HW problem week 3

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

In class, we derived the Bohr model of the atom following the assumption that the angular momentum $\vec{L}$ is quantized in units of $\hbar$. In this problem, you will repeat the derivation using the relativistic expressions for energy and momentum. In this problem (and always in life), you should try to combine physical constants into α, the fine structure constant, wherever possible.

1. Assume that the electron travels in a circular orbit with constant angular speed ω. Using Newton's law $\vec{F} = d\vec{p}/dt$ with the relativistic expression for the momentum, obtain an expression relating the radius r and the velocity v. Hint: Since the speed is constant, the relativistic answer for $d\vec{p}/dt$ is related in a simple way to the nonrelativistic answer.

   In the nonrelativistic case, we have $\vec{F} = md\vec{v}/dt = mv^2/r$. In the relativistic case, $F = d(\gamma mv)/dt = \gamma md\vec{v}/dt$ since γ doesn't depend on time (constant speed). So,

$$\frac{kZe^2}{r^2} = \frac{\gamma mv^2}{r} \implies r = \frac{kZe^2}{\gamma mv^2}$$

2. By quantizing the relativistic angular momentum (still $\vec{r} \times \vec{p}$) to integer multiples of $\hbar$, and using the result of part 1, show that the speed of the n'th Bohr orbit is the same as in the nonrelativistic case:

$$v_n = \frac{Z\alpha c}{n}.$$

$$n\hbar = \vec{L} = \vec{r} \times \vec{p} \underbrace{=}_{\text{circular orbit}} \gamma mvr \underbrace{=}_{\text{part 1}} \frac{kZe^2}{v}$$

$$v = \frac{kZe^2 c}{n\hbar c} = \frac{Z\alpha c}{n}$$

3. Using the result of parts 1 and 2, calculate the radius of the n'th Bohr orbit.

   Using parts 1 and 2,

$$r \underbrace{=}_{\text{angular momentum}} \frac{\hbar n}{mv}\sqrt{1-\frac{v^2}{c^2}} = \frac{\hbar n^2}{mZ\alpha c}\sqrt{1-\frac{Z^2\alpha^2}{n^2}} = \frac{\hbar n}{Z\alpha mc}\sqrt{n^2-Z^2\alpha^2} = \frac{a_0 n}{Z}\sqrt{n^2-Z^2\alpha^2}$$

4. The formula $E = \sqrt{(mc^2)^2 + p^2c^2} = \gamma mc^2$ is for a free particle; in the presence of a potential we add the potential energy U. Using the result of the previous parts, calculate the relativistic answer for the total energy.

   Putting everything together, $E = \gamma mc^2 - kZe^2/r$. The most convenient way to solve this is to use $r = \hbar n/\gamma mv$ from angular momentum quantization to get

$$E = \gamma mc^2 - \frac{kZe^2c}{c\hbar n}\gamma mv$$

   where we also multiplied by c in the numerator and denominator of the second term. Now recognize the second term as

$$\underbrace{\frac{Z\alpha c}{n}}_{=v,\text{ from part 2}} \times \gamma mv = \gamma mv^2$$

   so that the answer for the energy is

$$E = \gamma m(c^2 - v^2) = m\frac{c^2 - v^2}{\sqrt{1 - v^2/c^2}} = mc^2\sqrt{1-\frac{v^2}{c^2}}.$$

   To see how this depends on n, we now plug in $v = Z\alpha c/n$ to find

$$E = mc^2\sqrt{1 - (Z\alpha/n)^2}$$

5. By expanding in powers of α using the Taylor expansion

$$\sqrt{1+\epsilon} \approx 1 + \epsilon/2 - \epsilon^2/8 + O(\epsilon^3)$$

   for small ϵ, show that the energy levels are of the form

$$E_n = (\text{rest energy}) + (\text{bohr result}) - \frac{mc^2}{8}\left(\frac{\alpha Z}{n}\right)^4 + O(\alpha^6).$$

   Expanding the square root using the given formula,

$$(1 - (Z\alpha/n)^2)^{1/2} \approx 1 - (Z\alpha/n)^2/2 - (Z\alpha/n)^4/8.$$

So

$$E = \underbrace{mc^2}_{\text{rest energy}} - \underbrace{\frac{mc^2}{2}\frac{Z^2\alpha^2}{n^2}}_{\text{bohr answer}} - \frac{mc^2}{8}\left(\frac{\alpha Z}{n}\right)^4$$

6. The relativistic Bohr model of the atom actually makes a prediction for size of the largest stable element. By looking at the results derived in this problem and imposing some physical assumptions on the radius, velocity, or energy, find a condition on the atomic number Z of a hydrogen like atom.

   Possible answers are:

   (a) from the expression for the velocity, the condition that $v < c$ yields $Z < \alpha^{-1}$ for $n = 1$.

   (b) from the expression from the radius, the condition that r is real and ≥ 0 implies $n^2 - Z^2\alpha^2 > 0$ so for $n = 1$ obtains $Z < \alpha^{-1}$.

   (c) the energy should be real, so $1 - (Z\alpha/n)^2 > 0$. Again, for $n = 1$, find $Z <= \alpha^{-1}$.