

which replaces the usual relation for dE . All the thermodynamic calculations will go through as before¹ with $-n$ replacing V and μ replacing P . No instability occurs if C_μ becomes infinite along a λ line, but instability will occur if C_n tends to become infinite.

We speculate now that at the end of the λ line, where the phase separation starts to take place, an infinity in C_μ tends to go over to an infinity in C_n . This seems a fairly reasonable hypothesis. As long as ^3He is in low concentration, the interaction between rotons, which is responsible for the λ transition, would be expected to depend upon an intrinsic property of the ^4He part, such as its chemical potential. But as the concentration of ^3He increases and ^3He itself becomes more intimately concerned in the transition, the dependence might well go over into one on concentration.

Some measurements on the specific heat at constant composition C_n have been made on ^3He - ^4He mixtures.⁴ In general, C_n reaches a peak at the λ line, but does not appear at become infinite, as, of course, it should not. It would be much more difficult to obtain measurements of C_μ , which should become infinite. Relationships analogous to those with pressure and volume as the parametric variables should hold.

The peak of C_n along the λ line becomes lower as the temperature λ is lowered. This pre-

sumably occurs because the decrease in the temperature is correlated with a decrease in the concentration of ^4He . The λ transition thus becomes "diluted." Also, at a composition very close to that of the top of the phase-separation curve, the order-disorder changes in the phase separation and in the λ transition tend to overlap in the curve of C_n vs T , which gives an appearance of rounding off. These phenomena are probably not connected particularly with the change in the character of the transition. But it would be very interesting to have complete thermodynamic data in this region, in order to be able to test and analyze the thermodynamic relationships.

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NONLINEAR EVOLUTION OF A TWO-STREAM INSTABILITY*

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Calculations of a two-stream instability have been made by following the motion of the phase-space boundaries of an incompressible and constant-density phase-space fluid. Because of the condensation of holes, which to a good approximation act as gravitational particles, large-scale nonlinear pulses develop.

This Letter presents a picture of the nonlinear development of a two-stream instability that has resulted from numerical experiments with the one-dimensional Vlasov-Poisson equations. The motion of the electrons is described by Vlasov's equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{\partial \varphi}{\partial x} \frac{\partial f}{\partial v} = 0. \quad (1)$$

There is a uniform neutralizing background,

and the potential φ is determined from Poisson's equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \omega_p^2 \left[\int f \frac{dv}{v_0} - 1 \right], \quad (2)$$

where ω_p is the plasma frequency and v_0 is a constant chosen so that in the mean, $\int f dv / v_0 = 1$.

We picture the distribution function $f(x, v, t)$ as the density of an incompressible "phase

fluid" which moves in (x, v) space. This analogy is well known,¹⁻⁴ but the hydrodynamics, statistical mechanics,⁵ and thermodynamics^{6,7} of such a classical self-interacting fluid have not yet been examined in detail and are intrinsically well worth studying. Numerical computations on fluid motion in phase space are easier than for a real fluid, and our results bring out features of the nonlinear development of an instability that extend those found in previous numerical calculations.^{8,9} The method of calculation employed here can be used for a variety of one-dimensional Vlasov-Poisson problems, and the results may be relevant to other situations where similar equations occur, for example in gravitating systems,¹⁰ particle accelerators, electron tubes, microwave devices, and thermonuclear machines.

To study nonlinear phenomena in phase space in their simplest form, it is natural to assume that $f=1$ in certain regions and $f=0$ elsewhere. This model corresponds to the homogeneous incompressible fluid of classical hydrodynamics and has been used by several workers.¹⁻⁴ The state of a one-dimensional system is completely defined by specifying the boundary curves between the $f=0$ and $f=1$ regions. The equations of motion of each point on a curve are

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{\partial \varphi}{\partial x}, \quad (3)$$

and the potential at each instant is determined from Poisson's equation, the charge density being obtained by a geometrical construction. This type of calculation was employed by Dory² in a study of the negative mass instability in circular particle accelerators. The main computational problem is that the curves continually stretch, so that their representation by a fixed number of points becomes inaccurate. Our program therefore uses a list-processing technique which enables extra points to be inserted automatically, wherever they are needed. The calculation is then quite accurate and fast, although the practical duration of a run is still limited.

The example to be discussed is a two-stream instability, in which the electron plasma is slightly perturbed at $t=0$ from an equilibrium characterized by four straight lines in phase space: $f=1$ for $\frac{1}{2}v_0 < |v| < v_0$ and $f=0$ elsewhere. Periodic boundary conditions are imposed at $x=(0, L)$ and the parameters of the problem

are $v_0 \Delta t / \Delta x = 0.25$, $\omega_p \Delta t = 1/20$, and $\Delta x = L/64$, where Δx is the grid used for evaluating Poisson's equation. The unstable wave numbers are $k = 2m\pi/L$ with $n = (1, 2)$, and the linear growth rates are $\gamma/\omega_p = 0.30, 0.315$.

The most striking feature of the calculation is the behavior of the $f=0$ "cavity" which initially occupies the strip ($|v| < \frac{1}{2}v_0$) between the two plasma layers. This must preserve constant area as it deforms, and it is seen in Fig. 1 to coalesce into holes of roughly elliptical shape, so that a large-amplitude electrostatic wave is set up. Superimposed on this wave are coherent oscillations due to rotation of the holes in phase space, and also random fluctuations due to the motion of smaller elements of the hole "fluid." The two outer curves adjust almost adiabatically to the instantaneous potential function.

The holes are evidently being attracted to one another, and so behave like positively charged bodies with negative mass. One explanation is that the boundary of each hole is actually determined by the motion of negatively charged electrons, which are attracted towards a neighboring positively charged region occupied by another hole. Alternatively, we can construct a formal analogy between any plasma problem and an equivalent gravitational problem, in which the law of force is attractive and the $f=0$ and $f=1$ regions are interchanged. We can then show rather accurate-

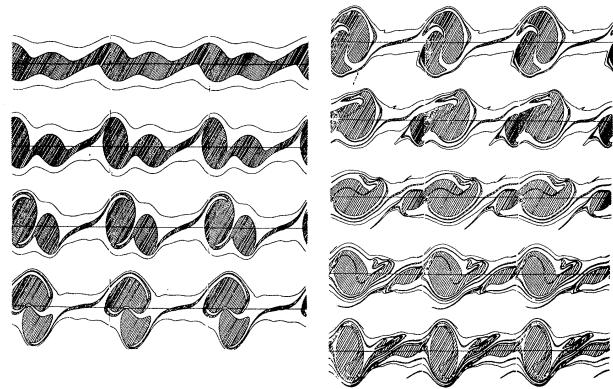


FIG. 1. Evolution in phase space of a two-stream instability from time steps 200 to 600 at intervals of 50 steps. Each step is $1/20$ of a plasma period, and the horizontal and vertical coordinates are x, v , respectively. Periodic boundaries have been imposed and three identical periods are shown along each row. The shaded area represents the $f=0$ region enclosed by the plasma fluid.

ly that the holes become globular masses that attract one another with a shielded Coulomb force. The boundaries of the two semi-infinite $f=1$ regions merely provide the shielding, since the phase velocity of any wave on these curves is much higher than the velocity of a hole. The roughly elliptical cavities in Fig. 1 thus correspond to the gravitational equilibria found by Hohl and Feix,¹⁰ who employed a similar model. A similar duality or "mass conjugation" has been demonstrated by Dory² for the negative mass instability in circular accelerators.

Figure 1 suggests that the system is relaxing to a Bernstein-Greene-Kruskal¹¹ (BGK) standing electrostatic wave, whose periodicity is the one that has been externally imposed. A similar result has been obtained by Armstrong¹² for a continuous two-humped distribution. However, since we have imposed periodic boundary conditions, this final state is artificial. If the periodicity is relaxed, the BGK mode may be unstable, since it follows from our gravitational analog that two adjacent holes should attract if slightly displaced towards one another.

The important structure that has formed is the solitary electrostatic pulse, which is supported by a hole in phase space. The BGK mode is a special alignment of these pulses, but the significant description of the system should be in terms of the number of holes, their distribution (shape, size, position, velocity), and their mutual interaction (polarization, hole-hole collisions). The total electrostatic field may be expressed as the sum of the quasistatic screened Coulomb fields due to the individual holes, together with a high-frequency component due to plasma oscillations.

The simplest example of an electrostatic pulse is a single hole ($f=0$) in an infinite electron fluid ($f=1$), and this corresponds exactly to the gravitational equilibrium found by Hohl and Feix.¹⁰ A more general case is a hole in a finite electron slab. Each phase-space boundary is an equal-energy contour in the frame of reference in which the hole is at rest, and these pulses may be constructed analytically. A computer program has been developed for carrying out the necessary quadratures, so that holes may be generated with any desired area and velocity, and used as the input data for numerical experiments on

hole-hole interactions. These experiments will be reported elsewhere.

There is an interesting correspondence with atomic theory. An isolated pulse is analogous to a Thomas-Fermi atom, and the invariant area of the hole is a classical action variable. A pulse may be either in its ground state, or excited by waves on the inner curve. Two pulses will interact with one another by hole-hole collisions, which may be inelastic, or almost elastic. A pair of holes may capture one another into a bound state, or may fuse together. The various possibilities are best envisaged with the help of the gravitational analogy.

Figure 2, which shows some results from a run with periodicity $4L$, was carried out to test the stability of the individual holes and of the BGK mode. The number and sizes of the initial holes result from interference between a group of fastest growing modes. Step 250 shows six holes of substantial size (to be labeled 1-6, starting from the left), together with four smaller cavities. The in-

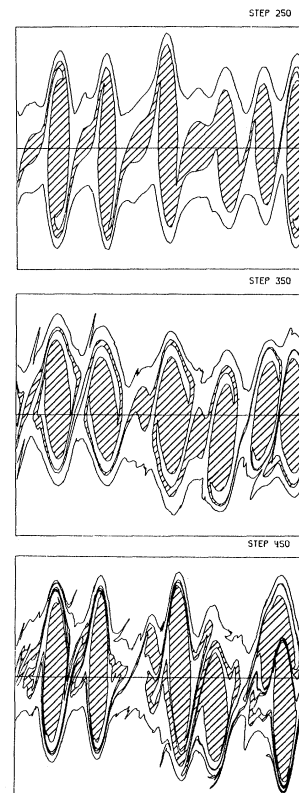


FIG. 2. Time development of a two-stream instability. The unperturbed equilibrium parameters and time steps are the same as Fig. 1, but the periodicity length has been extended four times and a different initial perturbation has been used. Only one period is shown at each time step.

teraction causes some holes to coalesce as the result of collisions, while others undergo tidal distortions but maintain their separate identities. In all cases small filaments of $f = 0$ "evaporate" from the main body and randomize throughout the system.

Holes 5 and 6 move together and coalesce by step 450, either because of their relative velocity or because of their mutual attraction. Hole 4 is moving towards 3 and will obviously coalesce shortly. The small cavity on the left of hole 3 has captured extra material, but is moving towards 3 and will coalesce with it. The orientation of holes 1 and 2 is remarkably stable; it appears that the screened attractive force is too weak to produce any appreciable relative motion during the course of the run. In addition to the secular motion of the holes, there is a vertical oscillation due to an $n = 1$ plasma wave of period 60 time steps which is generated by the initial conditions and persists with constant amplitude and frequency throughout the run.

It is desirable to develop a statistical description of the system, and this may be on more than one level. One task is to explain the "local" coarse-grained distribution $\bar{f}(x, v)$ or $\bar{f}(E)$ within the cavities and in the surrounding regions (here $E = \phi + \frac{1}{2}v^2$ is the total energy), and to follow its evolution in the presence of excitations. Another task is to predict the number, size distribution, and velocity dispersion of the holes themselves.

A suitable coarse-grained distribution function has been introduced by Lynden-Bell,⁵ who points out the relevance of the exclusion principle; i.e., two elements of fluid cannot occupy the same region in phase space. If f takes only the values (0, 1), the "most probable" final state for the electron fluid is a Fermi distribution. This result may be extended in an interesting way to our problem. Using the gravitational analogy, we can invert the $f = 0$ and $f = 1$ regions, so that the Lynden-Bell theory predicts a Fermi distribution for the holes, which is in qualitative agreement with the results. The remaining degrees of freedom of the system are only weakly excited.

Solitary pulses have been found to be a common occurrence in nonlinear theory.¹³ Nielson has observed solitary waves evolving from the negative-mass instability in experiments on the Cosmotron¹⁴ and DCX,¹⁵ and the persistence of electric field fluctuations found in Daw-

son's numerical experiment with the two-stream instability is consistent with our solitary pulse picture. In a gravitational example, Maxwell¹⁶ explained the stability of Saturn's rings in terms of the negative effective mass of the orbiting particles, which causes them to repel one another. Under suitable conditions, holes in a rotating gravitational system might therefore be expected to attract.

It will be interesting to determine to what extent solitary waves can describe the nonlinear phase of other instabilities, and to see how they fit in with quasilinear theory.^{17,18}

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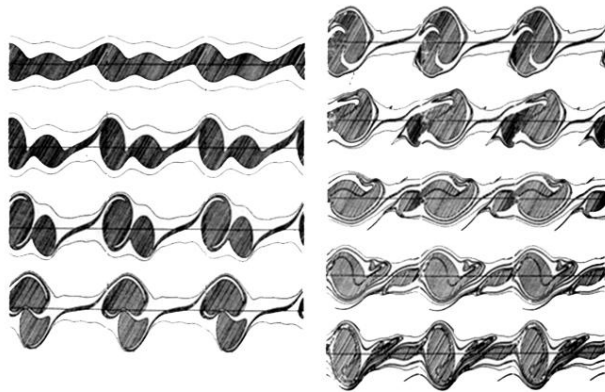


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