HW problem week 3

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

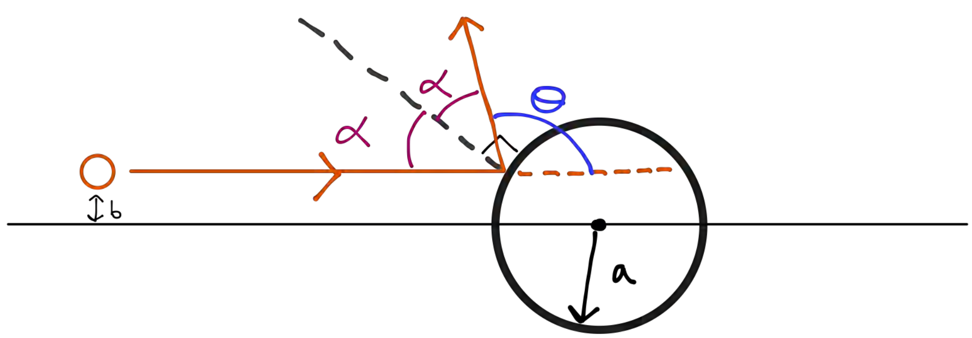
In class we learned about the theory behind Rutherford scattering, in which an alpha particle scatters off of an atomic nucleus due to the repulsive coulomb force. In this problem, we will consider scattering off of a 'hard sphere' potential.

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

A potential like this would correspond to, for example, two billiard balls bouncing off of each other. Here we will treat the incident particles as point particles. For scattering off of this potential, the rule is that the angle of 'reflection' is equal to the angle of incidence (from the normal).

1. Determine the relation between the scattering angle θ and the impact parameter b in the form of an equation for $b(\theta)$.

   As always, it's helpful to draw a diagram.

   The scattering angle is θ, the impact parameter is b, the radius of the hard sphere is a, and the angle of incidence is α. From this diagram we can find the following relations: $2\alpha + \theta = \pi$, and $b = a \sin \alpha$ (to see this, extend the dashed black line to the origin of the sphere, it makes a right triangle). So $b(\theta) = a \sin(\frac{\pi - \theta}{2}) = -a \cos(\theta/2)$.

2. By going through the same manipulations that we performed in the Rutherford scattering calculation, obtain the cross section $d\sigma$, and integrate it to obtain the total scattering cross section. Comment on your result: does it make sense? Why or why not?

   The cross section is defined as $d\sigma = 2\pi b; db$, the area of a ribbon of radius b and width db. Usually we prefer to give the differential cross section per solid angle, $d\Omega = 2\pi \sin\theta d\theta$ so we do some algebra to write

   $$d\sigma = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| d\Omega.$$

   In this problem $b(\theta) = -a\cos(\theta/2)$ so by plugging into the above equation we get

   $$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}.$$

   Don't worry about the missing minus sign: the area can't be negative after all. The cross section is independent of the angle so we can integrate it over the sphere

   $$\int d\Omega \frac{d\sigma}{d\Omega} = \pi a^2$$

   which is the transverse surface area of the sphere. The total scattering angle is just the total area presented by the hard sphere to the oncoming particles, great!

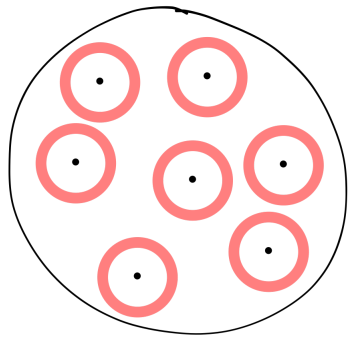
3. Does this hard sphere potential exhibit the same small angle scattering singularity as the coulomb potential did? Why or why not?

   No, it does not. In fact it doesn't even depend on the angle at all. The reason why we don't encounter that problem here is because the hard sphere potential is 'short ranged' whereas the coulomb force is a long range force. In the coulomb scattering problem, the divergence in the scattering cross section at small angles leads to an infinite total cross section: the point is that even particles which are miles away scatter very slightly off the nucleus (through a very small angle, effectively zero) because the coulomb force extends over infinite range. In the hard sphere scattering problem, if the incident particle does not hit the πa^2 surface area presented by the hard sphere, it just does not scatter. The hard sphere only exerts a 'force' over a limited region of space, so we get a finite total scattering cross section.

4. To obtain the expression in the book relevant for comparing our calculation to an experiment, we need to think about the statistical description of many incident particles scattering off of many targets. Assume that the targets are far enough apart that the cross section that one target provides for scattering does not overlap with the cross section provided by any other targets. For a target of area A, calculate the fractional area for scattering through an angle θ in terms of the density and thickness of the

target. Use this to determine the fraction of incident particles scattered through angle θ (per unit time per unit solid angle).

Here it's useful to draw another diagram.



This is a diagram of the head on view of the scattering surface (i.e. the gold foil) illuminated by the incident beam of particles. Each black dot represents a target (a nucleus, or a hard sphere) and the red highlighted region around each target represents the cross section *from that target* for scattering through some given angle θ. Note that in the hard sphere problem, the cross section for each target is literally 'on' the target, but that's hard to draw.

We want to calculate the fraction of incident particles scattered through angle θ. So the quantity we need is the fraction of the total incidence area occupied by the red ribbons. Then multiplying by the rate of incident particles will give us the fraction

$$R_{\text{scattered through angle }\theta}/R_{in}.$$

Suppose the total area illuminated by the incoming beam is A. The total area covered by the red ribbons is the area of one ribbon times the number of targets (this is where we need the assumption that the cross sections from different targets dont overlap).

In terms of the *number* density of the scattering material n, the number of targets is $n * d * A$ where d is the thickness of the material, and A is the area. Since we want the fraction of area covered by the cross section, divide by the area A to get the number of targets per area $n * d$.

Finally, the rate of particles scattered through angle θ, per solid angle, in terms of the rate of incident particles is

$$R_s = R_i * n * D * \frac{d\sigma}{d\Omega}.$$

To get the number of particles that our hypothetical detector would pick up, we would multiply by the solid angle covered by the detector.

5. **Optional, ungraded** Suppose a fraction x of the material you are scattering off of is composed of hard spheres of radius r_a and a fraction $1 - x$ have radius r_b. What is

the differential cross section for scattering through angle θ for this material? What if the material is composed of hard spheres of many different radii determined by some distribution $p(r)$?

From the diagram in the solution for part 4, we can imagine that if a fraction x of the particles have radius r_a and a fraction $1-x$ have radius r_b, the cross sections provided by either type of hard sphere contribute additively. So we could calculate the cross sections from the two kinds of hard spheres separately, and then add them together. If we have a continuous distribution of hard spheres of different radii, the sum becomes an integral $\int dr\ p(r)\ d\sigma(r)$ where $\sigma(r) = \frac{r^2}{4} d\Omega$.