HW problem week 4

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

In this homework problem we will look a little bit more at wave equations. Only problem 1 will be graded, but you should do them all!

1 Consider the differential equation

$$-\mathbf{i}\frac{\partial y}{\partial t} = \lambda \frac{\partial^2 y}{\partial x^2}.$$

Notice the $\mathbf{i}$ on the left hand side.

1. Find the dispersion relation for this differential equation by plugging in an ansats $y \sim e^{\mathbf{i}(kx-\omega t)}$.
2. Calculate the group and phase velocities.
3. Like we did in the discussion session, construct a gaussian wave packet using the amplitude $a(k) = e^{(k-k_0)^2/2\sigma^2}$ (hint: all of the integrals are gaussian). Using your favorite software, plot the absolute magnitude of the wave packet $|\Psi(x,t)|$ for some values of k_0, λ, and σ at a couple different times. Describe what happens to the wave packet as a function of time, and determine with what speed it moves.
4. If the factor of $\mathbf{i}$ was not there on the left hand side of the differential equation, what would the solutions look like? Would they still represent travelling waves? If you remove the $\mathbf{i}$, this differential equation is called the heat equation.

2, optional By rewriting the wave equation in terms of the variables $s_+ = x + vt$ and $s_- = x - vt$, show that it becomes

$$\frac{\partial^2 y}{\partial s_+ \partial s_-} = 0$$

implying that the solution is of the form $y(x) = af(s_+) + bg(s_-)$ where a and b are constants. This is sometimes called the d'Alembert formula or d'Alembert solution. Using the d'Alembert formula, find the solution to the one dimensional wave equation with the following intial conditions (a triangle pulse):

$$y(x, t = 0) = \begin{cases} 0, & |x| > 1 \\ 1 + x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \end{cases}$$

and $\dot{y}(x, t = 0) = 0$. Describe in words what your solution looks like. Draw a couple of snapshots at different times if you want to.

3, optional Consider the modified wave equation (here y_t means $\partial y/\partial t$)

$$y_{tt} = -v^2 y_{xx} - g y_t.$$

What does the extra term gy_t correspond to? If you dont know, you can wait until the end of the problem to come back and answer.

1. By plugging in a guess $y \sim e^{\mathbf{i}(kx-\omega t)}$, find the relation between ω and k. In general, an equation $\omega(k) = ...$ is called a *dispersion relation.*
2. By plugging $\omega(k)$ into our expression for y, convince yourself that something dramatic happens at a particular value of g. What is this value of g, and what happens there?
3. Calculate the group velocity of this wave.
4. If you didn't know what the extra term gy_t represented earlier, try to figure it out now based on how g affects the solution $y(x, t)$.