

PHYS 218A

Fall 2008

Quasilinear Theory

→ how instabilities / turbulence modify the profiles which drive them

→ Mean field theory → evolves $\langle F \rangle$

→ useful as device for calculating turbulent transport coefficients

i.e. anomalous resistivity.

→ special application: turbulent/anomalous resistivity

→ see also: posted supplementary notes
→ Chapter 3 of book manuscript

→ read Kadetad: 11.1, 11.2, 14.1, 14.2, 14.7

Quasilinear Theory - Vlasov Plasma

i) Motivation and Overview

→ linear theory determines 'instantaneous stability' of plasma

$$\text{ie. } \epsilon(k, \omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

⇒ growth/damping rate $\gamma_k = \gamma_k[\langle f \rangle]$

but $\langle f \rangle$ evolves... If $\langle f \rangle$ evolves slowly:

$$\text{"slowly"} \Rightarrow \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t} < \gamma_k$$

can consider: $\gamma_k = \gamma_k[\langle f^{(p)} \rangle] \rightarrow \left\{ \begin{array}{l} \text{evolution driven} \\ \text{by instabilities} \end{array} \right.$
 physics: mean distribution evolution ...
 ⇒ driven by relaxation.

⇒ quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f \rangle$

③ quasilinear theory is "mindless mean field theory", i.e.

$$\langle f \rangle = \langle f(v, t) \rangle \quad \text{where } \rightarrow \langle \rangle \text{ eliminates spatial dependence}$$

→ t understood "slow"

∞ f :

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} \left\langle \frac{q}{m} \tilde{E} f \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$)
for mean of conserved order parameter

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial J_v}{\partial v} = 0 \quad \rightarrow \text{phase space continuity equation}$$

($\langle f \rangle$)

$$J_v = \overline{J_v} = \left\langle \frac{q}{m} E f \right\rangle$$

$$= \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

for: $E = \tilde{E}$

$$f = \langle f \rangle + \tilde{f}$$

elementary closure problem
i.e. relate $\langle f \rangle$ to $\langle \tilde{E} \tilde{f} \rangle \rightarrow$ hierarchy!
How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(f) $\tilde{f} \rightarrow \tilde{f}$ linear (i.e. linear response of
plug in linear response \tilde{f})

$$\text{i.e. } \frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} + \frac{q}{m} E \frac{\partial \tilde{f}}{\partial v} = 0 \quad v/\omega \text{ or } v$$

Egn.

$$\Rightarrow -i(\omega - kv) \frac{\tilde{f}_k}{\omega} = -\frac{q}{m} \tilde{E}_k \frac{\partial \langle f \rangle}{\partial v}$$

$$\text{so } J_v = -\frac{q^2}{m^2} \sum_{k \neq 0} |\tilde{E}_k|^2 \frac{\partial \langle f \rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of
eigenmodes, only)

i.e. contrast approach to
criticality in usual
phase transitions (2nd
order)

Q.L. equation is:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial D}{\partial v} \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \frac{q^2}{m^2} \sum_k |\tilde{E}_k|^2 \frac{1}{\omega - kv}$$

→ here growth of order
parameter in broken symmet.
phase ... not noise driven

Q.L. equation

i.e. mindless mean field theory, ...

$$\text{with } \epsilon(k, \omega) = 0$$

$$\partial_t |E_k|^2 = 2\delta_k |E_k|^2$$

→ advance fields.

But

Surprisingly : Q.L.T. works quite well!

key issue : why ?

N.B. : In contrast to critical phenomena, external noise ignored → instability driven ...

④ Some questions to keep in mind : deterministic

i) why is Q.L. equation a diffusion equation ? when is this valid ?

↔ nature of "irreversibility" ...

ii) can Q.L. equation be derived from Fokker-Planck theory ?

↔ also "irreversibility" related ...

iii) how does Q.L. equation balance the energy-momentum budgets ?

iv) when } does Q.L. theory fail ?
how }

↔ related (i) ... What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated ?

v) what is dynamics of quasilinear relaxation ?

c.e. physics ?

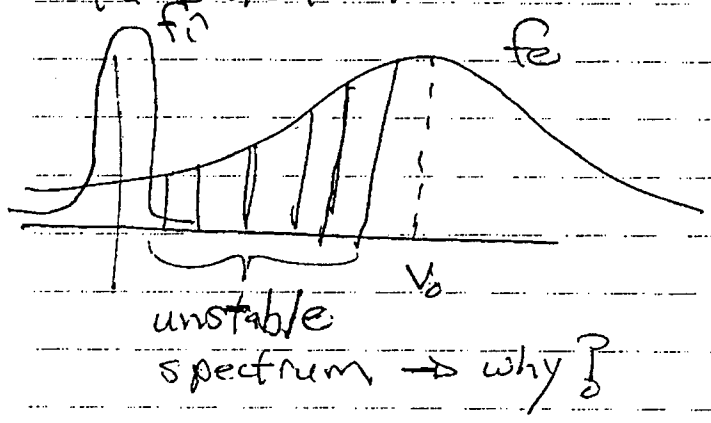
iii) Basic Scales / Regime Definition

① → Generally, Q, L, T , concerned with

i) 'broad' spectrum of:

ii) unstable waves

ie for current-driven ion-acoustic (G.O.I-A.) turbulence:



② → In finite system, k quantized, i.e.

$$k_m = m\pi/L, \text{ etc.}$$

- so, have spectrum of phase velocities

$$\omega_m/k_m = \omega(k_m)/k_m = v_{ph, m}$$

- wave-particle resonance occurs when

$$V = v_{ph, m}$$

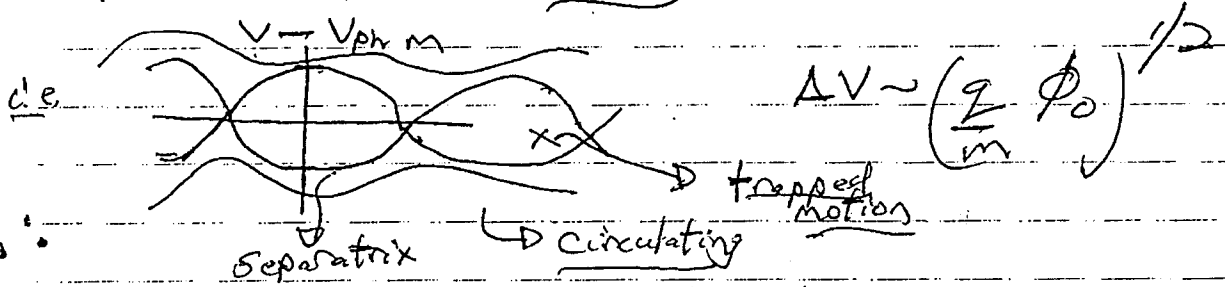
then $\sin \text{Isaac} \Rightarrow$

$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t) \quad \left. \begin{array}{l} \text{n.b.} \\ \text{deterministic,} \\ \text{no RPA} \end{array} \right\}$$

and 1 resonance dominant \Rightarrow

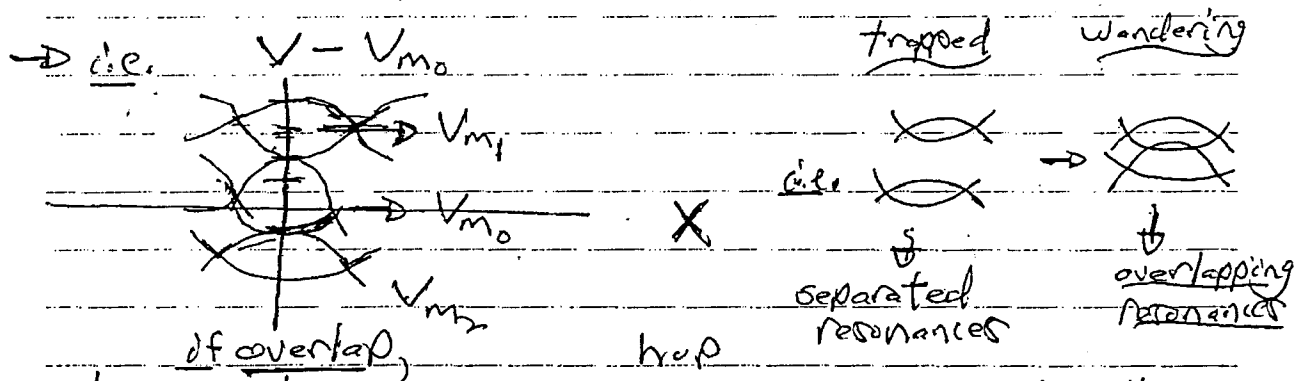
$$m\ddot{x} \approx q E_{m_0} \cos(k_{m_0} x_0 + (k_{m_0} v - \omega_{m_0}) t)$$

\Rightarrow each resonant velocity defines a phase space island



QLT is concerned with the case of:

\rightarrow multiple, overlapping resonances \rightarrow $\left. \begin{array}{l} \text{separatrix} \\ \text{proximity} \end{array} \right\} \rightarrow \text{destruction}$



particle can wander stochastically from resonance - to - resonance, i.e. hopping

\Rightarrow diffusion in v $\quad \frac{D_v}{v} \sim \frac{(\Delta v)^2}{\tau_{ac}}$ $\quad \left. \begin{array}{l} \Delta v = \text{resonance width} \\ \tau_{ac} \rightarrow \text{pattern time} \end{array} \right\}$

\hookrightarrow what is it?

Overlap condition (B.V. Chirikov) !

$$\frac{1}{2} (\Delta V_m + \Delta V_{m+1}) \gtrsim V_{ph_{m+1}} - V_{ph_m}$$

$\Delta V_m \sim \sqrt{e E_0}$

→ particle motion stochastic

→ fundamental irreversibility ⇒ orbit stochasticity (not dissipation, Landau damping ⇒ contrast critical phenomena)

→ underpinning of diffusion equation

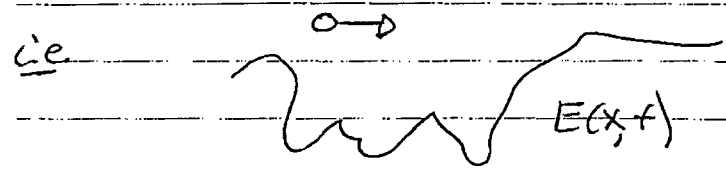
③ → But, a swindle! $\int \rho \rightarrow$ use of un-perturbed orbit in estimate!

i.e. $x \rightarrow x_0 + vt$ valid!

Consider: linear, un-perturbed orbit!

have: $E(x, t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

particle "sees" instantaneous pattern of electric field, from modal superposition



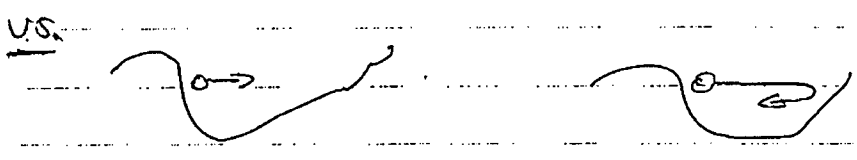
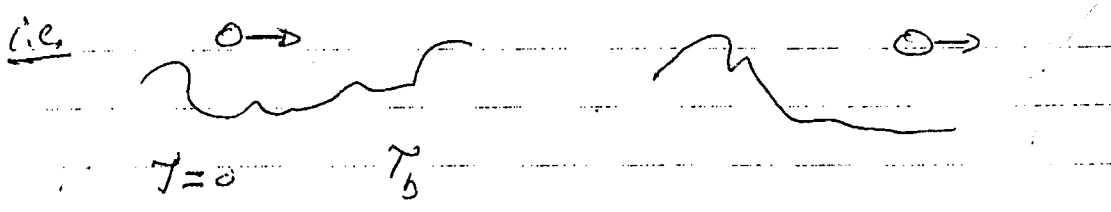
∴ relevant comparison is:

$T_L \rightarrow$ life time of 'instantaneous' pattern

$T_b \rightarrow$ 'bounce time' of particle in pattern

obviously, ① $T_L \ll T_b \rightarrow$ unperturbed orbit is satisfactory approximation
(pattern changes prior to bouncing)

② $T_L \gg T_b \rightarrow$ particle bounces prior pattern changes
so must consider orbit perturbation...



∴ quasilinear theory relevant to evolution when:

- ① \rightarrow orbits stochastic (Chirikov condition satisfied)
- ② $\rightarrow T_{Life} < T_{bounce} \rightarrow$ unperturbed orbits ① valid.

3)

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$$E(x,t) \Rightarrow \sum_k E_k e^{i(kx - \omega_k t)}$$

$$= \sum_k E_k \exp \left[i \left(k \left[x - \underbrace{\left(\frac{\omega_k}{k} \right)}_{v_{ph}(k)} t \right] \right) \right]$$

$\Delta(\omega_k/k) \equiv$ spread in phase velocities.
 \downarrow
 sets dispersal rate

so dispersal rate is (time)⁻¹ to disperse by one wavelength

$$1/T_{life} = k \Delta(\omega_k/k)$$

$$= k \left(\frac{d\omega_k}{dk} \frac{\Delta k}{k} - \frac{\omega_k}{k^2} \Delta k \right)$$

$$\left(v_g - v_{ph} \right)$$

$$= \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k} \right) \Delta k = (v_g(k) - v_{ph}(k)) \Delta k$$

n.b. $T_{life} \rightarrow \infty$ for non-dispersive waves!

Generally; QLT / weak turbulence

encounters trouble for $\left\{ \begin{array}{l} \text{non dispersive} \\ \text{weakly dispersive} \end{array} \right.$ waves.

How systematize?

$$\text{Consider: } \langle E(x_1, t_1) E(x_2, t_2) \rangle_{x,t} = C$$

electric field correlation function

$$C = C(x, T) \text{ , for } \left\{ \begin{array}{l} \text{homogeneous} \\ \text{stationary} \end{array} \right\} \text{ fluctuations}$$

$$x_1 = x_+ + x_- \quad t_1 = t_+ + t_-$$

$$x_2 = x_+ - x_- \quad t_2 = t_+ - t_-$$

$$\langle \rangle_{x,t} = \langle \rangle_{x_+, t_+}$$

so

$$C(x, T) = \left\langle \sum_{k, k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k + \omega_{k'})t_+} * \right. \\ \left. e^{i(k-k')x_-} e^{-i(\omega_k - \omega_{k'})t_-} \right\rangle_{x_+, t_+}$$

$$x_+, t_+ \text{ average } \Rightarrow k = -k' \quad \omega_k = -\omega_{k'}$$

so

$$C(x, T) = \sum_k |E_k|^2 e^{ikx} e^{-i\omega_k t}$$

Now:

→ assume continuous spectrum

→ for simplicity, take model

$$|E_k|^2 = E_0^2 / \left[\left(\frac{k-k_0}{\Delta k} \right)^2 + 1 \right]$$

→ evaluate on u.p.o.

$$x_- = x_0 + vT$$

$$\langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2} + 1 \right]} e^{ikx_0} e^{i(kv - \omega_k)T}$$

integrating:

phase info - irrelevant

$$\sim E_0^2 e^{ik_0 x_-} e^{-|\Delta k x_0|} *$$

$$e^{i(k_0 v - \omega_{k_0})T} e^{-|\Delta(kv - \omega_k)|T}$$

↓
oscillation
(→ on resonance)

↳ correlation decay
due dispersion
and its interplay
with resonance.

note: note that spread is doppler-shifted
ω is critical

$$\underline{\text{now}} \quad A(kv - \omega_k) = v \Delta k - v_{gr} \Delta k$$

$$= |(v - v_{gr}) \Delta k|$$

$$v_{gr} = \frac{d\omega}{dk}$$

$$\stackrel{\text{So}}{=} \langle E^2 \rangle = C(x, x)$$

$$= E_0^2 e^{i k_0 x} e^{i(k_0 v - \omega_{k_0})T} e^{-| \Delta k x |}$$

$$* \exp\left[-(v - v_{gr}) \Delta k |T|\right]$$

sets lifetime

$$\frac{1}{T_L} = |(v - v_{gr}(k)) \Delta k| \equiv (\text{Autocorrelation Time})^{-1}$$

$$\text{Note:} \quad \equiv 1/T_{ac}$$

- for resonant particles, $v = \omega_k/k$

$$\frac{1}{T_L} = |(v_{ph} - v_{gr}) \Delta k| \rightarrow \text{recovers earlier!}$$

- can think: $|v \Delta k| \rightarrow 1/T_{ac}^{\text{wave-particle}}$

$$|v_{gr} \Delta k| \rightarrow 1/T_{ac}^{\text{wave}}$$

generally, shorter time domains,
except for non-dispersive waves,

So, can enumerate key time scales

$$\tau_{ac} = |\Delta k (v_{ph} - v_{gr})|^{-1}$$

\equiv persistence of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles.

$\gamma^{-1} =$ growth/damping time

$$\tilde{\tau}_{Tr} = (k \sqrt{g \phi / m})^{-1} \equiv \text{trapping time}$$

$$\tilde{\tau}_{relax} = \left(\frac{1}{\langle F \rangle} \frac{\partial \langle F \rangle}{\partial t} \right)^{-1} \equiv \text{avg. distribution relaxation time}$$

So \sim \odot

$$\tau_{ac} < \tilde{\tau}_{Tr} \rightarrow \text{u.p.o. valid}$$

$$\tau_{ac} < \tilde{\tau}_{relax} \rightarrow \langle F \rangle \text{ closure meaningful.}$$

$$\tau_{ac} < \gamma^{-1} < \tilde{\tau}_{relax} \rightarrow \text{QL.T. valid.}$$

iii.) Energy - Momentum Budgets

→ Key Point: There are two ways of implementing the book-keeping and accounting

ie $\left\{ \begin{array}{l} \text{resonant particles} \\ \text{or} \\ \text{particles} \end{array} \right.$ vs. 'waves'
 vs. fields

keep in mind: Wave = Field + Non-resonant particles

ie for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial \omega} (\omega \epsilon) \Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \frac{\omega \partial \epsilon}{\partial \omega} \Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle

(show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = - \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} f \rangle$$

$$\begin{aligned} \frac{\partial}{\partial t} \int dv \frac{mv^3}{2} \langle f \rangle &= - \int dv \frac{mv^3}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} f \rangle \\ &= \int dv mv \frac{q}{m} \langle \tilde{E} f \rangle \end{aligned}$$

1. trying in $\tilde{f}_k^{\text{linear}}$ for \tilde{f} ?

$$\frac{\partial}{\partial t} \Sigma_{\text{kin}} = - i \int dv \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{1}{\omega - kv} - \text{c.c.} \right) \frac{\partial \langle f \rangle}{\partial v}$$

$$\frac{\partial}{\partial t} \Sigma_{\text{kin}}^{\text{res}} = - \int dv \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \delta(\omega/k - v) \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

↑
resonant only

$$= - \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \frac{\partial \langle f \rangle}{\partial v} \Big|_{\omega/k}$$

As resonant particles stabilize/destabilize waves, expect resonant particles conserve energy against waves.

In wave energy evolution:

$$\text{Recall: } \epsilon = 1 + \frac{\omega_p^2}{k} \int dV \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

$$E^r(\omega_n + i\gamma_n) + i\epsilon^{IM} = 0$$

$$i\gamma_n = -\frac{\epsilon^{IM}}{\partial \epsilon^r / \partial \omega}$$

$$i\gamma_n = -\frac{\epsilon^{IM}}{\partial \epsilon^r / \partial \omega} = -\epsilon^{IM} / \partial \epsilon^r / \partial \omega$$

Now, $W \equiv$ Wave Energy Density

$$W = \sum_k \frac{\partial (\omega \epsilon)}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$= \sum_k \omega_k \frac{\partial \epsilon^r}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k \omega_k \frac{\partial \epsilon^r}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$= \sum_k 2 \left(-\frac{\epsilon^{IM}}{\partial \epsilon^r / \partial \omega} \right) \omega_k \frac{\partial \epsilon^r}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon^{IM}(k, \omega_k) \omega_k \left(\frac{|E_k|^2}{4\pi} \right)$$

$$i \epsilon_{IM} = \frac{\omega_p^2}{k} \frac{\partial \langle \epsilon \rangle}{\partial \nu} \bigg|_{\omega/k} \frac{(-i\pi)}{|k|}$$

($n_0 = 1$)

$$\begin{aligned} \frac{dW}{dt} &= \sum_{\mathbf{k}} \frac{\pi q^2}{m} \frac{\omega_p^2}{k|k|} \frac{\partial \langle \epsilon \rangle}{\partial \nu} \bigg|_{\omega/k} \frac{|E_n|^2}{4\pi} \\ &= + \frac{\pi q^2}{m} \sum_{\mathbf{k}} \frac{\omega}{k|k|} \frac{\partial \langle \epsilon \rangle}{\partial \nu} \bigg|_{\omega/k} |E_n|^2 \end{aligned}$$

$$\boxed{\frac{d}{dt} \sum_{\text{kinetic}}^{\text{resonant}} + \frac{d}{dt} W = 0}$$

$$\frac{d}{dt} \int \mathbf{v} \cdot \mathbf{v} \frac{d^3v}{(2\pi)^3} = -\partial \langle \epsilon \rangle$$

Note:

- this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} + Q = 0$$

\downarrow wave energy \downarrow divergence of wave energy density flux \downarrow $\langle \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}} \rangle$ coupling

For homogeneous system: $\nabla \cdot \mathcal{S} = 0$

$\Rightarrow \frac{\partial W}{\partial t} + Q = 0$
 $\int_V \langle E \cdot J \rangle$ mediated by resonant particles (DC field)

$\Leftrightarrow \frac{\partial W}{\partial t} + \frac{\partial}{\partial t} \int_V (\text{RPKED}) = 0$
 resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$W = \int_V \text{NRPKED} + \int_V \text{FED}$
 non-resonant particle kinetic energy density field energy density

so, simply re-grouping terms:

$\frac{\partial}{\partial t} (\text{FED}) + \frac{\partial}{\partial t} (\text{RPKED} + \text{NRPKED}) = 0$
 $\int_V \text{PKED}$

So $\frac{d}{dt} F E D + \frac{d}{dt} (P K E D) = 0$

Energy Thm 1.

i.e. fields and particles conserve energy.

What is the physics of all this?

$$D = \frac{pe}{k} \sum \frac{q^2}{m^2} |E_k|^2 (c/\omega - kv)$$

QL diffusion

for general, weakly non-stationary state ...

$$= \sum_k \frac{q^3}{m^2} |E_k|^2 \left(\frac{|\gamma_k|}{(\omega - kv)^2 + |\gamma_k|^2} \right)$$

n.b. causality \Rightarrow no negative diffusion for damped waves

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \underbrace{\pi c(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega^2}}_{\text{non-resonant diffusion}} \right\}$$

resonant diffusion non-resonant diffusion

Resonant diffusion \rightarrow irreversible - resonance overlap is underpinning

\rightarrow rooted in particle stochasticity

→ Resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy...?)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2 |E_k|^2}{m^2} \frac{\gamma_k}{\omega_k^2}$$

ponderomotive energy

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \quad \text{where} \quad |V_k|^2 = \sum \frac{|E_k|^2}{m^2 \omega_k^2}$$

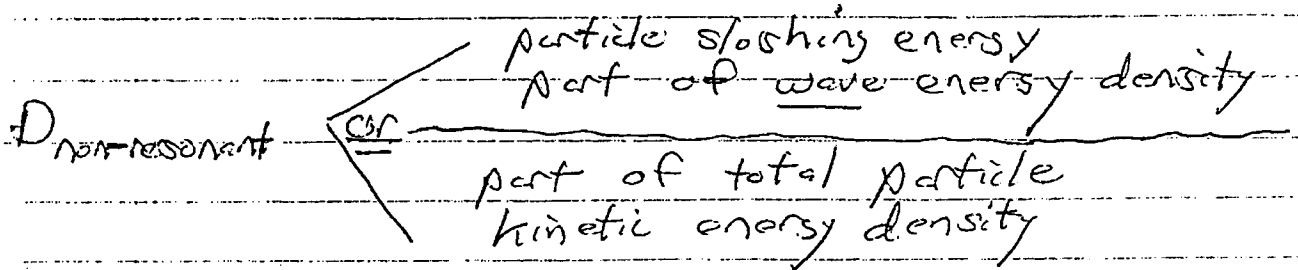
→ corresponds to "slashing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t E_{\text{quiver}}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka! "fake diffusion"

→ vanishes in stationary state

Point is that can count resonant diffusion as:



so two forms of energy conservation!

Note: Physically, the picture of plasma as gas $\left\{ \begin{array}{l} \text{- resonant particles} \\ \text{- waves} \end{array} \right.$ or equivalently

resonant particles + quasi-particles

waves $\left\{ \begin{array}{l} N(k, \omega, t) \\ WKE, \text{ etc.} \end{array} \right.$

is appealing and will pervade this course.

N.B.: Direct Proof of $\partial_t (PKED + FED) = 0$

From Q.L equation:

$$\frac{\partial}{\partial t} (PKED) = - \sum_k \int dV \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{c}{\omega - kv} \frac{\partial \langle f \rangle}{\partial V}$$

$$\epsilon(k, \omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dV}{\omega - kv} \frac{\partial \langle f \rangle}{\partial V}$$

$$\frac{\partial}{\partial t} (PKED) = -c \sum_k \frac{|E_k|^2}{4\pi} \int dV \frac{\omega_p^2}{k} \left(\underbrace{kv - \omega}_{\downarrow} + \omega \right) \frac{c}{\omega - kv} \frac{\partial \langle f \rangle}{\partial V}$$

↳ {cancels denom
residual add on
k

$$= -c \sum_k \frac{|E_k|^2}{4\pi} \int dV \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial V}$$

using $\epsilon(k, \omega) = 0$

$$= c \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

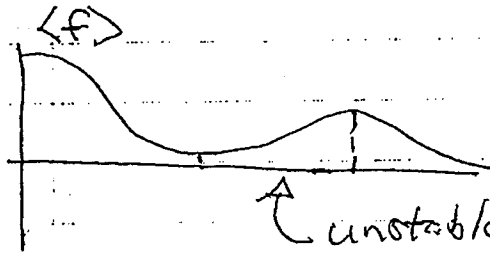
$$\omega_k = \omega_k^r + i\gamma_k$$

$$= - \sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= - \dot{\partial}_t (FED) \quad \checkmark$$

cu.) Applications of Quasilinear Theory

→ Bump on Tail



unstable phase velocities. (bump on tail)
 $\omega_n = \omega_{pe} \left(1 + \frac{3}{2} k^2 \lambda_D^2\right)^{1/2}$

Quasi-linear Equations:

$$E(k, \omega_n) = 0 \Rightarrow \omega(k), \gamma(k) \text{ from } \langle f \rangle$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_n \frac{q^2}{m^2} |E_n|^2 \left\{ \pi \delta(\omega_n - kv) + \frac{\gamma_n}{\omega_n^2} \right\}$$

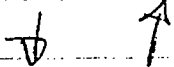
$$\frac{\partial}{\partial t} (|E_n|^2 / 8\pi) = 2\gamma_n |E_n|^2 / 8\pi$$

observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening

with



- adjustment of core/bulk profile (i.e. effective temperature)

we first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v}$$

* $\langle f \rangle$ and $\partial \langle f \rangle / \partial v = 0$



$$\frac{\partial}{\partial t} \int_{res} \frac{\langle f \rangle^2}{2} = - \int_{res} dv D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2$$

{ generalization \Rightarrow Zeldovich Thm.

stationarity \Rightarrow

$$D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2 = 0$$

Now "res" \rightarrow some finite interval of phase velocities

so

stationarity $\Rightarrow DR = 0$; i.e. fluctuations decay and damp

$$\frac{\partial \langle F \rangle}{\partial V} = 0 ; \text{ plateau forms, removing growth}$$

N.B.: - In 1D \rightarrow plateau
- can generalize

To resolve:

$$DR = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\approx \frac{16\pi^2 q^2}{m^2} \int dk \Sigma_F(k) \delta(\omega - kv)$$

$$DR = \frac{16\pi^2 q^2}{m^2 v} \Sigma_F(\omega_{pe}/v)$$

so

$$\partial_t DR = \frac{16\pi^2 q^2}{m^2 v} (\partial_t \gamma_{\omega_{pe}/v}) \Sigma(\omega_{pe}/v)$$

Now, $\gamma_H = -E_{FM} / \frac{\partial \phi}{\partial \omega} \Big|_{\omega_H}$

$$\gamma_H = \gamma_{\omega_p} = \pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial V}$$

so $\frac{\partial D^R}{\partial t} = \frac{16\pi^2 \hbar^2}{m^2 v} \left(2\pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial V} \right) \mathcal{E}(\omega_p/v)$

$$= \left(\pi \omega_p v^2 \frac{\partial \langle f \rangle}{\partial V} \right) D^R \quad , \text{ using } D^R \text{ defn.}$$

B

$$D^R(v, t) = D^R(v, 0) \exp \left[\pi \omega_p v^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial V} \right]$$

and:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial V} D^R \frac{\partial \langle f \rangle}{\partial V}$$

$$= \frac{\partial}{\partial t} \frac{\partial}{\partial V} \left[\frac{D^R}{\pi \omega_p v^2} \right] \quad \left\{ \begin{array}{l} \text{using } \gamma_H, D \\ \text{definitions} \end{array} \right.$$

$$\langle f(v,t) \rangle - \langle f(v,0) \rangle = \frac{\partial}{\partial v} \left[\frac{D^R(v,t) - D^R(v,0)}{\pi \omega_p v^2} \right]$$

∴ have:

$$D^R = D^R(v,0) \exp \left[\pi \omega_p v^2 \int_0^t dt \frac{\partial \langle f \rangle}{\partial v} \right]$$

$$\langle f(v,t) \rangle = \langle f(v,0) \rangle + \frac{\partial}{\partial v} \left[\frac{D^R(v,t) - D^R(v,0)}{\pi \omega_p v^2} \right]$$

Now, recall seek to know if:

$$i.) D^R \rightarrow 0 \Rightarrow \left. \frac{\partial \langle f \rangle}{\partial v} \right|_{t \rightarrow \infty} < 0 \quad (\text{Fluctuations damp})$$

$$ii.) \frac{\partial \langle f \rangle}{\partial v} \rightarrow 0 \Rightarrow \text{finite } D^R, \text{ distribution plateaus.}$$

Now, if $D^R \rightarrow 0$,

$$\langle f(v,t) \rangle = \langle f(v,0) \rangle - \frac{\partial}{\partial v} \left[\frac{D^R(v,0)}{\pi \omega_p v^2} \right]$$

$$D^R(0) = \frac{16 \pi^2 \epsilon^2}{m^2 v} \sum (\omega_p / v, 0)$$

Fluctuation energy

$$\text{but } \frac{16\pi^2 g^2}{m^2 v} \frac{E_c(0)}{\pi \omega v^2} = 2 E_F(0) / (m v_0^2 / 2)$$

$$\ll 1, \text{ as } n \gg n_0$$

$\therefore \langle f(v, t) \rangle \approx \langle f(v, 0) \rangle$, to good approx.

but, for resonant velocities,

$$\rightarrow \text{linear instability} \Rightarrow \partial \langle f \rangle / \partial v > 0$$

$$\rightarrow \begin{matrix} D^R \rightarrow 0 \\ t \rightarrow \infty \end{matrix} \Rightarrow \partial \langle f \rangle / \partial v < 0$$

but have (for $D^R \rightarrow 0$) $\langle f(t) \rangle = \langle f(0) \rangle$ ↓

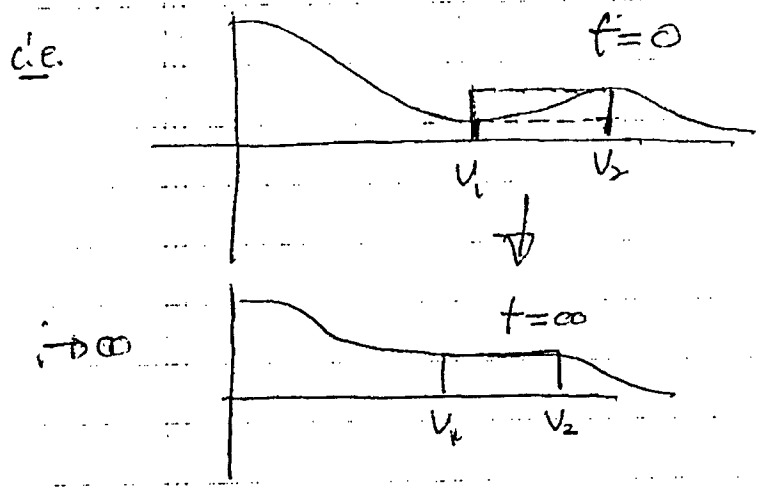
\therefore contradiction follows from assumption of $D^R(v, t) \rightarrow 0$

\therefore have established that

$$\left. \partial \langle f \rangle / \partial v \right|_{v_0} \rightarrow 0 \Rightarrow \text{plateau forms!}$$

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t} (R P K E \Delta) + \frac{\partial}{\partial t} (W E \Delta) = 0$$



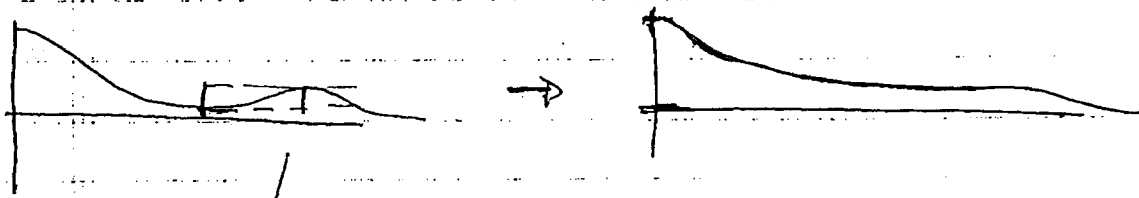
$$k = \frac{mv}{\hbar}$$

$$\Delta \left(\int_{v_1}^{v_2} \frac{mv^3}{2} \langle f \rangle \right) = - \Delta \int_{k_1}^{k_2} W_k dk$$

but $W_k = 2 \epsilon(k)$

$$\Rightarrow \Delta \left(\int_{v_1}^{v_2} dv \frac{mv^3}{2} \langle f \rangle \right) = -2 \Delta \int_{k_1}^{k_2} \epsilon(k) dk$$

→ can estimate A (RPAKE) analytically, via construction



i.e. beam slows down

but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial V} \frac{\partial NR}{\partial V} \frac{\partial \langle F \rangle}{\partial V}$$

$$= \frac{\partial}{\partial V} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\delta n_k}{(\omega - kv)^2} \frac{\partial \langle F \rangle}{\partial V}$$

$$\approx \frac{8\pi q^2}{m^2} \int dk \epsilon(k) \frac{\delta n_k}{\omega_{pe}^2} \frac{\partial^2 \langle F \rangle}{\partial v^2}$$

is, using γ definition:

$$\frac{\partial \langle F \rangle}{\partial t} = \left(\frac{1}{nm} \frac{\partial}{\partial t} \int dk \epsilon(k) \right) \frac{\partial^2 \langle F \rangle}{\partial v^2}$$

now define $T(A) = \frac{2}{n_e} \int dk \epsilon(k, t)$

so
 \Rightarrow

$$\frac{\partial \langle F \rangle}{\partial T} = \frac{1}{2m} \frac{\partial^2 \langle F \rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle F \rangle = \left[\frac{m}{2\pi} [T + T(A) - T(0)] \right]^{1/2} \exp \left[\frac{-mv^2/2}{[T + T(A) - T(0)]} \right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \rightarrow T/2 + \frac{1}{n} \int dk [\epsilon(k, \infty) - \epsilon(k, 0)]$$

ie. electrons 'heated' by net increase in field energy

- can also notes

$$\frac{\partial}{\partial t} (R P K E D) + \frac{\partial}{\partial t} (W E D) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t} (R P K E D) = -2 \frac{\partial}{\partial t} (F E D)$$

so $A (R P K E D) = -2 A (F E D)$

but

$$A (P K E D) = -A (F E D)$$

so $A (R P K E D) = +2 (A (P K E D))$

$$\Rightarrow 0 = A (R P K E D) + 2 A (N R P K E D) \quad \checkmark$$

and

$$A (P K E D) - A (R P K E D) = -A (F E D) - (-2) A (F E D)$$

$$\boxed{A (N R P K E D) = A (F E D)}$$

as shown
above

→ heating is one-sided, to conserve momentum.

A.) Mean Field Theory

i.) Quasilinear Theory

a.) Review of 1D/Basis

Refs:

① "Nonlinear Plasma Theory", by A. A. Galeev and R. Z. Sagdeev

→ Reviews of Plasma Physics, Vol. 7
(Leontovich)

→ much better than book by same authors, with same title

② "Plasma Turbulence", by B. B. Kadomtsev (1966)

③ "Cooperative Effects in Plasmas", by B. B. Kadomtsev

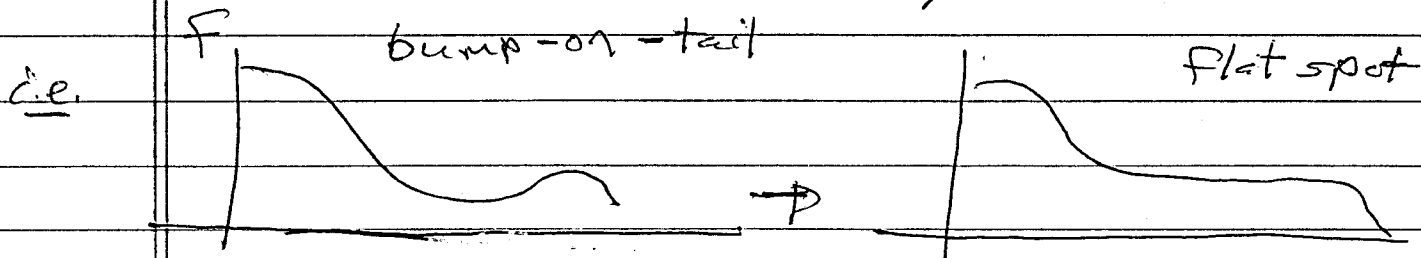
→ Reviews of Plasma Physics, Vol. 22

④ "Regular and Chaotic Dynamics", A. J. Lichtenberg and M. A. Leiberman; Springer-Verlag (1991).

⑤ "Hamiltonian Chaos and Fractional Dynamics" G. M. Zaslavsky, Oxford (2005)

①

→ Seek: How does $\langle F \rangle$ evolve due to turbulence instability?



em \rightarrow nonlinear saturation
 \rightarrow transport (v, χ, \dots) \Rightarrow turbulent transport coefficients
 \rightarrow relaxation, structure formation

\Rightarrow tend toward uniform, most probable, etc. state

if $\omega, k \gg 1/T, 1/L$
 \downarrow \downarrow
 turbulence scales mean scales

then can employ 2-scale strategy

i.e.
$$\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} + \frac{e}{m} E \frac{\partial F}{\partial v} = 0$$

\Rightarrow

$$\frac{\partial \tilde{f}^2}{\partial t} + v \frac{\partial \tilde{f}^2}{\partial x} + \frac{q}{m} \tilde{E} \frac{\partial \tilde{f}^2}{\partial v} = - \frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

\downarrow linear propagation \downarrow nonlinearity
 - mode-mode coupling
 - trapping
 \downarrow drive from mean
 \Rightarrow fluctuations

and

$$\frac{\partial \langle f \rangle}{\partial t} = - \frac{q}{m} \frac{\partial \langle \tilde{E} f \rangle}{\partial v}$$

\downarrow slow evolution \downarrow better
 \Rightarrow mean

- reasonable, but closure problem?
 What to do about \tilde{f}^2

- QLT:
 - ① assume all spectral constituents are eigen modes
 i.e. all $\omega = \omega(k)$
 - ② \tilde{f} treated as linear response

i.e. plug:

$$\frac{\tilde{f}^2_H}{\omega - kv} = - \frac{q}{m} \tilde{E}_H \frac{\partial \langle f \rangle}{\partial v} \rightarrow \text{linear response}$$

into $\langle F \rangle$ equation, yielding:

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle F \rangle}{\partial v}$$

$$D = n_0 \frac{q^3}{m^2} \sum_n |E_n|^2 \frac{c}{\omega - kv}$$

$$= \frac{q^3}{m^2} \sum_n |E_n|^2 \left\{ \frac{|\gamma_n|}{(\omega_n - kv)^2 + \gamma_n^2} \right\}$$

Note:

→ $|\gamma_n|$ → causality (i.e. γ from spectrum of damped modes)

→ resonant, non-resonant

$$\frac{|\gamma_n|}{(\omega_n - kv)^2 + \gamma_n^2} \rightarrow \pi \delta(\omega_n - kv) + \frac{|\gamma_n|}{\omega_n^2}$$

→ can formulate full set of "quasi-linear equations" → describes evolution via mean relaxation

i.) $\langle F \rangle \rightarrow$ linear stability, via

$$\epsilon(k, \omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle F \rangle / \partial v}{\omega - kv} \Rightarrow \frac{\omega_H}{\delta_H}$$

ii.) spectral evolution

$$\frac{\partial |E_H|^2}{\partial t} = 2\gamma_H |E_H|^2 \Rightarrow |E_H|^2$$

iii.) $\langle F \rangle$ modification / evolution

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle F \rangle}{\partial v}$$

$$D = \nu \sum_n \frac{q^2}{m^2} |E_n|^2 \frac{c}{\omega - kv} = \langle F \rangle$$

\Rightarrow convergence to marginal profile (?! \rightarrow plateau)

\rightarrow Amazingly QLT works (as description of transport) quite well

WHY?

} though not universally

②
 ⇒ Numbers and Ratios

- what is assumed in QLT?

→ linear response adequate - no NL distortion

→ resonant diffusion - Markov process,
 aka FPE

→ stochasticity/irreversibility
 → RPA (I)

Exercise:

a.) Derive QL (resonant diffusion)
 equation from Fokker-Planck theory

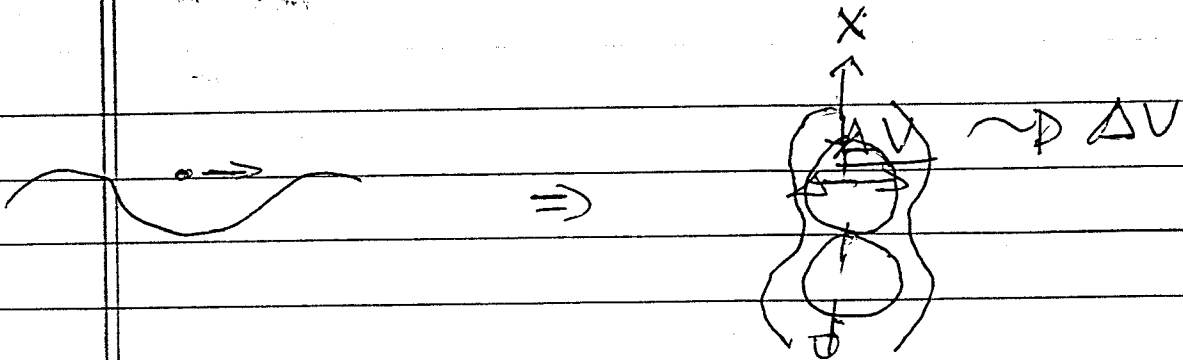
b.) use Hamiltonian structure of
 dynamics to eliminate dynamical
 friction term (cf. Lichtenberg
 and Leiberman)

⇒ 2 dimensionless numbers:

D

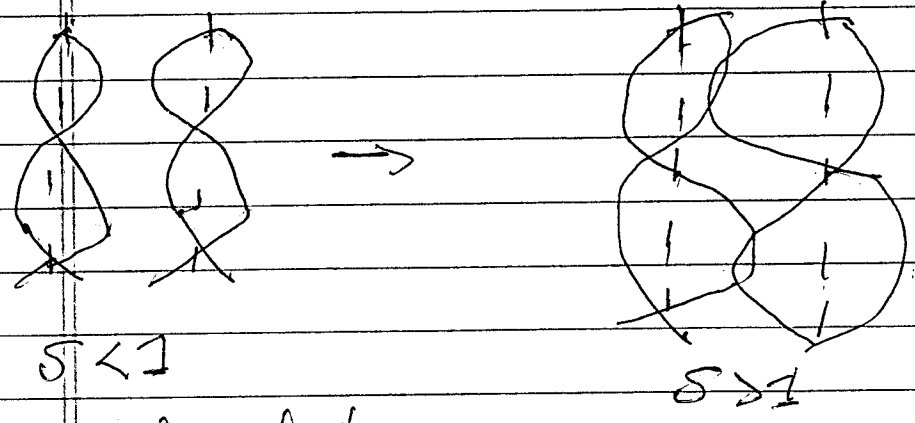
$S_e = \frac{\Delta V}{|V_{\phi i} - V_{\phi i+1}|}$ → measures
 stochasticity
 of particle
 orbits
 Chirikov #

ie. resonances overlap



so if multiple waves:

$$\omega/k$$



⇒ independent resonant behaviors

⇒ resonance overlap ⇒ diffusive "kicking" from 1 resonance to another.

② Kubo # > Strohal # → measures memory of flow

→ field correlation time

de

$$K = \frac{(qE/m) \tau_c}{(\Delta V)_c}$$

↳ velocity correlation length

for spatial scattering:

→ scatterer correlation time

$$S^2 = \frac{\tilde{V} \gamma_0}{l_c}$$

↳ correlation length

Note: For $\Delta V_0 \sim \Delta V_T \sim \sqrt{2\epsilon/m}$

$$K \sim K(\Delta V_T) \gamma_0$$

$$\sim \omega_b \gamma_0$$

↓
bounce freq.

⇒

$k_b S > 1 \rightarrow \begin{cases} \text{ordered flow with memory} \\ \text{persistent scatterer pattern} \end{cases}$

$k_b S < 1 \rightarrow \begin{cases} \text{random, short lived scatterers} \\ \text{random flow - suggests RPA} \end{cases}$

usually $k_b S > 1 \Rightarrow$ linear unperturbed trajectory
poor approximation

⇒ need circumpolar scattering field.

usual wisdom is that QLT valid if:

→ $\delta_c > 1$ → need stochastic orbits

→ $k \ll 1$ → need avoid trapping, string distortion, etc.

but: \int

① → is field/scatter correlation time the relevant time scale?

② → what of $\delta_c > 1$, $k \ll 1$?

③ → what of non-resonant piece?

Regarding ①,

$$D = \sum_{\mathbf{k}} \frac{q^2}{m^2} |E_{\mathbf{k}}|^2 \int_0^{\infty} e^{-i(\omega - k v) \tau} d\tau$$

$$|E_{\mathbf{k}}|^2 = E_0^2 \Delta k / [(k - k_0)^2 + (\Delta k)^2]$$

⇒

$$\begin{aligned}
 \chi &= \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty d\tau e^{i(\omega - kv)\tau} \\
 &= \int dk \frac{q^2}{m^2} E_0^2 \Delta k \int_0^\infty d\tau e^{i(\omega - kv)\tau} \\
 &\approx \frac{q^2}{m^2} E_0^2 (2\pi) \int_0^\infty d\tau e^{i(\omega_{k_0} - kv)\tau}
 \end{aligned}$$

$$\exp\left[-\left|\frac{d\omega}{dk} - v\right| \Delta k \tau\right]$$

↳ sets correlation decay

$$\Rightarrow \text{if } \chi \sim \frac{q^2}{m^2} \langle E^2 \rangle \tau_c$$

↳ wave-particle correlation time

then:

$$\frac{1}{\tau_{ac}} \sim \left| \frac{d\omega}{dk} - v \right| \Delta k \rightarrow \text{wave-particle decorrelation rate}$$

$$\text{if } v \sim \omega/k - \text{resonance}$$

$$\frac{1}{\tau_{ac}} \sim \left| \frac{d\omega}{dk} - \frac{\omega}{k} \right| \Delta k \rightarrow \text{packet dispersy rate}$$

Generally: $\tau_{ac}^{\omega-p} \neq \tau_{ac}^{pk}$ → differences more pronounced (in 3D)

So, really need:

→ specify velocity for wave-particle decorrelation being considered.

→ need: $1/\tau_{sc} > \omega_i$, $1/\tau_{sc} > \sqrt{\frac{\omega_i}{k^2 D}}^{1/3}$

→ $\tau_{sc} \sim \tau_c$ only for resonant particles in 1D.

→ broad spectrum alone is not sufficient to justify QLT → effective dispersion significant!

$1/\tau_{sc} |_{res} \sim \left| \frac{d\omega}{dk} - \frac{\omega}{k} \right| \Delta k \rightarrow 0$; for non-dispersive waves

→ $k \ll 1$ criterion most accurate if one takes $\tau_c \sim \tau_{sc, pk}$

$k = \frac{q}{m} E \tau_{sc, pk} / \Delta v_i = (k \Delta v_i) \tau_{sc} \ll 1$

agrees with intuition.

→ "phase randomization" irrelevant \Rightarrow

- can have $\sigma > 1$, $k < 1$ with coherent phases

- QLT known to well describe stochastic trajectory divergence in standard map/magnetic field/chaos, even for static fields/fixed phases.
c.f.: Rechester, Rosenbluth, White PRL '80.

- phases fixed in Tsunoda/Malmberg experiments

→ often, QLT seems to work reasonably well in limit of $\sigma > 1$, $k \sim 1$
- unclear why,

- corrections due granulation needed $\left[\frac{P}{\rho} \right]$

c.e. $k \sim 1 \Rightarrow \rightsquigarrow \Rightarrow \rightarrow$

phase space eddy formed

∴ strong non-stationarity can boost applicability of QLT.

Exercise :

⇒ consider, starting from $\vec{B} \cdot \nabla \psi = 0$,
 a heuristic equation for the
 density of magnetic field lines
 in a "cylindromak":

$$\vec{B}_T \frac{\partial N}{\partial z} + \frac{B_0(r)}{r} \frac{\partial N}{\partial \theta} + \tilde{\vec{B}} \cdot \nabla N = 0$$

$$\frac{\partial N}{\partial z} + \frac{B_0(r)}{r B_T} \frac{\partial N}{\partial \theta} + \frac{\tilde{\vec{B}} \cdot \nabla N}{B} = 0$$

from $\frac{dr}{B_r} = \frac{r d\theta}{B_\theta + \tilde{B}_\theta} = \frac{dz}{B_T}$

⇒ derive the QL equation for
 radial diffusion of N

⇒ what is the QL diffusion
 coefficient

⇒ derive expressions for $k_y S$

⇒ what is the analogue of
 wave-particle resonance here ...

③ Energetics

- easy to show (c.f. supplementary notes)

RPKED \rightarrow resonant particle kinetic energy density

$$= \epsilon_R = \int_{res} dV \frac{1}{2} m v^2 F$$

WED \rightarrow wave energy density

$$= \int dk \omega_k N_k$$

$$N_k = \frac{\partial \epsilon / \partial \omega}{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial}{\partial t} \left(\epsilon_R + \int dk \omega_k N_k \right) = 0$$

- since $WED = FED + NRPKED$

\downarrow
field energy density
 $\sim |E_k|^2 / 8\pi$

\downarrow
non-resonant particle kinetic energy density

then, equivalently can write:

$$\frac{\partial}{\partial t} \left(\epsilon_p + \epsilon_F \right) = 0$$

Exercise: Prove these

- why important?

- basic consistency requirements

- physical picture \rightarrow i.e. "quasi-particle picture"

i.e. represent plasma as:

+ {
 - resonant particles $\Rightarrow \int V / \omega \omega v \omega \eta + \delta(\omega - kv)$
 - waves \rightarrow quasiparticles \Rightarrow WKE

$$\frac{\partial N}{\partial t} + (\underline{v}_g + \underline{v}) \cdot \nabla N - \frac{\partial}{\partial x} (\omega + \underline{k} \cdot \underline{v}) \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

\Rightarrow natural, multi-component picture

\Rightarrow useful for accounting for { momentum
 energy
 transport by waves.

i.e. $\underline{P}_w = \int d\underline{k} \underline{k} N$

$$\Sigma_w = \int d\underline{k} \omega_{\underline{k}} N$$

\rightarrow can compute wave-induced stresses, etc.
 { formulate radiation hydro, etc.

Anomalous Resistivity →

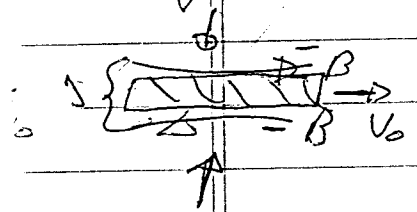
Application of QLT

Follows/expands on Galeev/Sagdeev Rev. Pl Phys 87.

→ an instructive and important example of quasilinear theory is anomalous resistivity

→ here, try approach of a classic current-driven ion acoustic instability (CDIA) model of anomalous resistivity via coupled micro-macro dynamics

- consider Sweet-Parker model, i.e.



$$VL = v_{out} L$$

$$v_{out} = v_A$$

$$\langle E \rangle = \langle \frac{v \times B}{c} \rangle \ll 0$$

(into page)

$$2 \frac{v B^2 L}{8\pi} = \eta \bar{J}^2 L \Delta$$

} CF 218B notes

$$\frac{\Delta}{L} = \frac{v}{v_A} \Rightarrow \Delta^2 = \frac{L \eta}{v_A} \Rightarrow \text{layer width } \frac{\Delta}{L} \sim \sqrt{\eta / R_m}$$

What happens as η decreased?

$$\frac{c B}{4\pi \Delta} = J = n_e \bar{v}_e$$

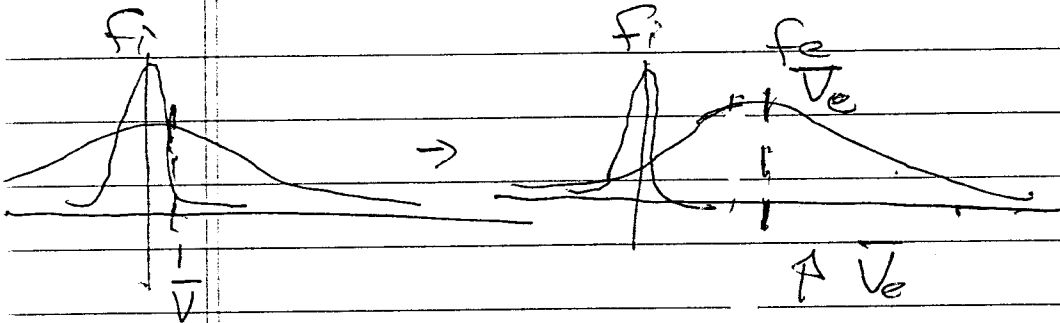
electron drift speed

so $\bar{v}_e = c B / 4\pi n_e \Delta = \frac{d_{skin}^2}{\Delta} \Omega_e$

Now, $\bar{v}_e \sim B/\Delta n \Rightarrow \bar{v}_e \uparrow$ as

$\Delta \downarrow \Rightarrow$ narrower
 $n_b \rightarrow$ few charge carriers
 $B \uparrow \Rightarrow$ stronger field (drive)

$\Delta \uparrow \Rightarrow$

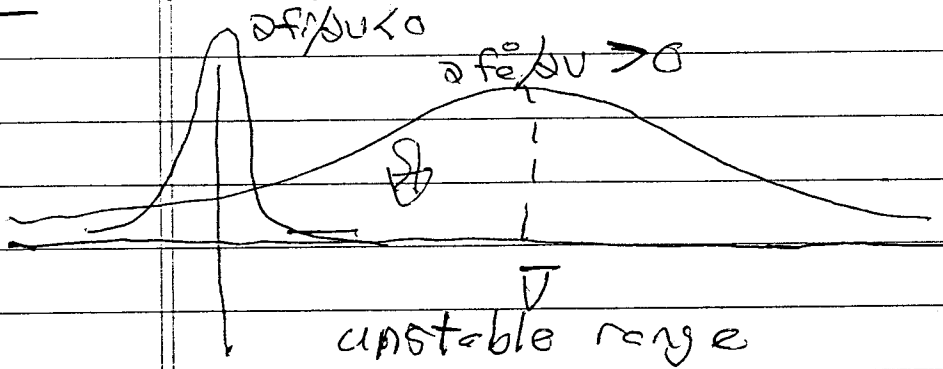


\Rightarrow decreasing Δ raises \bar{v}

\rightarrow up-shifts f_e centroid relative to f_i

\rightarrow destabilizes CDIA!

ie. classic scenario of CDIA



\therefore expect CDIA will:

\rightarrow exchange momentum between electrons and waves

so \rightarrow slow down electrons, reduce \bar{v}_e

\rightarrow act as "anomalous turbulent" resistivity

$$\text{c.e. } \left\{ \begin{aligned} \Delta^2 &= \frac{L}{V_A} \left(\eta + \eta_A(\bar{v}) \right) \\ &\rightarrow \text{anomalous resistivity} \\ \bar{v} &= \frac{cB}{4\pi n e A} \end{aligned} \right.$$

How calculate:

① Brute Force

- confining oneself to 1D model, ignoring layer structure, have:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{e}{m} E \frac{\partial f}{\partial v} = e(f) \quad \begin{array}{l} \text{here } x \rightarrow \text{vertical} \\ v \rightarrow \text{vertical} \\ \text{velocity} \end{array}$$

MeV * \Rightarrow

vertical \rightarrow
+ to layer

$$\frac{\partial \langle p_e \rangle}{\partial t} = -e \langle E \int v f \rangle = -\gamma_{e,d} n_0 m_e \bar{v}_e$$

collisional loss to comp

$$\frac{\partial \langle p_e \rangle}{\partial t} = -e n_0 \langle E \rangle - e \langle \tilde{E} \tilde{n} \rangle = -\gamma_{e,d} n_0 m_e \bar{v}_e$$

So

$$\langle E \rangle + \langle \tilde{E} \tilde{n} \rangle \frac{1}{n_0} = \frac{1}{n_0 e} \frac{\partial \langle p_e \rangle}{\partial t} = + \frac{\gamma_{e,d} m_e n_0 e \bar{v}_e}{n_0 e^2}$$

at (a) stationary state,

$$\langle E \rangle + \langle \tilde{E} \tilde{n} \rangle_{n_0} = \eta \langle J \rangle$$

↓
driving field
 $\sim \langle V B \rangle$

↓
electron
acceleration
by turbulence

↓
collisional
resistivity

↳ "anomalous resistivity"

to calculate:

$$\langle \tilde{E} \tilde{n} \rangle_{n_0} = \sum_{\mu} +ik \tilde{\phi}_{-\mu} \tilde{n}_{\mu}^e$$

$$= \int dV \sum_{\mu} +ik \tilde{\phi}_{-\mu} \tilde{n}_{\mu}^e$$

↑
electron
density perturbation

$$f_{\mu}^e \rightarrow f_{\mu}^{eL}$$

- quasilinear calculation

- stationarity \Rightarrow resonant transport.

b) Conservation Argument

- as in (a) anticipate stationarity \Rightarrow
resonant quasilinear evolution

- recall,

$$\frac{\partial}{\partial t} (\mathcal{E}^{RP} + \sum \text{wave}) = 0$$

$$\frac{\partial}{\partial t} (p^{RP} + p^{\text{wave}}) = 0$$

$$\sum_n^w = \omega_n \frac{\partial \epsilon}{\partial \omega} \bigg|_n \frac{|E_n|^2}{8\pi} \equiv \omega_n N_n \quad \rightarrow \# \text{ particles}$$

$$p_n^w = \frac{k}{\omega} \sum_n^w = k N_n$$

as water (CIRA) electrostatic, can ignore field momentum.

so, for resonant electrons

$$\frac{\partial}{\partial t} p_e^{RP} = - \frac{\partial}{\partial t} p^w = - \sum_n (\gamma_n^e) \frac{k}{\omega_n} \sum_n^w$$

$\gamma_n^e \equiv$ electron (resonant) growth rate

but

$\frac{\partial p_{\text{electron}}^{\text{PRD}}}{\partial t} \rightarrow \text{slowing down}$

\rightarrow macro-representation as effective collision

$$150 \quad \frac{\partial p_{\text{electron}}^{\text{PRD}}}{\partial t} = -n m_e \nu_{\text{eff}} \bar{v} \quad \text{Frequency}$$

ν_{eff}
effective collision frequency

slowing down by resonant scattering
(resonant particle interaction)

$$n m_e \nu_{\text{eff}} \bar{v} = \sum_k (2\gamma_k^e) \frac{k}{\omega_k} \Sigma_k^{\omega}$$

- defines \bar{v}

- for macro-micro link

$$\bar{v} = \frac{cB}{4\pi n q A}$$

* - n.b. of 2D, 3D theory, i.e. 1 dynamics \Rightarrow non-resonant scattering \Rightarrow wave driven momentum flux
i.e. $\Pi_{\perp, \parallel} \Rightarrow$ 1 radiation // momentum. Relation to whatler interpretation of Bellan? There need include wave radiation in energy balance.

so now have

$$\Delta M V_{\text{eff}}(R, A) \bar{V} = \sum_k \left(\frac{F^e}{\omega_k} \right) \frac{k}{\omega_k} \Sigma_k^w \quad (1)$$

$$\Delta^2 = \frac{L}{V_A} \left(1 + \frac{e^3 V_{\text{eff}}}{\omega_p^2} \right)$$

\Rightarrow need γ_k^e , Σ_k^w and $\langle F^e \rangle$ evolution

at simplest level, proceed via linear/quasilinear theory in 1D

- at more advanced level:

- consider 1D phase space structures

\rightarrow electron/ion clumps, momentum exchange

\rightarrow electron scattering off ion hole

- consider 3D $J_{||}^*$ driven instability with electron viscosity

Now, proceed in usual fashion:

$\gamma_k^e \rightarrow$ linear theory

$\Sigma_k^w \rightarrow$ nonlinear saturation

$\langle F^e \rangle \rightarrow$ QL equation - flattening

For linear theory of CIA ;

$$\nabla^2 \hat{\phi} = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$\frac{\hat{n}_i}{n_0} = \frac{k^2 R_s^2}{\omega^2} \frac{|e| \hat{\phi}}{T}$$

$$\frac{\hat{n}_e}{n_0} = \frac{|e| \hat{\phi}}{T} \left[\gamma - i \Gamma(k) \right]$$

in $\hat{n}(k)$,

$$\frac{\partial \hat{F}^0}{\partial t} + v \frac{\partial \hat{F}^0}{\partial x} = -\frac{|e|}{m_e} \hat{E} \frac{\partial \langle F \rangle}{\partial v}$$

$$\hat{F}^0 = \frac{|e| \hat{\phi}}{T} \langle F \rangle + g$$

$$\frac{\partial g}{\partial t} + v \frac{\partial g}{\partial x} = -v \frac{\partial}{\partial x} \left(\frac{|e| \hat{\phi}}{T} \langle F \rangle \right) + \frac{|e|}{m_e} \frac{\partial \hat{\phi}}{\partial x} \frac{\partial \langle F \rangle}{\partial v} - \frac{\partial}{\partial t} \left(\frac{|e| \hat{\phi}}{T} \langle F \rangle \right)$$

$$= v \frac{\partial \hat{\phi}}{\partial x} \frac{|e|}{T} \langle F \rangle + \frac{|e|}{m_e} \frac{\partial \hat{\phi}}{\partial x} - \frac{(v - \bar{v})}{T/m_e} \langle F \rangle - \frac{\partial}{\partial t} \frac{|e| \hat{\phi}}{T} \langle F \rangle$$

$$= -\frac{\partial}{\partial t} \frac{|e| \hat{\phi}}{T} \langle F \rangle + v \frac{\partial}{\partial x} \frac{|e| \hat{\phi}}{T} \langle F \rangle$$

$$\Rightarrow \rho_H = \frac{c(\omega - kV)}{-c(\omega - kV)} \frac{1}{T} \hat{\phi}_H \langle F \rangle$$

$$= - \left(\frac{\omega - kV}{\omega - kV} \right) \frac{1}{T} \hat{\phi}_H \langle F \rangle$$

$$\omega_H^2 = \frac{k^2 c^2}{1 + k^2 \lambda_D^2}$$

$$-i r(k) = \int dV \frac{(\omega - kV)}{(\omega - kV)} \langle F \rangle$$

$$= - \frac{(\omega - kV)}{|k| v_{Th}} \frac{c\pi}{\omega/kv_{Th}} \bar{F}$$

$$\bar{F} = \pm \exp \left[- \frac{(\omega/k - V)^2}{v_{Th}^2} \right]$$

$$1 + k^2 \lambda_D^2 = \frac{k^2 c^2}{\omega^2} + \frac{(\omega - kV)}{|k| v_{Th}} \frac{c\pi}{\omega/kv_{Th}} \bar{F}$$

$$\omega \rightarrow \omega + \delta\omega$$

$$0 = - \frac{2\delta\omega}{\omega} + \frac{(\omega - kV)}{|k| v_{Th}} \frac{c\pi}{\omega/kv_{Th}} \bar{F}$$

$$\frac{\delta\omega}{\omega} = \frac{c\pi}{2} \frac{(\omega - kV)}{|k| v_{Th}} \frac{\bar{F}}{\omega/kv_{Th}}$$

$$\delta\omega \Rightarrow \gamma_H \quad \text{growth rate}$$

$$\textcircled{2} \quad \gamma_H \sim \frac{c\pi}{2} \frac{\omega_H}{|k| v_{Th}} \frac{(\omega - kV)}{\omega/kv_{Th}} \bar{F} \Rightarrow \gamma > 0 \text{ for } v > c_s \Rightarrow \text{critical velocity}$$

For $\langle F \rangle$ evolution,

$$\frac{\partial \langle F \rangle}{\partial t} = + \frac{\partial}{\partial V} \sum_{\mu} \frac{1}{m_0} \tilde{E}_{-\mu} \tilde{J}_{\mu}$$

$$= + \frac{\partial}{\partial V} \sum_{\mu} \frac{1}{m_0} \tilde{E}_{-\mu} \left(\frac{-\omega - k \bar{v}}{\omega - k v} \frac{1}{T} |\phi_{\mu}| \langle F \rangle \right)$$

$$= \frac{\partial}{\partial V} \sum_{\mu} \frac{1}{m_0} \frac{1}{T} \tilde{E}_{-\mu} \left(-(\omega - k v) \pi \delta(\omega - k v) \right) \frac{1}{T} |\phi_{\mu}| \langle F \rangle$$

$$= \frac{\partial}{\partial V} \sum_{\mu} \frac{(-v_{th}^2)}{T} \frac{|\phi_{\mu}|^2}{T} \kappa(\omega - k v) \pi \delta(\omega - k v) \langle F \rangle$$

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial V} \sum_{\mu} \frac{(-v_{th}^2)}{T} \frac{|\phi_{\mu}|^2}{T} \kappa(\omega - k v) \pi \delta(\omega - k v) \langle F \rangle \quad (3)$$

- mean evolution

Note:

- really only assumed $\langle F \rangle = \langle F \left(\frac{v - \bar{v}}{2v_{th}^2} \right) \rangle$

$$\frac{\partial \langle F \rangle}{\partial V} = \left(\frac{v - \bar{v}}{v_{th}^2} \right) \langle F \rangle$$

and

$$\langle F \rangle' = - \langle F \rangle$$

→ minimal assumption on structure

- can write as \bar{v} evolution

$$\bar{v} = \int dv v \langle F \rangle / \int dv \langle F \rangle$$

$$\frac{d\bar{v}}{dt} = + \int dv \sum_{\underline{h}} v_{\underline{h}}^2 \frac{e^{\beta h}}{T} \frac{1}{k} \left(\frac{\omega}{k} - \bar{v} \right) \pi \rho(\omega - kv) \langle F \rangle$$

$\omega/k < \bar{v} \rightarrow d\bar{v}/dt < 0$
 $\omega/k > \bar{v} \rightarrow d\bar{v}/dt > 0$

Remains to determine fluctuation intensity level

Generically, can write:

$$\frac{d}{dt} \Sigma_{\underline{h}}^{\omega} = \gamma_{\underline{h}} \Sigma_{\underline{h}}^{\omega} - \left(\sum_{\underline{h}'} \omega_{\underline{h}'} C_1(\underline{h}, \underline{h}') \frac{\Sigma_{\underline{h}'}^{\omega}}{NT} \right) \Sigma_{\underline{h}}^{\omega} - \left(\sum_{\underline{h}', \underline{h}''} \omega_{\underline{h}} C_2(\underline{h}, \underline{h}', \underline{h}'') \frac{\Sigma_{\underline{h}'}^{\omega}}{NT} \frac{\Sigma_{\underline{h}''}^{\omega}}{NT} \right) \Sigma_{\underline{h}}^{\omega}$$

spectral equation constituents:

(a) - linear growth

(b) - quadratic nonlinearity \rightarrow $\begin{cases} \nearrow$ 3 wave coupling
 \searrow NL ion-wave interaction

(c) - cubic NL \rightarrow wave coupling

Now, for ion-acoustic wave:

- wave coupling effects negligible
 \Rightarrow can't satisfy resonance
- NL wave-particle effects weak \rightarrow
 intrinsically
 \Rightarrow consider 4 wave process

$$\frac{\partial \epsilon_{\perp}^w}{\partial t} = \left[\gamma_{\perp} - \omega_{\perp} B(\omega, k) \left(\frac{\epsilon_{\perp}^w}{nT} \right)^2 \right] \epsilon_{\perp}^w \quad (5)$$

- 'cartoon' NL saturation equation

Now, (1)-(5) \Rightarrow { coupled, @-stationary
 } micro-macro system

\Rightarrow describe anomalous resistivity dynamics
 and its effect on reconnection

\Rightarrow coupled solution corresponds to
 solution of the problem

$$\textcircled{1} \begin{cases} n m_e \gamma_{\text{eff}}(B, \Delta) \bar{v} = \sum_k 2 \gamma_k^e \frac{k}{\omega_k} \epsilon_{\omega_k}^{\omega} \\ \Delta^2 = \frac{L}{V_A} \left(-\eta + \frac{c^2}{4\pi^2} \gamma_{\text{eff}} \right), \quad \bar{v} = cB/4\pi n q \Delta \end{cases}$$

$$\textcircled{2} \gamma_k^e = -\frac{\pi}{2} \omega_k \frac{(\omega - kv\bar{v})}{|k|v_{Th}} \bar{F} / \omega_k v_{Th}$$

$$\textcircled{4} \frac{\partial \bar{v}}{\partial t} = \int dv \left(\sum_k v_{Th}^2 \left| \frac{e\phi_k}{T} \right|^2 k^2 \left(\frac{\omega}{k} - \bar{v} \right) \pi C(\omega - kv) \langle F \rangle \right)$$

$$\textcircled{5} \frac{\partial \epsilon_{\omega_k}^{\omega}}{\partial t} = \left[\gamma_k - \omega_k B(\omega, k) \left(\frac{\epsilon_{\omega_k}^{\omega}}{nT} \right)^2 \right] \epsilon_{\omega_k}^{\omega}$$

Now, stationarity \Rightarrow

$$\epsilon_{\omega_k}^{\omega} = nT \left(\gamma_k / \omega_k B \right)^{1/2}$$

$$\gamma_k = \frac{+\pi}{2} \frac{(\bar{v} - c_s) k \omega_k}{|k|v_{Th}} \bar{F} / \omega_k v_{Th}$$

so, for scalings:

$$\gamma_{\text{eff}} = \frac{1}{nm\bar{v}} \sum_{\mu} 2 \gamma_{\mu} \frac{k}{\omega_{\mu}} \epsilon_{\mu}^{\omega}$$

$$\sim \frac{1}{nm\bar{v}} \frac{(\bar{v}-c_s)}{|k|v_{Th}} k \omega_{\mu} \bar{F} \frac{k (nT)}{k v_{Th}} \left(\frac{\gamma_{\mu}}{\omega_{\mu} B} \right)$$

$$\sim \frac{(1 - c_s/\bar{v})}{nm} \frac{k^2 nT}{|k| v_{Th}} \left(\frac{\gamma_{\mu}}{\omega_{\mu} B} \right) \bar{F} \frac{\omega_{\mu}}{k v_{Th}}$$

$$\sim (1 - c_s/\bar{v}) \bar{F} \frac{\omega_{\mu}}{|k| v_{Th}} \left(\frac{k^2 v_{Th}}{|k|} \right) \left(\frac{\gamma_{\mu}}{\omega_{\mu} B} \right)$$

$$\sim (1 - c_s/\bar{v}) \left(\frac{k \omega_{\mu}}{|k| v_{Th}} \frac{(\bar{v}-c_s)}{\omega_{\mu} B} \bar{F} \right)^{1/2} \bar{F} \frac{k^2 v_{Th}}{|k|}$$

$$\sim \left[(\bar{v}-c_s) k \right]^{3/2} \bar{F}^{3/2} \frac{k (v_{Th}/\bar{v})^{1/2}}{|k| |k| v_{Th}}$$

$\gamma_{\text{eff}} \sim \frac{\left[(\bar{v}-c_s) k \right]^{3/2}}{ k v_{Th}^{1/2}} \frac{v_{Th}}{\bar{v}} \bar{F} \left(\frac{k}{ k } \right)$	(c) Turbulent collision's frequency
--	--

so have

$$\Delta^2 = \frac{L}{L_A} \left(1 + \frac{C^2}{\omega_{pe}^2} Y_{eff} \right)$$

$$\bar{V} = c B_0 / 4\pi n_0 q L$$

$$Y_{eff} = \frac{\left[(\bar{V} - C_0) k \right]^{3/2}}{|k v_{th}|^{1/2}} \frac{v_{th}}{\bar{V}} \bar{F} \left(\frac{k}{|k|} \right)$$

with:

$$\frac{Y_{th}}{C_0} = \frac{\pi}{2} (\bar{V} - C_0) \frac{k}{|k| v_{th}} \bar{F}$$

$$\bar{F} = \frac{1}{\sqrt{\pi}} \exp \left[- \frac{(\omega/k - \bar{V})^2}{2 v_{th}^2} \right]$$

→ characterize micro-macro coupling with anomalous resistivity

→ now, can envision situation
 - finite current, $\Delta \sim (L_A/L_A)^{1/2}$
 $Y_{eff} = 0$

so \bar{F} :

- decrease $\eta \Rightarrow \Delta$ decreases

- Δ decreases $\Rightarrow \bar{V}$ increases

- \bar{V} increases $\Rightarrow \delta_H > 0$

- $\delta_H > 0 \Rightarrow \begin{cases} \sum \omega > 0 \\ \gamma_{eff} > 0 \end{cases}$

- $\gamma_{eff} > 0 \Rightarrow \begin{cases} A \text{ increases} \\ \bar{V} \text{ decreases} \end{cases}$

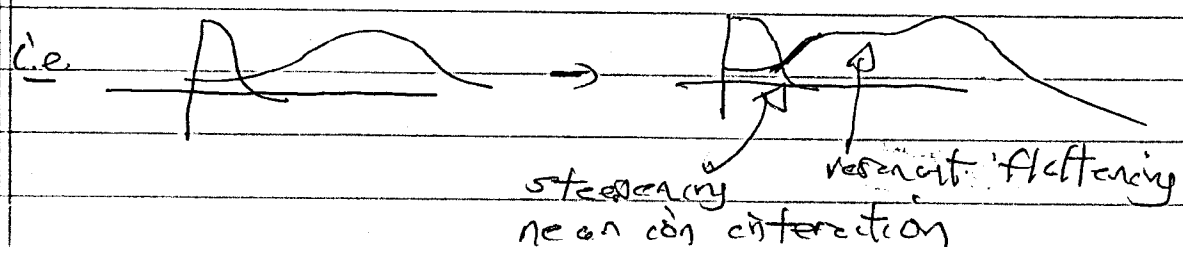
\Rightarrow decreasing μ so A decreases triggers feedback
 so A increases \rightarrow self-regulation / \oplus feedback
 " in this model, can expect:

- at low μ collisional, so $\bar{V} \sim c B_0 / 4\pi n_0 e^2 A_c$

\Rightarrow COIA "hovers" near marginal stability $\sim c_s$

- for stronger drive (above $B_0 \uparrow$)

- \Rightarrow ion interaction important
- \Rightarrow strong ion distortion possibly significant
- \Rightarrow granulation formation important
- \Rightarrow distortion of electron distribution function need by considered



⇒ Useful extensions:

- 1D avalanche model → avalanche ⇒ jitter effects on Δ , \bar{V}

- nonlinear noise effects via fluctuations

- 2D, 3D ⇒ wave radiation, esp. wave momentum flux \perp layer.

* - granulation effects ⇒ strong distortion hole (cf. later in course)

Comment:

This simple problem is surprisingly poorly understood. Excellent example of:

→ micro-macro feedback

→ self-regulation

→ marginal stability