HW problem week 6

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you've solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

The normalized eigenfunctions of the infinite square well potential are $\psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$ inside the well.

1. Use these eigenstates to solve the time dependent schrodinger equation to find $\Psi_n(x,t)$. Using the full time dependent wave function, calculate $\langle x\rangle(t)$ and $\langle p\rangle(t)$ in the n'th energy eigenstate.

   Once we have an eigenstate ψ_n with energy ϵ_n, the time dependence is just an oscillating exponential. So $\Psi_n(x,t) = \psi_n(x)e^{-\mathbf{i}\epsilon_n t/\hbar} \equiv \psi_n(x)e^{-\mathbf{i}\omega_n t}$. To calculate the expectation value:

$$\langle x\rangle = \int dx\ \Psi_n^*(x,t)x\Psi_n(x,t) = \int dx\ \psi_n(x)^* e^{+\mathbf{i}\omega_n t} x\psi_n(x)e^{-\mathbf{i}\omega_n t} = \int dx\ \psi_n^*(x)x\psi_n(x)$$

$$\langle p_{op}\rangle = \int dx\ \psi_n(x)^* e^{+\mathbf{i}\omega_n t}(-\mathbf{i}\hbar\partial_x)\psi_n(x)e^{-\mathbf{i}\omega_n t} = \int dx\ \psi_n^*(x)(-\mathbf{i}\hbar\partial_x)\psi_n(x)$$

   We've calculated both of the expressions at the right hand side in class/discussion/problem session: $\langle x\rangle = L/2$ and $\langle p\rangle = 0$. Neither of them have any time dependence. This is generally true whenever we calculate an expectation value in an energy eigenstate.

2. Calculate the average energy $\langle E\rangle(t)$ and the average square energy $\langle E^2\rangle(t)$ to find the uncertainty in the energy $\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$. Hint: use the fact that the ψ_n are eigenstates of $\hat{H}$.

   Following the hint, $\hat{H}\psi_n = \epsilon_n\psi_n$ so $\hat{H}^2\psi_n = \hat{H}(\hat{H}\psi_n) = \hat{H}\epsilon_n\psi_n = \epsilon_n\hat{H}\psi_n = \epsilon_n^2\psi_n$. The average energy is

$$\langle E\rangle = \int dx \Psi^*\hat{H}\Psi = \epsilon_n \int dx\ \Psi^*\Psi = \epsilon_n$$

   and the average square energy is

$$\langle E^2\rangle = \int dx\ \Psi^*\hat{H}^2\Psi = \epsilon_n^2 \int dx\ \Psi^*\Psi = \epsilon_n^2.$$

So the energy uncertainty is $\sqrt{\langle E^2\rangle - \langle E\rangle^2} = 0$. There is no uncertainty in the energy, because we are in an energy eigenstate.

3. Now consider the superposition state

$$\Phi(x,t) = \frac{1}{\sqrt{2}}\left(\Psi_1(x,t) + \Psi_4(x,t)\right).$$

Verify that this state is normalized.

$$\int dx\ \Phi^*\Phi = \frac{1}{2}\int dx\ (\psi_1(x)^* e^{\mathbf{i}\omega_1 t} + \psi_4(x)^* e^{\mathbf{i}\omega_4 t})(\psi_1(x)e^{-\mathbf{i}\omega_1 t} + \psi_4(x)e^{-\mathbf{i}\omega_4 t})$$

$$= \frac{1}{2}\left(\underbrace{\int dx\ |\psi_1|^2}_{1} + \underbrace{\int dx\ |\psi_2|^2}_{1} + e^{\mathbf{i}(\omega_1-\omega_4)t}\underbrace{\int dx\ \psi_1^*\psi_4}_{0\text{,check it}} + e^{-\mathbf{i}(\omega_1-\omega_4)t}\underbrace{\int dx\ \psi_1\psi_4^*}_{0}\right) = 1$$

4. For this new state Φ, calculate $\langle x\rangle(t)$ and $\langle p\rangle(t)$. Would you call the state $\Phi(x,t)$ a stationary state? Why or why not?

Look up at the solution for part 3, and imagine sandwiching an x or a p_{op} in between $\Phi*$ and Φ. We can save ourselves a little bit of effort since we already know that $\langle x\rangle = L/2$ and $\langle p\rangle = 0$ if we evaluate their expectation value in a given eigenstate, so we only have to calculate the cross terms $\int \psi_1\ (x,\ \hat{p})\ \psi_4$ and $\int \psi_4\ (x,\ \hat{p})\ \psi_1$. The integrals for $\langle x\rangle$ turn out to be the same so I will just do one of them: they can be done by using trig product formulas to turn $\sin A \sin B$ into a combination of $\cos A \pm B$ and then using integration by parts. This is what is sometimes called 'straightforward yet tedious':

$$\int dx\ \psi_1 x \psi_4 = \frac{2}{L}\int dx\ x \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{4\pi x}{L}\right) = -\frac{32L}{225\pi^2}$$

We also need

$$\int dx\ \psi_1(-\mathbf{i}\hbar\partial_x)\psi_4 = -\mathbf{i}\hbar\frac{2}{L}\frac{4\pi}{L}\int dx\ \sin(\pi x/L)\cos(4\pi x/L) = -\frac{16\hbar}{15\mathbf{i}L},$$

$$\int dx\ \psi_4(-\mathbf{i}\hbar\partial_x)\psi_1 = -\mathbf{i}\hbar\frac{2}{L}\frac{\pi}{L}\int dx\ \cos(\pi x/L)\sin(4\pi x/L) = \frac{16\hbar}{15\mathbf{i}L}$$

Finally, putting it all together,

$$\langle x\rangle(t) = \frac{L}{2} - \frac{64L\hbar}{225\pi^2}\cos(\omega_{14}t)$$

where $\omega_{14} = \omega_1 - \omega_4$ and

$$\langle p\rangle(t) = \frac{32\hbar}{15L}\sin\omega_{41}t.$$

5. Repeat part 2 for the state Φ. Hint: be careful in how you apply the hint from part 2.

   The equation $\hat{H}\psi_n = \epsilon_n \psi_n$ applies to each energy eigenstate individually (note that the function Φ is *not* an energy eigenstate). So when we calculate $\langle H \rangle$, the cross terms will drop out again. Explicitly:

   $$\frac{1}{2}\int dx\, (\psi_1(x)^* e^{\mathbf{i}\omega_1 t} + \psi_4(x)^* e^{\mathbf{i}\omega_4 t})\hat{H}(\psi_1(x)e^{-\mathbf{i}\omega_1 t} + \psi_4(x)e^{-\mathbf{i}\omega_4 t})$$

   $$= \frac{1}{2}\int dx\, (\psi_1(x)^* e^{\mathbf{i}\omega_1 t} + \psi_4(x)^* e^{\mathbf{i}\omega_4 t})(\epsilon_1\psi_1(x)e^{-\mathbf{i}\omega_1 t} + \epsilon_4\psi_4(x)e^{-\mathbf{i}\omega_4 t})$$

   $$\rightsquigarrow \frac{1}{2}(\epsilon_1 + \epsilon_4)$$

   (the squiggly arrow includes using the fact that the wave functions ψ_n are normalized, among other things). By a similar story (use $\hat{H}^2 = \hat{H}\hat{H}$ and apply them one after another to the functions on the right), find

   $$\langle \hat{H}^2 \rangle = \frac{1}{2}(\epsilon_1^2 + \epsilon_4^2).$$

   Now, $(\Delta E)^2 = \frac{1}{2}(\epsilon_1^2 + \epsilon_4^2) - \left(\frac{1}{2}(\epsilon_1 + \epsilon_4)\right)^2 = \frac{1}{4}(\epsilon_1 - \epsilon_4)^2$.

6. Using the time-energy uncertainty principle $\Delta E \Delta t > \hbar/2$, estimate approximately how much time the particle spends in a particular eigenstate state before flipping to the other one.

   Setting $\Delta E \Delta t = \hbar/2$ we get

   $$\Delta t = \frac{\hbar}{2}\frac{2}{|\epsilon_1 - \epsilon_4|} = \frac{\hbar}{|\Delta E_{14}|}.$$

   There is a good lesson to learn here which is that there is a relationship in quantum mechanics between energy difference of a process, and the characteristic time scale over which that process occurs, as $\tau \sim \frac{\hbar}{\Delta E}$ or $\frac{1}{\omega}$. In this problem, we see that the energy difference between the two states *literally is* the frequency of the oscillations in the nonstationary state.