Stochastic population genetics: homework 5
To be returned on May 31st

May 25, 2017

1 Population dynamics with fitness dilution

We consider the dynamics of a population of cells $C(t)$ with a basal division rate μ and basal death rate ν ($\mu - \nu < 0$). The fitness of the population $f(t)$ fluctuates in time with damped fluctuations for large populations. The dynamics are represented by the equations

$$\begin{cases} \partial_t C(t) = (\mu-\nu)C(t) + f(t)C(t) + \sqrt{(\mu+\nu)C(t)}\xi(t), \\ \partial_t f(t) = -\lambda f(t) + \sqrt{\dfrac{2}{C(t)}}\gamma\eta(t), \end{cases} \tag{1}$$

where ξ and η are both Gaussian white noises and they are independent. $C = 1$ is an absorbing boundary. $\sqrt{(\mu+\nu)C(t)}\xi(t)$ is an approximation of demographic noise and is taken with the Itô convention.

a. Write down the Fokker-Planck equation associated with Eq. 1. Check that it is independent of the choice of Itô or Stratonovitch for η. What is the mean value of f if you don't include the condition of $C > 1$? In what regime is $\sqrt{(\mu+\nu)C(t)}$ a good approximation for the complete expression of the amplitude of the noise?

b. Make the change of variable $x = \ln C$. What is the Fokker-Planck (FP) version of the equation after the change of variable? Is it equivalent to changing variables in the FP Eq. 1?

c. Show that

$$f(t) = \sqrt{2}\gamma \int_0^t e^{-\lambda u} e^{-x(t-u)/2}\eta(t-u)du. \tag{2}$$

We are interested in the limit of short-time correlation for the fitness noise (i.e $\lambda \to \infty$). To keep the noise relevant, we take at the same time the limit of $\gamma \to \infty$ while keeping the ratio λ/γ constant.

For any $k \in [0, \lambda t]$ we can rewrite

$$f(t) = \sqrt{2}\gamma \int_0^{k/\lambda} e^{-\lambda u} e^{-x(t-u)/2}\eta(t-u)du + \sqrt{2}\gamma \int_{k/\lambda}^t e^{-\lambda u} e^{-x(t-u)/2}\eta(t-u)du. \tag{3}$$

d. Show that the second integral in Eq. 3 vanishes for $k = \sqrt{\lambda}$ in the limit of large λ and γ with constant ratio λ/γ. Conclude that in that limit

$$f(t) \simeq \sqrt{2}\frac{\gamma}{\lambda} e^{-x(t_-)/2}\eta(t)\,, \tag{4}$$

where $t_- = \lim_{u \to 0 \& u \geqslant 0}(t-u)$.

e. We write the equation of the dynamics for the limit of short-time correlation in fitness noise (that is now one-dimensional)

$$\partial_t x(t) = \mu - \nu + \sqrt{\mu+\nu}e^{-x/2}\xi - e^{-x}\frac{\mu+\nu}{2} + \sqrt{2}\frac{\gamma}{\lambda}e^{-x/2}\eta. \tag{5}$$

Based on Eq. 4, what is the convention which is being used for η? Change variables back to $C = e^x$ in Eq. 5. How is that result different from the naif limit of short-time correlations in Eq. 1?