HW problem week 4

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

In this homework problem we will look a little bit more at wave equations. Only problem 1 will be graded, but you should do them all!

1 Consider the differential equation

$$-\mathbf{i}\frac{\partial y}{\partial t} = \lambda \frac{\partial^2 y}{\partial x^2}.$$

Notice the $\mathbf{i}$ on the left hand side.

1. Find the dispersion relation for this differential equation.

   By plugging in the ansatz $y \sim \exp\left(\mathbf{i}(kx - \omega t)\right)$, you should find $\omega = \lambda k^2$.

2. Calculate the group and phase velocities.

   The definitions are $v_{ph} = \omega/k = \lambda k$ and $v_g = d\omega/dk = 2\lambda k$. So here, $v_g = 2v_{ph}$.

3. Like we did in the discussion session, construct a gaussian wave packet using the amplitude $a(k) = e^{(k-k_0)^2/2\sigma^2}$ (hint: all of the integrals are gaussian). Using your favorite software, plot the absolute magnitude of the wave packet $|\Psi(x,t)|$ for some values of k_0, λ, and σ at a couple different times. Describe what happens to the wave packet as a function of time, and determine with what speed it moves.

   Put the integral into our standard gaussian integral format e^{-ax^2+bx+c}:

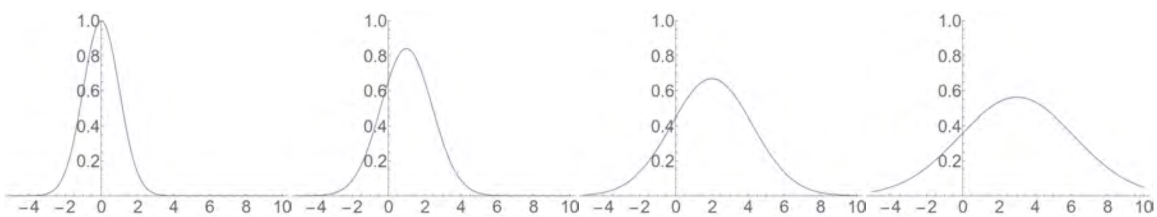
$$\int dk\, e^{-\frac{(k-k_0)^2}{2\sigma^2}} e^{\mathbf{i}(kx-\lambda k^2 t)} = \int dk\, e^{-k^2\left(1/2\sigma^2+\mathbf{i}\lambda t\right)+k\left(k_0/\sigma^2+\mathbf{i}x\right)-k_0^2/2\sigma^2}$$

   So the answer is

$$\Psi(x,t) = \sqrt{\frac{2\pi}{1/\sigma^2 + 2\mathbf{i}\lambda t}} \exp\left[\frac{\left(k_0/\sigma^2 + \mathbf{i}x\right)^2}{2/\sigma^2 + 4\mathbf{i}\lambda t}\right]$$

which could be simplified a little bit further according to your preference.

Below is plotted $|\Psi(x,t)|$ (rescaled so that $|\Psi(0,0)| = 1$) for $\lambda = 0.5, k_0 = 1, \sigma = 1$ at $t = 0, 1, 2, 3$.



The wave packet moves to the right with the group velocity (not the phase velocity!) and it also spreads out as it moves. This spreading out effect is called dispersion and generically occurs when ω depends nonlinearly on k.

4. If the factor of **i** was not there on the left hand side of the differential equation, what would the solutions look like? Would they still represent travelling waves? If you remove the **i**, this differential equation is called the heat equation.

   In this case, we would find $\omega = -\mathbf{i}\lambda k^2$ and the solutions would be dying exponentials $y \sim e^{\mathbf{i}kx-\lambda k^2 t}$. If you make a wave packet out of these solutions, they will not move; they just sit at the origin and decay in amplitude.

2, optional By rewriting the wave equation in terms of the variables $s_+ = x + vt$ and $s_- = x - vt$, show that it becomes

$$\frac{\partial^2 y}{\partial s_+ \partial s_-} = 0$$

implying that the solution is of the form $y(x) = af(s_+) + bg(s_-)$ where a and b are constants. This is sometimes called the d'Alembert formula or d'Alembert solution. Using the d'Alembert formula, find the solution to the one dimensional wave equation with the following intial conditions (a triangle pulse):

$$y(x, t = 0) = \begin{cases} 0, & |x| > 1 \\ 1 + x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \end{cases}$$

and $\dot{y}(x, t = 0) = 0$. Describe in words what your solution looks like. Draw a couple of snapshots at different times if you want to.

Our solution is of the form $af(x + vt) + bg(x - vt)$ where a and b are constants. The initial conditions say $af(x) + bg(x) = y_0(x)$ and $avf'(x) - bvg'(x) = 0$. A solution to these intial conditions is $y(x,t) = \frac{1}{2}y_0(x - vt) + \frac{1}{2}y_0(x + vt)$: two triangles of height 1/2 moving away from the origin at velocity v.

3, optional Consider the modified wave equation (here y_t means $\partial y/\partial t$)

$$y_{tt} = -v^2 y_{xx} - g y_t.$$

What does the extra term $g y_t$ correspond to? If you dont know, you can wait until the end of the problem to come back and answer.

1. By plugging in a guess $y \sim e^{\mathbf{i}(kx-\omega t)}$, find the relation between ω and k. In general, an equation $\omega(k) = ...$ is called a *dispersion relation.*

   Find the quadratic equation $-\omega^2 = -v^2k^2 + \mathbf{i}g\omega$ which has the solutions

   $$\omega = \frac{-\mathbf{i}g}{2} \pm \sqrt{v^2k^2 - g^4/4}.$$

2. By plugging $\omega(k)$ into our expression for y, convince yourself that something dramatic happens at a particular value of g. What is this value of g, and what happens there?

   When $g^2/4 > g_c = v^2k^2$, the frequency ω becomes purely imaginary. In general, we can write ω as a combination of its real and imaginary part $\omega = \omega' + \mathbf{i}\omega''$. Then since it appears in y as $e^{-\mathbf{i}\omega t}$, we see that the real part of ω corresponds to oscillations (complex exponential) and the imaginary part of omega corresponds to exponential growth or decay. When ω has both real and imaginary parts, the behavior is oscillations whose amplitudes decreases with time. When ω becomes purely imaginary there are no longer any oscillations, only exponential decay.

3. Calculate the group velocity of this wave.

   $$v_g = d\omega/dk = \pm\frac{v^2k}{\sqrt{v^2k^2 - g^2/4}}.$$

   Notice that this only makes sense for $g < g_c$, otherwise there are no travelling waves to begin with.

4. If you didn't know what the extra term $g y_t$ represented earlier, try to figure it out now based on how g affects the solution $y(x,t)$.

   The extra term represents damping, it is some kind of frictional force (notice it's proportional to the velocity). When $g < g_c$, the string can oscillate back and forth with an amplitude which decreases exponentially in time. The name for this is 'underdamped'. When $g > g_c$, the friction is so strong that the string does not even complete one oscillation; this is called the 'overdamped' case. The transition between the two at the value of g_c is called critical damping.