HW problem week 2

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

Problem 0, ungraded Redo the derivation of the number of modes of oscillation in a cavity for the situation where the boundary condition on the electric field is that the derivative vanishes at the boundary, rather than that the field vanishes on the boundary (this is what's done in the supplementary pdf posted on the class website). Show that the conclusion about the number of modes in a range $d\lambda$ is unchanged.

Problem 1 A 2000 W incandescent bulb radiates from a tungsten filament. The distribution of energy radiated per wavelength has a maximum at $\lambda_{max} = 483$ nm.

1. By expressing Planck's law in terms of frequency (see problem 3-14 in the book), and using that a photon of frequency f has energy hf, obtain the distribution function $n(f)$ for number of photons with frequency between f and $f + df$ emitted per second by the filament. You will need to use the analog of equation 3-7:

$$R(\lambda) = \frac{c}{4}u(\lambda)$$

for u as a function of f, and you will need the surface area of the filament.

The power radiated per frequency per unit area in terms of the energy density is $R(f) = \frac{c}{4}u(f)$, where

$$u(f) = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/k_bT} - 1}.$$

We find the surface area of the filament by using Stefan's law and Wien's displacement law. From Wein's displacement, we find $T = 6000$ K, and then from Stefan's law we find $R = 7.35 * 10^7$ J/m^2. The total power radiated is $R *$ (Surface Area) so we find $(SA) = 27.2 * 10^{-6}$ m^2.

So, The total power radiated per frequency is now $R(f) * (SA)$. Since a photon has energy hf, the number of photons radiated at a given frequency per second is

$$n(f) = R(f) * (SA)/hf = \frac{c}{4}\frac{8\pi f^2}{c^3}\frac{(SA)}{e^{hf/k_bT} - 1}.$$

2. Consider calculating the total number of photons emitted by the wire by integrating the distribution $n(f)$ over all frequencies.

$$N_{tot} = \int_0^\infty df\ n(f)$$

By adimensionalizing, show that the total number of photons goes like T^3. (You don't have to actually do the integral to compute N_{tot}, but if you want to, I recommend using mathematica or wolfram alpha to help).

The total number of photons is found by integrating the number of photons per frequency $n(f)$ over all frequencies $N_{tot} = \int n(f) df$. Factoring out some constants like 8, π, c, and surface area, this integral is given by

$$N_{tot} = (\text{constants}) \times \int_0^\infty df\ \frac{f^2}{e^{hf/k_BT} - 1}.$$

Defining the dimensionless variable $x = \frac{f}{(k_BT/h)}$, we find

$$N_{tot} = (\text{constants}) \times \left(\frac{k_BT}{h}\right)^3 \int_0^\infty dx\ \frac{x^2}{e^x - 1}.$$

The integral is now just a number (which happens to be equal to $2\zeta(3)$ where $\zeta(x)$ is the riemann zeta function), so the total number of photons depends on the temperature as T^3.

3. How many photons are emitted between frequencies $4 * 10^{14}$ and $4.01 * 10^{14}$ Hz?

Using the result of part 1, $N \approx n(f_0)\Delta f$ because the range of frequencies is small. Plugging in all the numbers, I get $1.29 * 10^{19}$ photons.

4. The peak of the energy distribution $u(f)$ occurs near $f \approx 2.82\ k_bT/h$, but the peak of the number distribution $n(f)$ occurs near $f \approx 1.59\ k_bT/h$. Why are these numbers different?

The number of photons has a maximum at $f_{ph,max}$ and decreases past that, but the energy of the photons increases as hf. The energy density is $n_{ph}(f) * \epsilon(f)$, so there is a competition between the decrease in number of photons and increase in the energy of the photons, which leads to the maximum in the energy density at the value $f_{en,max}$.