HW problem week 8

A 3d quantum harmonic oscillator is described by the potential

$$V(x) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2.$$

1. In cartesian coordinates, what would be the first four energy eigenstates if all of the ω were equal? For unequal ω, what are the corresponding wave functions and energies?

   Since the Hamiltonian is separable, we can write the wavefunction as $\Psi_{\mu,\nu,\lambda} = \psi_\mu(x)\psi_\nu(y)\psi_\lambda(z)$ where ψ_n is the $n'th$ 1-d harmonic oscillator wave function with the appropriate frequency. The ground state is

   $$\Psi_{000} \sim e^{-\frac{m}{2\hbar}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)}$$

   and it has energy $\hbar/2(\omega_x + \omega_y + \omega_z)$. The first couple excited states are $\sim x\Psi_{000},\ y\Psi_{000},\ z\Psi_{000}$ with energies $3\hbar\omega_x/2 + \hbar/2(\omega_y + \omega_z)$, $3\hbar\omega_y/2 + \hbar/2(\omega_x + \omega_z)$, and $3\hbar\omega_z/2 + \hbar/2(\omega_x + \omega_y)$ respectively.

2. In words (i.e., something other than $\omega_x = \omega_y$), what is the condition on the potential for L_z to be a sharp observable? Show by explicit calculation the the ground state is an eigenstate of $L_z = -\mathbf{i}\hbar\partial_\phi$ only if $\omega_x = \omega_y$.

   L_z will be a sharp observable if the potential has rotational symmetry in the x-y plane, which will be the case if $\omega_x = \omega_y$. For the total angular momentum to be a good quantum number requires complete rotational symmetry, which means all ω must be equal.

   Using $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ I find $\partial_\phi\Psi_{000} \sim r^2\sin^2\theta\sin 2\phi(\omega_x - \omega_y)\Psi_{000}$. So Ψ_{000} will be an eigenstate (with eigenvalue zero) only if $\omega_x = \omega_y$.

3. If $\omega_x = \omega_y = \omega_z$, the potential has full spherical symmetry. In this case, give the angular dependence of the ground state, and the three first excited states. For each of these states, what is their total angular momentum, and what is their L_z?

   In this case, we know that in spherical polar coordinates, the angular part of the Shroedinger equation are solved by the spherical harmonics $Y_{lm}(\theta,\phi)$. (this is just like the hydrogen atom) So the full wavefunctions are $\Psi_{nlm} = R_{nlm}(r)Y_{lm}(\theta,\phi)$ where R is some unknown function of r which could in principle depend on l and m as well. As for the hydrogen atom, the state Ψ_{nlm} has total angular momentum $|\boldsymbol{L}| = \hbar\sqrt{l(l+1)}$ and has $L_z = \hbar m$.

4. Your answers for the first excited wave functions in part 1 were different than your answer in part 3. By taking linear combinations of the wave functions from part 1 you can put the wavefunctions into the form you found in part 3. Using the table on page 282 of the textbook, find explicitly the linear combinations which produce the three $l = 1$ states. What (up to some overall constant) are the functions $R_{2,l,m}$?

   In polar coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. So apparently, the wavefunction $z\Psi_{000}$ is already in the desired form:

   $$ze^{-m\omega r^2/2\hbar} = \left(re^{-m\omega r^2/2\hbar}\right)\cos\theta.$$

   This is the state with $l = 1$ and $m = 0$, and has $R_{2,1,0} \sim re^{-m\omega r^2/2\hbar}$. To get the $m = \pm 1$ states, use the fact the $\cos\phi \pm \mathbf{i}\sin\phi = e^{\pm\mathbf{i}\phi}$. So

   $$x + \mathbf{i}y = r\sin\theta(\cos\phi + \mathbf{i}\sin\phi) = r\sin\theta e^{\mathbf{i}\phi} \sim rY_{1,1}.$$

   Similarly, $Y_{1,-1}$ is obtained by taking the other combination $x - \mathbf{i}y$. These states have thus have $m = +1$ and -1 respectively. The radial functions are the same for all three.

5. For the spherically symmetric case, find the next state which has zero total angular momentum. What is its energy?

   For this we need to use the second excited state wavefunction of the 1d QHO which are proportional to $(1 - 2m\omega x^2/\hbar)e^{-m\omega x^2/\hbar}$, with a similar expression for y and z.

   If we form the linear combination $\psi_2(x)\psi_0(y)\psi_0(z)+\psi_0(x)\psi_2(y)\psi_0(z)+\psi_0(x)\psi_0(y)\psi_0(z)$, you can check that this wavefunction is proportional to

   $$(3 - 2m\omega r^2/\hbar)e^{-m\omega r^2/2\hbar}.$$

   This wavefunction has no angular dependence (compare to Y_{00}) so it has $l = 0$. It's energy is $2\hbar\omega + 3/2\hbar\omega$.