Physics 4E   Quiz 1 Solutions   Spring 2017

Problem 1

$$F(v_x, v_y) = \frac{m}{2\pi k_B T} e^{-m\frac{v_x^2+v_y^2}{2k_BT}} = \frac{m}{2\pi k_B T} e^{-\frac{1}{2}\frac{mv_x^2}{k_BT}} e^{-\frac{1}{2}\frac{mv_y^2}{k_BT}}$$

$$\int_{-\infty}^{\infty} dv_x dv_y F(v_x, v_y) = \left[\left(\frac{m}{2\pi k_BT}\right)^{1/2}\int_{-\infty}^{\infty} dv_x\, e^{-\frac{1}{2}\frac{mv_x^2}{k_BT}}\right]^2$$

$$\lambda = \frac{1}{2}\frac{m}{k_BT}, \quad \int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\pi}{\frac{1}{2}\frac{m}{k_BT}}} = \sqrt{\frac{2\pi k_BT}{m}} = \boxed{\left(\frac{m}{2\pi k_BT}\right)^{-\frac{1}{2}}}$$

$\Rightarrow$ distribution is normalized.

(b) $$g(v)dv = \int_{v_x^2+v_y^2=v} dv_x dv_y \frac{m}{2\pi k_BT} e^{-\frac{mv^2}{2k_BT}} = 2\pi v dv \cdot \frac{m}{2\pi k_BT} e^{-\frac{mv^2}{2k_BT}}$$

$$\Rightarrow \boxed{g(v) = \frac{m}{k_BT} v\, e^{-mv^2/2k_BT}}$$

Check normalization: $$\int_0^\infty dv\, g(v) = \frac{m}{k_BT}\int_0^\infty dv\, e^{-\frac{mv^2}{2k_BT}} \cdot v$$

$$\lambda = \frac{m}{2k_BT}, \quad \int_0^\infty dx\, x e^{-\lambda x^2} = \frac{1}{2\lambda} \Rightarrow \int_0^\infty dv\, g(v) = \frac{m}{k_BT}\frac{1}{2m}\cdot 2k_BT \boxed{= 1}$$

(c) $$\langle v\rangle = \int_0^\infty dv\, v\, g(v) = \frac{m}{k_BT}\int_0^\infty dv\, v^2 e^{-\frac{mv^2}{2\pi k_BT}}$$

using $$\int_0^\infty dx\, x^2 e^{-\lambda x^2} = -\frac{\partial}{\partial\lambda}\frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}} \Rightarrow$$

$$\Rightarrow \langle v\rangle = \frac{m}{k_BT}\cdot\frac{1}{4}\cdot\sqrt{\frac{\pi\cdot 8(k_BT)^3}{m^3}} = \sqrt{\frac{\pi}{2}\frac{k_BT}{m}} \Rightarrow \boxed{A = \frac{\pi}{2}}$$

(d) There is several ways to find $v_{rms} = \sqrt{B \frac{k_B T}{m}}$:

1) equipartition:

$$\frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} k_B T \Rightarrow \frac{1}{2} m \langle v^2 \rangle = k_B T \Rightarrow$$

$$\Rightarrow \langle v^2 \rangle = \frac{2 k_B T}{m} \Rightarrow v_{rms} = \left(\frac{2 k_B T}{m} \right)^{1/2} \Rightarrow \boxed{B = 2}$$

2) Calculate directly

$$\langle v^2 \rangle = \int_0^\infty dv \cdot v^2 \cdot g(v) = \frac{m}{k_B T} \int_0^\infty dv\, v^3 e^{-m v^2 / 2 k_B T}$$

3) Calculate directly

$$\langle v_x^2 \rangle = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \int_{-\infty}^{\infty} d v_x\, v_x^2 e^{-\frac{1}{2} \frac{m v_x^2}{k_B T}}$$

Problem 2

Wavelength where it emits maximum power:

$$\lambda_m = \frac{hc}{4.965 k_B T} = \frac{12{,}400 \times 11{,}600}{4.965 \times 4674} = \boxed{6200\,\mathring{A}}$$

Power emitted: $P(\lambda) = \frac{c}{4} u(\lambda) \cdot A$; A = area, $u(\lambda) = \frac{8\pi}{\lambda^4} \frac{hc/\lambda}{e^{hc/\lambda kT} - 1}$

For $\lambda_0 = 620{,}000\,\mathring{A}$, $P(\lambda_0) = 1\mu W$.

Also, for $\lambda_0 \gg \lambda_m$, $\frac{hc/\lambda_0}{e^{hc/\lambda_0 kT} - 1} \simeq kT$. Using this, and $\frac{\lambda_0}{\lambda_m} = 100$,

$$P(\lambda_m) = \frac{u(\lambda_m)}{u(\lambda_0)} P(\lambda_0) = \left(\frac{\lambda_0}{\lambda_m}\right)^4 \frac{hc/\lambda_m}{e^{4.965} - 1} \frac{1}{k_B T} \times 1\mu W =$$

$$= (100)^4 \frac{12{,}400/6200}{e^{4.965} - 1} \times \frac{11{,}600}{4{,}674} \times 1\mu W = \boxed{3.49\,W}$$

Or, if we don't use that approximation,

$$P(\lambda_m) = \frac{u(\lambda_m)}{u(\lambda_0)} P(\lambda_0) = \left(\frac{\lambda_0}{\lambda_m}\right)^5 \cdot \frac{e^{0.04965} - 1}{e^{4.965} - 1} \cdot 1\mu W = 100^5 \times \frac{0.051}{142.31} \cdot 1\mu W = \boxed{3.58\,W}$$

(b) For $\lambda \gg \lambda_m$, $u(\lambda) = \frac{8\pi}{\lambda^4} \cdot k_B T$

So for $u(\lambda) \sim 16\mu W = 16\, u(\lambda_0) \Rightarrow \boxed{\lambda \simeq \frac{\lambda_0}{2} = 310{,}000\,\mathring{A}}$

(c) For example, $\lambda_1 = \lambda_m/10$, then $e^{49.65} = 3.6 \times 10^{20}$, and

$$P\left(\lambda_1 = \frac{\lambda_m}{10}\right) = \frac{u(\lambda_1)}{u(\lambda_m)} P(\lambda_m) = 10^5 \frac{e^{4.965} - 1}{e^{49.65}} \times 3.58\,W \; \boxed{\sim 10^{-14}\,W < 1\mu W}$$

Problem 3

$$\lambda' = \lambda + \lambda_c(1-\cos\theta) \quad ; \quad \lambda_c = \frac{h}{m_e c} = 0.0243\,\mathring{A}$$

$K_e = \frac{hc}{\lambda} - \frac{hc}{\lambda'}$. Maximum K_e is for $\theta = \pi$, $\lambda' = \lambda + 2\lambda_c \Rightarrow$

$$K_e = \frac{hc}{\lambda} - \frac{hc}{\lambda+2\lambda_c} = \frac{hc}{\lambda(\lambda+2\lambda_c)}\cdot 2\lambda_c \Rightarrow \qquad (K_e = 248\,eV)$$

$$\Rightarrow \lambda(\lambda+2\lambda_c) - \frac{2hc}{K_e}\lambda_c = 0 \Rightarrow \lambda^2 + 2\lambda\lambda_c - \frac{2hc}{K_e}\lambda_c = 0$$

$$\Rightarrow \lambda = -\lambda_c + \sqrt{\lambda_c^2 + \frac{2hc}{K_e}\lambda_c} = -0.0243\,\mathring{A} +$$

$$+ \sqrt{0.0243^2 + 2\times\frac{12{,}400}{248}\times 0.0243}\,\mathring{A} = 1.535\,\mathring{A}$$

$$\Rightarrow \boxed{\lambda = 1.535\,\mathring{A}},\ \lambda' = 1.584\,\mathring{A},\quad \frac{hc}{\lambda} - \frac{hc}{\lambda'} = 247.9\,eV$$

(b) The minimum kinetic energy of the scattered electron is $\boxed{0}$, corresponding to $\theta = 0$, $\lambda' = \lambda$.

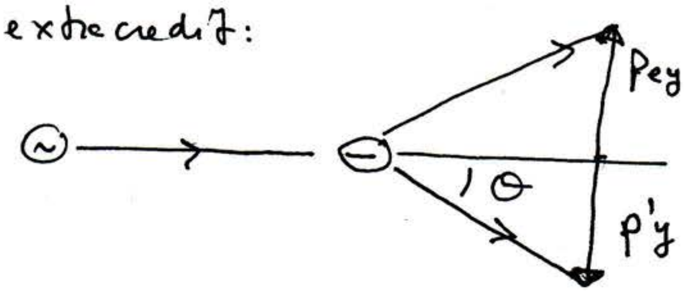
(c) For $K_e = 100\,eV$, find λ' and θ

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = K_e \Rightarrow \frac{hc}{\lambda'} = \frac{hc}{\lambda} - K_e \Rightarrow \lambda' = \frac{hc}{\frac{hc}{\lambda} - K_e} = 1.554\,\mathring{A}$$

$$\lambda' = \lambda + \lambda_c(1-\cos\theta) \Rightarrow \frac{\lambda'-\lambda}{\lambda_c} = 1-\cos\theta \Rightarrow \cos\theta = 0.218 \Rightarrow$$

$$\Rightarrow \boxed{\theta = 77.40^\circ}$$

(d) extra credit:



$p_{ey} = p'_y \Rightarrow$ maximum p_{ey} is for maximum p'_y. $p = \frac{h}{\lambda}$ for photon.

$p'_y = \frac{h}{\lambda'} \sin\theta = \frac{h \cdot \sin\theta}{\lambda + \lambda_c(1-\cos\theta)}$. Max p'_y is for $\frac{d p'_y}{d\theta} = 0 \Rightarrow$

$$\Rightarrow \frac{\cos\theta}{\lambda + \lambda_c(1-\cos\theta)} - \frac{\lambda_c \sin^2\theta}{[\lambda + \lambda_c(1-\cos\theta)]^2} = 0 \Rightarrow$$

$$\cos\theta\,(\lambda + \lambda_c(1-\cos\theta)) = \lambda_c \sin^2\theta \Rightarrow$$

$$\Rightarrow \lambda\cos\theta + \lambda_c\cos\theta - \lambda_c\cos^2\theta = \lambda_c\sin^2\theta \Rightarrow$$

$$(\lambda + \lambda_c)\cos\theta = \lambda_c \Rightarrow \boxed{\cos\theta = \frac{\lambda_c}{\lambda + \lambda_c}}$$

$\lambda = 1.535\,Å$
$\lambda_c = 0.0243\,Å$

$$\Rightarrow \cos\theta = 0.0156 \Rightarrow \boxed{\theta = 89.1^\circ}$$