HW problem week 1

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

An experimentalist has a gas of N ionic molecules confined in a cubic box whose sides are length L. By applying an electric field in the z direction, they subject the molecules to a potential $V(z) = \alpha z$. The electric charge of the molecules is small and the gas is dilute enough so that you can neglect the coulomb force between different molecules and they are well described by statistical mechanics (i.e. they follow the Boltzmann distribution). If you need to plug in numbers in the following, let $L = 2$ mm, $\alpha = 10^{-19}$ J/m, and $N = 10^8$.

1. Derive the normalized distribution function for the height of the molecules $\rho(z)$.

   The boltzmann distribution is $\rho(\boldsymbol{r}, \boldsymbol{v}) \sim e^{-\beta E(\boldsymbol{r},\boldsymbol{v})}$. In this case, the relevant part is $\rho(z) \sim e^{-\alpha\beta z}$ so all we need to do is normalize it.

$$\rho(z) = N \frac{\alpha\beta e^{\alpha\beta(L-z)}}{e^{\alpha\beta L} - 1}.$$

2. The experimentalist measures the number of particles in the lower half of the box and the upper half of the box separately, and finds that one of these quantities is twice as big as the other. Which one is larger? Use this information to determine the temperature.

   Let $f_1 = \int_0^{L/2} \rho(z) dz$ and $f_2 = \int_{L/2}^{L} \rho(z) dz$ be the probability for a particle to be in the lower and upper half of the box respectively. Then on average there are $N f_1$ particles in the bottom half and $N f_2$ particles in the upper half. Using the above result for ρ, we find the ratio $f_1/f_2 = e^{L\alpha\beta/2}$ and set it equal to two. So we have $e^{L\alpha\beta/2} = 2$ or $\beta = 2\log 2/L\alpha$. For numbers given above, $\beta = 6.93147 \times 10^{21} J^{-1}$ and $T = 10.4543$ K. Thats kind of cold but definitely achievable in a lab.

3. Now, the experimentalist studies the speed distribution of the particles. After some careful measurements, they find that the number of particles with speeds between 4 and 4.0001 m/s is three times the number of particles with speeds between 4.2 and 4.2002 m/s. Use this information to calculate the mass of the molecules.

Use the maxwell speed distribution. Let $v_1 = 4$ and $v_2 = 4.2$. Then since the velocity range dv is small we can approximate the integrals and just write

$$v_1^2 e^{-\frac{\beta m v_1^2}{2}} \Delta v_1 = 3 \times v_2^2 e^{-\frac{\beta m v_2^2}{2}} \Delta v_2.$$

where $\Delta v_1 = 0.0001$ and $\Delta v_2 = 0.0002$. Rearranging, taking a log, and rearranging again one finds

$$m = -\frac{2}{\beta(v_1^2 - v_2^2)} \log\left(3\frac{v_2^2 \Delta v_2}{v_1^2 \Delta v_1}\right).$$

Plugging everything in using β calculated in part 2, I find $m = 3.32 * 10^{-22}$ kg. You may have noticed that the total number of particles N never appeared in any calculation we did. That's because everything was given in terms of ratios, where N cancels out. If you were asked to find the number of particles in a certain window of speeds or velocities or heights, then you would need to carry along N.

4. Bonus question (ungraded: for your enjoyment, or extra practice): What is the average energy of the molecules?

   Equipartition theorem gives $\frac{3}{2}k_B T$ for the average kinetic energy. The average potential energy is calculated as

$$\int dz\ \rho(z)\alpha z = \frac{\alpha\beta L - e^{\alpha\beta L} + 1}{\beta - \beta e^{\alpha\beta L}}.$$

   So

$$\bar{E} = \frac{3}{2}k_B T + \frac{\alpha\beta L - e^{\alpha\beta L} + 1}{\beta - \beta e^{\alpha\beta L}}.$$