 0$. If $v \sim 0$, i.e.

$$\partial_t u = Du_{xx} + f(u) \tag{26}$$

where $f(u) = u(a - u)(u - 1)$ and $0 < a < 1$.

If v finite:

$$\partial_t u = Du_{xx} + f(u) - v_C \tag{27}$$

where v_c changes effective reaction function. So

$$c_- = (D/2)^{1/2}(u_C - 2u_p + u_D) \tag{28}$$

where u_C, u_P, u_D are roots of $f(u) = v_C$. Speed is function of amplitude of v. Then the pulse condition is:

$$\boxed{c_+ = c_-(v_c)} \tag{29}$$

This condition guarantees that pulse will not disperse, i.e., forward and backward transtions propagate together at the same speed. This sets a critical amplitude v_C for pulse to be excited.

The slow evolution links the up, back transitions. So the FitzHugh-Nagumo model is the basic simple model of pulse in excitable media, and is constructed from basic element of bistable front.

If time in included, then more phenomena will occur, such as spirals, etc. In the Magnetic Fusion Energy community, there's a saying: "Reaction is in the diffusion". There are more akin models, such as Cahn-Hilliard.

References

[1] Wikipedia contributors. "FitzHugh-Nagumo model." *Wikipedia, The Free Encyclopedia.* Wikipedia, The Free Encyclopedia, 31 Mar. 2015. Web.

[2] Murray, J. D. "Mathematical Biology I: An Introduction, Interdisciplinary Applied Mathematics, Mathematical Biology." (2002).