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ON THE INSTABILITY OF SUPERPOSED FLUIDS IN A GRAVITATIONAL FIELD*

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ABSTRACT

In this paper an approximate analytic solution is obtained for the following problem: A perfect, incompressible fluid occupies the upper half of a vertical tube, being supported against gravity by a rigid diaphragm. The lower half of the tube is empty. At time t_0 the diaphragm is removed, and an infinitesimal disturbance of a simple kind is impressed on the free surface of the fluid. The problem is to describe the subsequent flow, on the assumption that the fluid at sufficiently great heights above the free surface is permanently at rest. The initial disturbance is so chosen that the fluid rises in the center of the tube and runs down at the sides. The range of validity of the approximate solution obtained is discussed. It is shown that the vertex height ζ increases exponentially, in agreement with the linearized theory, until $\zeta \approx 0.2\ \lambda/2\beta_1$, where λ is the diameter of the tube and β_1 is the first zero of the Bessel function $J_1(r)$. When $\zeta \gtrsim 1.5\ \lambda/2\ \beta_1$, it increases at a nearly constant rate. The method of solution described is applied to an analogous problem involving two-dimensional flow between parallel plane walls and also to spatially periodic flows.

I. INTRODUCTION

The study of small oscillations at the interface between superposed fluids in a gravitational field was initiated by Stokes (see Lamb 1932). In the simplest case both fluids are inviscid and incompressible and extend to infinity on either side of the interface. If the mean level of the interface is the plane $y = 0$, and the gravitational potential is $-gy$, the equation of the interface is

$$\eta(x, t) = \int_{-\infty}^{\infty} A_k(t)\, e^{ikx} dk \qquad (A_{-k} = A_k^*), \tag{1}$$

where

$$A_k(t) = A_k e^{\sigma(k)t}, \qquad \sigma = \left[gk \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right]^{1/2}. \tag{2}$$

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The subscripts 1 and 2 refer to the lower and upper fluids, respectively. The initial condition,

$$\eta(x, 0) = \int_{-\infty}^{\infty} A_k e^{ikx} dk, \tag{3}$$

determines the coefficients A_k.

When $\rho_1 > \rho_2$, the frequency σ/i is real, and the interface oscillates periodically. When $\rho_1 < \rho_2$, however, σ itself is real, and small disturbances of the interface grow exponentially. The instability tends to bring the heavy fluid to the bottom and the light fluid to the top.

In the hydrodynamic equations the constant g need not represent a true gravitational acceleration; it can equally well represent an "inertial" acceleration (see Taylor 1950). For example, if two incompressible fluids occupy a tube fitted with a moving piston and no gravitational field is present, the hydrodynamic equations in a frame of reference moving with the piston are identical with those for superposed fluids at rest in a gravitational field that is equal in magnitude and opposite in sign to the acceleration of the piston. A more interesting example is that of a liquid separated from a gas by a rigid diaphragm. If the pressure of the gas exceeds the pressure of the liquid, the interface will be unstable when the diaphragm is taken away. Evidence for this kind of instability appears in photographs of underwater explosions (Cole 1948).

Spitzer (1954; see also Frieman 1954) has suggested that something analogous to an underwater explosion may occur when an O star or a B star is formed in a medium of neutral hydrogen. The ionizing radiation from the star produces a spherical volume of gas in which the pressure and temperature are much higher than in the surrounding medium. As the hot gas expands, it becomes less dense than its surroundings. One would therefore expect the interface between the regions of neutral and ionized hydrogen to become unstable—provided that the star is formed sufficiently rapidly.

The early work of Stokes has been generalized in several ways:

a) *Viscosity and interfacial tension.*—Stokes himself discussed the influence of viscosity and surface tension on the oscillations at the free surface of a liquid (Lamb 1932). Harrison (1908) worked out an approximate theory of oscillations at the interface between superposed viscous liquids, taking into account the effects of interfacial tension. Pennington (1952), Chandrasekhar (1955), and Hide (1955) have extended Harrison's results. Viscosity and interfacial tension both exert a stabilizing influence on the interface. When $\rho_1 < \rho_2$, a critical wave length exists, whose value depends on the viscosity and the interfacial tension. The interface is unstable against disturbances whose wave lengths exceed the critical frequency, but stable against disturbances of shorter wave length.

b) *Compressibility.*—Wheeler, Carter, Frieman, and Pennington (1952) have modified Stokes's theory to include the effects of compressibility when the Mach number is not too large.

c) *Boundary conditions.*—The formulae that have been obtained for two-dimensional flow can be extended in a straightforward way to axially symmetric flow. But the problem of small oscillations (or instability) at a spherical interface presents some new features. It has recently been solved by Chandrasekhar (1955).

The investigations mentioned so far all proceed from a set of approximate hydrodynamic equations from which the nonlinear terms have been omitted. Consequently, they apply only to the initial development of an unstable interface. Now in astronomical applications the subsequent development is also of interest. In the present paper we shall derive an approximate analytic solution of the exact, nonlinear equations for an especially simple problem, which may be formulated in the following terms:

Let an incompressible, inviscid fluid occupy the region above the plane $z = 0$ inside a right circular cylinder centered on the z-axis. The rest of the cylinder is empty; but the fluid is supported against gravity by a diaphragm. At time t_0 the diaphragm is taken away, and a small disturbance of a simple kind is impressed on the free surface. We assume that the fluid at very great heights above the interface is permanently at rest. The problem is to describe the ensuing flow.

For the initial conditions that we shall use, the fluid rises near the center of the tube and descends near the sides (see Fig. 1). After enough time has elapsed, the flow takes on a quasi-steady aspect: the vertex (i.e., the highest point of the interface) rises at a constant rate, and in a frame of reference moving with the vertex the flow is identical with the steady flow around a solid of revolution in the absence of a gravitational field—except that the descending fluid, having traveled only a finite distance, will not fill the space between the solid and the walls of the tube.

Davies and Taylor (1950) have given an approximate theory of the steady state. For the speed of the vertex they find

$$V = 0.464\, (gR)^{1/2}\,, \tag{4}$$

where R is the radius of the tube.

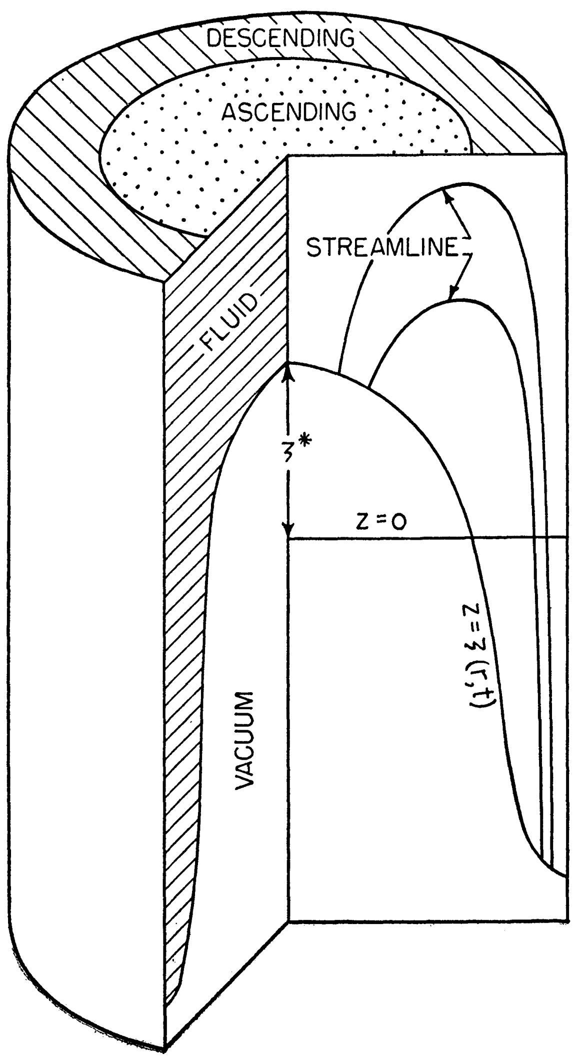

FIG. 1.—Illustrating the case of axially symmetric flow treated in Section II, 1

The theory of Davies and Taylor holds for $t \gg t_0$. The linear theory holds for $t \approx t_0$. One can bridge the gap between the two in a rough sort of way by using the linear theory until the predicted vertex speed attains the value (4) and then switching to the steady-state theory. As it happens (see Fig. 2), this simple approximation, which was originally suggested by Fermi, is never in error by more than about 25 per cent. For many purposes this accuracy will suffice. However, the more precise theory described later may be of some interest in its own right, because very few nonsteady flows with a free boundary have been successfully treated analytically, even in an approximate way.

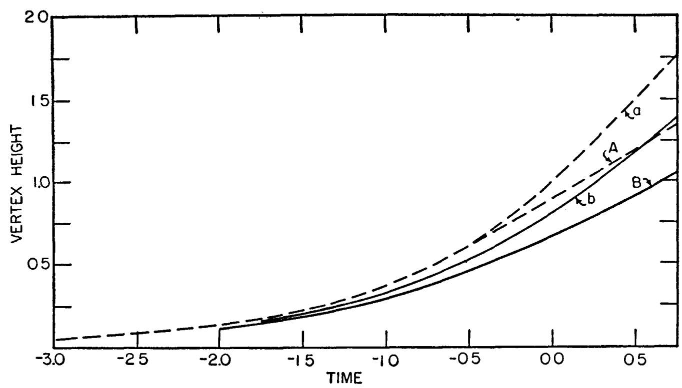

FIG 2.—Vertex height as a function of time, in the dimensionless units defined by equations (5) and (52). *A*, Fermi's approximation for two-dimensional flow. *B*, Fermi's approximation for tubular flow. *a*, Present approximation for two-dimensional flow. *b*, Present approximation for tubular flow.

II. APPROXIMATE SOLUTIONS

1 TUBULAR FLOW

We shall find it convenient to adopt the units of length and time defined by the equations

$$\frac{R}{\beta_1} = 1 , \qquad g = 1 , \tag{5}$$

where R is the radius of the tube, g is the acceleration of gravity, and $\beta_1 (= 3.83 \ldots)$ is the first zero of the Bessel function $J_1(r)$.

Since the fluid is inviscid and is initially at rest, the motion is permanently irrotational. The velocity field can therefore be derived from a scalar velocity potential

$$q = -\nabla\phi . \tag{6}$$

Since the fluid is also incompressible, the velocity field is solenoidal, so that ϕ must satisfy Laplace's equation,

$$\nabla^2\phi = 0 . \tag{7}$$

We seek an axially symmetric solution $\phi(z, r, t)$ of Laplace's equation that satisfies the boundary conditions,

$$\phi_r(z, \beta_1, t) = 0 \tag{8}$$ [1]

(no radial flow at the walls) and

$$\phi_z(\infty, r, t) = 0 \tag{9}$$

[1] Here and in what follows we shall use subscripts to indicate partial derivatives. Thus $\phi_r \equiv \partial\phi/\partial r$, etc.

(vanishing vertical flow at large positive values of z). In addition, ϕ must satisfy Bernoulli's equation,

$$\phi_t - \tfrac{1}{2}(\phi_z^2 + \phi_r^2) - z = \alpha(t) \tag{10}$$

on the free surface. Here α is an arbitrary function.

The simplest nonconstant function that satisfies equations (7)–(9) is

$$\phi = F(t)\, e^{-z} J_0(r)\,, \tag{11}$$

where $J_0(r)$ is the Bessel function of order zero. The function $F(t)$ must be chosen in such a way that equation (10) is satisfied as nearly as possible on the free surface.

Now the equations of motion for a fluid particle are

$$\dot{z} = -\phi_z = F(t)\, e^{-z} J_0(r)\,, \tag{12}$$

$$\dot{r} = -\phi_r = F(t)\, e^{-z} J_1(r)\,, \tag{13}$$

since

$$J_0'(r) = -J_1(r)\,. \tag{14}$$

Suppose that we have found a solution of equations (12) and (13):

$$z = z(t;\, r_0,\, z_0)\,, \qquad r = r(t;\, r_0,\, z_0)\,, \tag{15}$$

where (r_0, z_0) are the co-ordinates of the particle at time t_0. By eliminating t between these equations, we evidently obtain the equation of the trajectory of the particle. Similarly, to find the equation $z = \zeta(r, t)$ of a surface that moves with the fluid, given $z = \zeta(r, t_0)$—i.e., $z_0 = z_0(r_0)$—we insert $z_0(r_0)$ in equations (15) and eliminate r_0. In this way we can find the equation of the free surface for any given initial condition. Before doing this, let us consider briefly the nature of the flow defined by the velocity potential (11).

The Stokes stream function ψ is defined by the equations

$$\psi_z = -r\phi_r\,, \qquad \psi_r = r\phi_z\,. \tag{16}$$

Since

$$J_1'(r) = J_0(r) - \frac{J_1(r)}{r} \tag{17}$$

and $J_0'(r) = -J_1(r)$,

$$\psi(z, r, t) = F(t)\, r\, e^{-z} J_1(r)\,, \tag{18}$$

and the streamlines $\psi =$ Constant are given by

$$e^z = C r J_1(r)\,. \tag{19}$$

Since the pattern of streamlines does not change with time, the streamlines coincide with the trajectories of fluid particles. We can also obtain this result by noticing that equation (19) is the integral of the equation

$$\frac{dz}{dr} = \frac{J_0(r)}{J_1(r)}\,, \tag{20}$$

which follows from the equations of motion, (12) and (13).

According to equation (19), one can generate all the stream surfaces by displacing any one of them parallel to the z-axis. The section of a stream surface by a plane through the z-axis resembles an inverted **U** whose legs asymptotically approach the z-axis and the wall and whose apex lies directly above the first zero of $J_0(r)$. The flow has an upward component on the axis side and a downward component on the wall side (see Fig. 1).

To find the equation of the free boundary, we integrate the equations of motion (12) and (13). Set

$$Z = e^z , \tag{21}$$

$$v = r^2 , \tag{22}$$

$$T(t) = \int_t^t F(t)\,dt + 1 , \tag{23}$$

$$K(v) = \frac{2J_1(r)}{r} . \tag{24}$$

In terms of the new variables, equations (12) and (13) become

$$\dot{Z} = J_0(v^{1/2}) , \tag{25}$$

$$Z = \frac{vK(v)}{\dot{v}} , \tag{26}$$

where the dot now denotes differentiation with respect to T. Eliminating Z between these equations and using the identity (17), one obtains, after a simple calculation,

$$\frac{v}{v_0} = \frac{(T-1)\,K(v_0)}{Z_0+1} , \tag{27}$$

$$\frac{Z}{Z_0} = \frac{K(v)}{K(v_0)}\,\frac{(T-1)\,K(v_0)}{Z_0+1} . \tag{28}$$

These are the parametric equations of the free surface.

Since the velocity potential (11) is not exact, it cannot satisfy Bernoulli's equation (10) over the entire free surface. Now we are interested primarily in the flow near the vertex, for when that is known, we can find the flow at the sides—at least in an approximate way—by another method (see Sec. II, 3). We shall therefore choose the function $F(t)$ in such a way that Bernoulli's equation is satisfied in a first-order neighborhood of the vertex.

We shall assume that initially the free surface is perfectly flat, so that

$$z_0(r_0) = 0 , \qquad Z_0(v_0) = 1 . \tag{29}$$

To obtain the equation of the free surface, we insert this value of Z_0 in equations (27) and (28) and eliminate v_0. Neglecting terms nonlinear in v (since we are concerned only with a first-order neighborhood of the vertex), we obtain:

$$v = v_0 T , \tag{30}$$

$$e^{\zeta} = T\left[1 - \frac{v}{8}(1 - T^{-2})\right] . \tag{31}$$

We now substitute the velocity potential (11) in Bernoulli's equation (10), replacing z by ζ, as given by equation (31), and $F(t)$ by $T'(t)$. Equating the coefficient of v in the resulting equation to zero, we obtain a second-order, nonlinear differential equation for the function $T(t)$:

$$T(T^2+1)T'' - T'^2 - T^2(T^2-1) = 0 . \tag{32}$$

Let us verify that the solution of equation (32) has the proper asymptotic behavior. When $t \gg t_0$, $T \gg 1$, and equation (32) becomes, approximately,

$$T'' - T = 0 . \tag{33}$$

Hence

$$F(t) = T'(t) = e^{t+c} \quad (t \gg t_0) , \tag{34}$$

and

$$\phi = e^{-(z-t-c)} J_0(r) \quad (t \gg t_0) . \tag{35}$$

From equations (31) and (24) we find

$$\left.\frac{d\zeta}{dt}\right|_{r=0} = \frac{d \log T}{dt} = 1 \quad (t \gg t_0) . \tag{36}$$

Thus the velocity potential (35) describes a steady state of flow in which the vertex advances with unit velocity. In c.g.s. units the vertex speed is

$$V = \left(\frac{gR}{\beta_1}\right)^{1/2} \approx 0.511\,(gR)^{1/2} . \tag{37}$$

The experimental values of the numerical coefficient in this formula (Davies and Taylor 1949) lie between this value and that derived by Davies and Taylor (eq. [4]). Although Davies and Taylor used a velocity potential that is equivalent to equation (35), they chose to satisfy Bernoulli's equation at two distinct points ($r = 0$, $r = \frac{1}{2}R$) rather than in a first-order neighborhood of the vertex.

When $t \approx t_0$,

$$T = 1 + \tau(t) \quad [\tau(t) \ll 1] , \tag{38}$$

and equation (32) becomes, approximately,

$$\tau'' - \tau = 0 . \tag{39}$$

The initial condition (29) requires that

$$\tau(t_0) = 0 . \tag{40}$$

Hence the appropriate solution of equation (39) is

$$\tau(t) = a\,e^{t_0} \sinh(t - t_0) \quad (a\,e^{t_0} \ll 1) , \tag{41}$$

so that

$$F(t) = T'(t) = \tau'(t) = a\,e^{t_0} \cosh(t - t_0) \tag{42}$$

and

$$\phi = a\,e^{t_0} \cosh(t - t_0)\, e^{-z} J_0(r) \quad (t \approx t_0) . \tag{43}$$

The velocity potential (43) is the one given by the linearized theory. We conclude that the solutions of equation (32) have the proper asymptotic form.

To obtain the complete solution of equation (32), we set

$$X(T) = T'^2 . \tag{44}$$

Equation (32) may be written in the form

$$\frac{d}{dT}\left[X\left(1 + T^{-2}\right)\right] = 2\left(T - T^{-1}\right) , \tag{45}$$

whence

$$X = T'^2 = F^2 = \left(1 + T^{-2}\right)^{-1}\left(T^2 - 2 \log T + \text{const.}\right) \tag{46}$$

the constant in equation (46) being determined by the asymptotic form of $T(t)$ for $t \approx t_0$. For the sake of simplicity, we choose

$$t_0 = -\infty , \qquad a = 2 . \tag{47}$$

Then equation (41) becomes

$$\tau (t) = e^t , \tag{48}$$

and the constant in equation (46) = -1.

Let $\zeta^*(t)$ denote the co-ordinate of the vertex. The height of the vertex is given by equation (31):

$$\zeta^* (t) = \log T (t) . \tag{49}$$

Combining this equation with equation (46), we obtain, for the vertex speed,

$$V (t) = \frac{d\zeta^*}{dt} = \left(\frac{e^{2\zeta^*} - 2\zeta^* - 1}{e^{2\zeta^*} + 1} \right)^{1/2} . \tag{50}$$

The height of the vertex at time t is given implicitly by the formula

$$t(\zeta_2^*) - t(\zeta_1^*) = \int_{\zeta_1^*}^{\zeta_2^*} \left(\frac{e^{2x} + 1}{e^{2x} - 1 - 2x} \right)^{1/2} dx , \tag{51}$$

and is plotted in Figure 2.

2. TWO-DIMENSIONAL FLOW BETWEEN PARALLEL WALLS

The extension of the preceding theory to the case of two-dimensional flow between parallel walls presents no new features. We shall therefore quote the results for two-dimensional flow, but we shall omit the working.

We assume that the flow is identical in every plane parallel to the xy-plane. The gravitational field is in the negative y-direction, the walls are at $x = \pm R$, the plane $y = 0$ represents the free surface at time t_0, and the equation of the free surface is $y = \eta(x, t)$. The units of length and time are so chosen that

$$\frac{R}{\pi} \equiv k^{-1} = 1 , \qquad g = 1 . \tag{52}$$

The velocity potential is

$$\phi = F (t) \, e^{-y} \cos x . \tag{53}$$

The streamlines and the trajectories of fluid particles are given by the equation

$$e^y = C \sin x . \tag{54}$$

The auxiliary quantity T is defined as before and satisfies the equation

$$T (2T^3 + 1) T'' + (T^3 - 1) T'^2 - T^2 (T^3 - 1) = 0 , \tag{55}$$

which has the asymptotic solutions

$$T = e^{3^{-1/2} t + c} \qquad (t \gg t_0) \tag{56}$$

and

$$T = 1 + a \, e^{t_0} \sinh (t - t_0) \qquad (t \approx t_0) . \tag{57}$$

The corresponding velocity potentials are

$$\phi = e^{-(y - 3^{-1/2} t - c)} \cos x \qquad (t \gg t_0) , \tag{58}$$

$$\phi = a\, e^{t_0} \cosh (t - t_0)\, e^{-y} \cos x \qquad (t \approx t_0)\,. \tag{59}$$

The vertex speed in the steady state is

$$V = 3^{-1/2} \left(\frac{gR}{\pi}\right)^{1/2} \approx 0.326\,(gR)^{1/2}. \tag{60}$$

The vertex speed for arbitrary t is

$$V(t) = \left[\frac{e^{3\eta^*} - 3\eta^* - 1}{3\,(e^{3\eta^*} + \frac{1}{2})}\right]^{1/2}, \tag{61}$$

and the vertex height is given implicitly by the formula

$$t(\eta_2^*) - t(\eta_1^*) = \int_{\eta_1^*}^{\eta_2^*} \left[\frac{3\,(e^{3x} + \frac{1}{2})}{e^{3x} - 1 - 3x}\right]^{1/2} dx \tag{62}$$

and is plotted in Figure 2.

3. VALIDITY OF THE APPROXIMATION; NATURE OF THE FLOW NEAR THE WALLS

We shall explicitly consider the case of tubular flow; but the following discussion, with some obvious modifications, also applies to two-dimensional flow. In the quasi-steady state, the flow at depths greater than about $1.5R$ below the vertex is nearly parallel to the wall of the tube. Using this fact, together with Bernoulli's equation and the equation of continuity, Davies and Taylor (1950) wrote down the equations

$$\pi R^2 V = \pi\,(R^2 - r^2)\, q = \pi\,(R^2 - r^2)\,[2g\,(\zeta^* - z)]^{1/2}, \tag{63}$$

where V is the vertex speed. These equations, rather than the theory of Section II, 1, should be used to calculate the shape of the interface in the region where the fluid runs parallel to the wall.

According to the second of equations (63), $q \propto (-z)^{1/2}$ for large negative values of z. The approximate theory of Section II, 1, gives $q \propto \exp(-z)$. Unfortunately, it does not seem possible to develop a simple unified approximation that will yield both the correct vertex speed and the correct asymptotic flow for large negative values of z. The following discussion will bring out the nature of the difficulty.

The velocity potential (11) describes the simplest kind of axially symmetric flow that satisfies the fixed-boundary conditions of the present problem *and is free of singularities in the entire tube.* The most general flow of this kind is given by the velocity potential,

$$\phi = \sum_{k=1}^{\infty} F_k(t)\, e^{-\beta_k z/\beta_1} J_0 \frac{\beta_k r}{\beta_1}, \tag{64}$$

where the β_k's are the zeros of the Bessel function $J_1(r)$. We see at once that this velocity potential cannot have the proper asymptotic form for large negative values of z and $t \gg t_0$. Consequently, the exact velocity potential cannot be free of singularities in the entire tube: it must contain terms that represent sources in the region not actually occupied by the fluid.

Symmetry requires the sources to lie on the z-axis. Let $z^*(t)$ denote the co-ordinate of the highest source (or the limit point of a continuous distribution of sources). Then expansion (64) holds only for $z \geq z^*$.

We can form a rough idea of how z^* varies with time by examining the limiting cases $t \gg t_0$ and $t \approx t_0$. In the quasi-steady state the quantity $(\zeta^* - z^*)$ is constant in time. Inspection of the flows that have been calculated for various distributions of sources (Milne-Thomson 1950) suggests that $\zeta^* - z^* \approx R$. At the other limit, $t \to t_0$, the velocity potential (11) becomes exact. Hence $\zeta^* - z^* \rightarrow +\infty$ as $t \rightarrow t_0$. These two limiting cases suggest the approximate rule

$$\zeta^* - z^* \approx -\zeta_{rr}^{-1}(0, t) . \tag{65}$$

That is, the highest singularity coincides approximately with the center of curvature of the vertex.

The theory of Section II, 1, probably describes the flow above the plane $z = z^*$ with good accuracy. To describe the rest of the flow, one can use equations (63), together with the requirement that the volume of fluid below the plane $z = 0$ equals the volume not occupied by fluid above the plane $z = 0$.

III. INSTABILITY OF AN INFINITE PLANE INTERFACE

1. PERIODIC FLOWS

In Section II we considered flows confined by tubes and by parallel plane walls. We shall now extend the results to unconfined, but spatially periodic, flows. In the case of two-dimensional flow between parallel walls the extension is trivial: we may evidently regard the walls as mathematical planes of symmetry. To extend the results for tubular flow, consider an infinite array of right cylinders whose walls intersect any plane $z =$ Constant in a network of regular hexagons. We may reasonably assume that the flow in a hexagonal cylinder of mean radius R resembles the flow in a circular cylinder of radius R, except in the neighborhood of the walls. We may therefore use the expression derived in Section II, 1, for the vertex speed. Near the walls, however, the fluid will tend to flow into the corners, forming long spikes rather than a uniform curtain. If the flow is the same in every cylinder, it will not be affected by the walls, which we may therefore regard as mathematical planes of symmetry.

The fluid displaced by a rising bubble runs down into six spikes, each of which serves three bubbles; hence there are two spikes for every bubble. Assuming that the spikes have circular cross-sections of radius $a(z)$, we can determine $a(z)$ approximately from Bernoulli's equation and the equation of continuity. Considerations of continuity approximately fix the rate of growth of the spikes (see the final paragraph of Sec. II, 3).

2. APERIODIC FLOWS

The periodic flows arise from initial conditions that are very unlikely to be realized in nature. However, the theoretical description of more general flows presents great difficulty. In the present section we shall try to get some insight into the nature of aperiodic flows by qualitative arguments based on our knowledge of special flows.

Consider an arbitrary two-dimensional disturbance. In the initial phase of growth $\eta(x, t)$ is a superposition of independent Fourier amplitudes,

$$A_k(t) = A_k \exp[(gk)^{1/2}t] , \tag{66}$$

where

$$A_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x, 0)\, e^{-ikx} dx . \tag{67}$$

According to equation (66), the Fourier amplitudes corresponding to large values of k (i.e., small wave lengths) grow faster than those corresponding to small values of k (i.e., large wave lengths). On the other hand, the phase of exponential growth lasts longer for the latter than for the former. Now the Fourier amplitudes begin to interfere badly with one another when the mean amplitude of the disturbance becomes comparable with the mean wave length. From the preceding remarks it is clear that at this stage the predominant wave numbers will come from a relatively narrow subrange of the range of wave numbers for which the coefficients A_k are appreciable. Fourier amplitudes corresponding to the largest wave numbers will have been outstripped; those corresponding to the smallest wave numbers will not have had time to develop. Thus the initial competition between Fourier amplitudes tends to establish a nearly periodic pattern of flow.

As the mean amplitude of the disturbance becomes comparable with the mean wave length, the competition between noninterfering Fourier amplitudes goes over into a competition between individual bubbles. To see what effects are introduced by small departures from periodicity at this stage, let us consider a simple example. Figure 3 shows three well-developed bubbles, the middle one being slightly narrower than its companions. In general, the pattern of flow will resemble that for periodic flow. However, the flow pattern associated with the middle bubble will be slightly compressed at the top, while the flow patterns associated with the bubbles on either side will be correspondingly dilated. This is indicated in Figure 3 by the bending of the streamlines SS and $S'S'$. That the channel defined by these streamlines must, in fact, narrow from bottom to top follows from the requirement that the flow be continuous on SS and $S'S'$ and the fact that the vertex speed of the two large bubbles exceeds that of the small

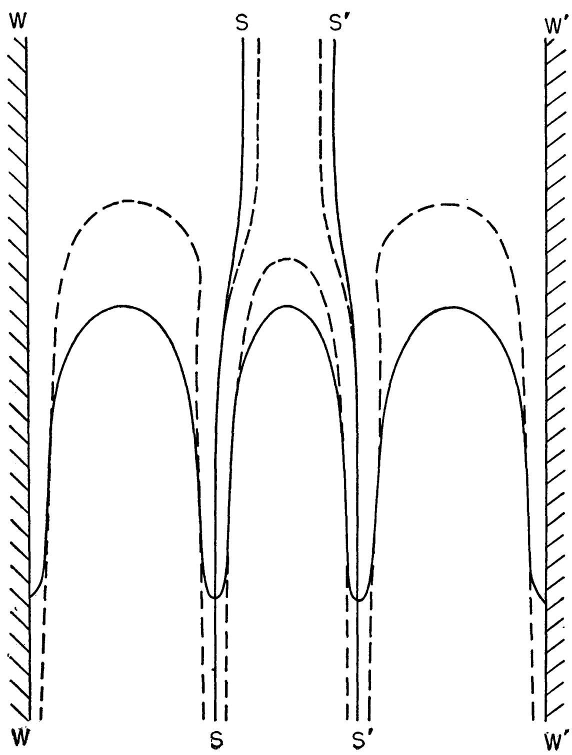

FIG. 3.—Illustrating the competition between well-developed bubbles in two-dimensional flow. The middle bubble is crowded out by the larger bubbles on either side. SS and $S'S'$ are streamlines.

bubble. The large bubbles will therefore expand, thereby acquiring a higher vertex speed, while the small bubble will shrink and slow down. This development, which is indicated in Figure 3 by the broken lines, shows up clearly in some experiments by D. J. Lewis (1950). Ultimately, the two large bubbles will fill the entire channel, the middle bubble having been washed downstream. Since perfectly periodic flow can never be attained in practice, the number of bubbles per unit length will continually diminish. The foregoing remarks apply, with some obvious modifications, to the more general case of three-dimensional flow.

Finally, we shall consider briefly the question whether an arbitrary disturbance tends to develop into a two-dimensional system of cylindrical troughs and crests or a three-dimensional system of bubbles and spikes. From equations (37) and (60) we see that the final vertex speed of a bubble in a tube of diameter $2R$ is higher, by a factor $(3\pi/\beta_1)^{1/2}$, than the final vertex speed of a cylindrical wave in a channel of width $2R$. This suggests that a well-developed two-dimensional disturbance would tend to break up into a system of bubbles and spikes.

I am indebted to Capt. Ralph Pennington for pointing out to me that equation (32) has a first integral, and to Capt. Pennington and Mr. Robert Goerss for performing the quadratures on which Figure 2 is based. Professor John A. Wheeler directed my attention to the problems discussed herein and made several valuable suggestions for improving the readability of the paper. The qualitative description of aperiodic flow contained in Section III, 2, was largely worked out in the course of a conversation with Professor Wheeler.

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