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Test Particles in a Completely Ionized Plasma

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Starting from the Liouville equation, a chain of equations is obtained by integrating out the coordinates of all but one, two, etc., particles. One "test" particle is singled out initially. All other "field" particles are assumed to be initially in thermal equilibrium. In the absence of external fields, the chain of equations is solved by expanding in terms of the parameter $g = 1/nL_D^3$. For the time evolution of the distribution function of the test particle, an equation is obtained whose asymptotic form is of the usual Fokker-Planck type. It is characterized by a frictional-drag force that decelerates the particle, and a fluctuation tensor that produces acceleration and diffusion in velocity space. The expressions for these quantities contain contributions from Coulomb collisions and the emission and absorption of plasma waves. By consideration of a Maxwell distribution of test particles, the total plasma-wave emission is determined. It is related to Landau's damping by Kirchoff's law. When there is a constant external magnetic field, the problem is characterized by the parameter g, and also the parameter $\lambda = \omega_c/\omega_p$. The calculation is made by expanding in terms of g, but all orders of λ are retained. To the lowest order in g, the frictional drag and fluctuation tensor are slowly varying functions of λ.

When $\lambda \ll 1$, the modification of the collisional-drag force due to the magnetic field, is negligible. There is a significant change in the properties of plasma waves of wavelength greater than the Larmor radius which modifies the force due to plasma-wave emission. When $\lambda \gg 1$, the force due to plasma-wave emission disappears. The collisional force is altered to the extent that the maximum impact parameter is sometimes the Larmor radius instead of the Debye length, or something in between. In the case of a slow ion moving perpendicular to the field, the collisional force is of a qualitatively different form. In addition to the drag force antiparallel to the velocity of the particle, there is a collisional force antiparallel to the Lorentz force. The force arises because the particle and its shield cloud are spiralling about field lines. The force on the particle is equal and opposite to the centripetal force acting on the "shield cloud." It is much smaller than the Lorentz force.

I. INTRODUCTION

THE usual kinetic theory of gases does not apply to a plasma because of the long-range character of Coulomb forces. In the treatment of Rosenbluth, MacDonald, and Judd,[1] the infrequent large-angle scatterings are treated as collisions. The effect of the many small-angle deflections is accounted for by means of a macroscopic field. Gasiorowicz, Neuman, and Riddell[2] have considered a test-particle problem. The frictional drag was determined from the response of the plasma to the test particle. Collisions between plasma particles that produce large deflections were neglected. Fluctuation effects were obtained by a Holtzmark-type calculation. Plasma-wave effects were not obtained with either of these methods. Making use of the random-phase approximation, Bohm and Pines[3] obtained a frictional drag for fast particles that is due to plasma-wave emission. It is of the same order of magnitude as the collisional effects.

In this paper an ensemble of plasmas will be considered. The density in phase space satisfies the Liouville equation. By integrating out the coordinates of all particles but one, but two, etc., one may obtain a chain of equations for the one-body, two-body, etc., functions. Kadomtsev[4] and Chan-Mou Tchen[5] have previously discussed plasma kinetics in terms of this chain. They consider the general problem of transport theory. In this paper only the test-particle problem will be considered. The loss in generality is compensated for by a substantial gain in tractability.

[1] Rosenbluth, MacDonald, and Judd, Phys. Rev. **107**, 1 (1957).
[2] Gasiorowicz, Neuman, and Riddell, Phys. Rev. **101**, 922 (1956).
[3] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[4] B. B. Kadomtsev, Soviet Phys.—JETP **6**, 117 (1958).
[5] Chan-Mou Tchen, Phys. Rev. **114**, 394 (1959).

II. REDUCTION OF THE LIOUVILLE EQUATION

Consider a gas of electrons and infinite-mass randomly distributed ions. It is a simple matter to generalize the results for this model to include finite-mass ions. Let $X_i = (\mathbf{x}_i, \mathbf{v}_i)$ be the position and velocity coordinates of the ith electron. N electrons and N ions are contained in a volume V. For an ensemble of similar plasmas, the density in phase space $D(X_1, X_2, \cdots X_N; t)$ satisfies the Liouville equation,

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{N}\left[\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m}\left(\mathbf{E}(\mathbf{x}_i, t) + \frac{1}{c}\,\mathbf{v}_i\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}_i}\right]\right\} D = 0, \qquad (1)$$

where $\mathbf{B}$ is a constant magnetic field applied externally.

Only Coulomb forces are considered, so that

$$\mathbf{E}(\mathbf{x}_i, t) = e \sum_{j=1}^{N}{}' \frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}, \qquad (2)$$

where the prime indicates that the term $i = j$ is to be omitted. The Liouville operator is symmetric with respect to the interchange of the coordinates of any two electrons. Therefore if D is initially symmetric, it will remain so. In the test-particle problem, one electron is singled out initially so that D will not be symmetric with respect to interchange of its coordinates with any other electron. If ions are included, D will not be symmetric with respect to the interchange of the coordinates of an electron and an ion.

The s-body function is defined as

$$f_s(X_1, X_2, \cdots X_s; t) = V^s \int D\, dX_{s+1} \cdots dX_N. \quad (3)$$

When moments of Eq. (1) are taken, a chain of equations is obtained. Assuming D has complete symmetry, it is

$$\left\{\frac{\partial}{\partial t} + \sum_{i=1}^{s}\left(\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{mc}\,\mathbf{v}_i\times\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}_i}\right) - \frac{e^2}{m}\sum_{i,j=1}^{s}{}' \frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}\cdot\frac{\partial}{\partial \mathbf{v}_i}\right\} f_s - \frac{ne^2}{m}\sum_{i=1}^{s}\int\left(\frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|}\right)\cdot\frac{\partial f_{s+1}}{\partial \mathbf{v}_i}\, dX_{s+1} = 0, \quad (4)$$

where $n = N/V$ and both N and V are large. In the limit that $|\mathbf{x}_i - \mathbf{x}_j| \rightarrow \infty$, the third term becomes negligible. If this term is omitted, it can be verified by direct substitution that a solution of Eq. (4) is

$$f_s^{(0)} = \prod_{i=1}^{s} f^{(0)}(X_i, t), \qquad (5)$$

where

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{e}{m}\left(\mathbf{E}_M^{(0)} + \frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f^{(0)} = 0, \quad (6)$$

and

$$\mathbf{E}_M^{(0)} = ne \int \left(\frac{\partial}{\partial \mathbf{x}}\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) f^{(0)}(X', t)\, dX'$$

is the macroscopic field. If, instead of taking this limit, we take the limit $e \rightarrow 0$, $m \rightarrow 0$, $n \rightarrow \infty$ such that e/m and ne remain constant, the result is the same. In this limit the plasma becomes a continuous fluid. Equation (6), which is the collisionless Boltzmann equation, is then a precise description. If the effects due to particle individuality are small, a suitable calculation procedure is to make an expansion in terms of some parameter proportional to e, where m and $1/n$ are considered to be of the same order. The natural units of the problem are $1/\omega_p$ for time and L_D for length, where $\omega_p = (4\pi ne^2/m)^{\frac{1}{2}}$ is the plasma frequency and $L_D = (\Theta/4\pi ne^2)^{\frac{1}{2}}$ is the Debye length. The mean energy per electron is $3\,\Theta/2$.

If Eq. (4) is rewritten in these units, the significant dimensionless parameters of the problem are self-evident:

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{s}\left(\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i} - \lambda\mathbf{v}_i\times\mathbf{e}_z\cdot\frac{\partial}{\partial \mathbf{v}_i}\right) - \frac{g}{4\pi}\sum_{i,j=1}^{s}{}' \frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}\cdot\frac{\partial}{\partial \mathbf{v}_i}\right] f_s - \frac{1}{4\pi}\sum_{i=1}^{s}\int\left(\frac{\partial}{\partial \mathbf{x}_i}\frac{1}{|\mathbf{x}_i - \mathbf{x}_{s+1}|}\right)\cdot\frac{\partial f_{s+1}}{\partial \mathbf{v}_i}\, dX_{s+1} = 0, \quad (4a)$$

where $\mathbf{e}_z$ is a unit vector in the direction of $\mathbf{B}$; $\lambda = \omega_c/\omega_p$, where $\omega_c = eB/mc$ is the cyclotron frequency; and $g = 1/nL_D^3 = O[e]$. The calculation procedure is to expand in powers of g and retain λ to all orders.

The expansion in powers of g is similar to the Mayer[6] cluster expansion of equilibrium statistical mechanics,

$$f_s = \prod_{i=1}^{s} f(X_i, t) + \sum_{p} \left[\prod f(X_i, t)\right] P(X_j, X_k; t)$$

[6] J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1940).

$$+ \sum_{pp} [\prod f(X_i, t)] P(X_j, X_k; t) P(X_l, X_m; t)$$
$$+ \sum_{t} [\prod f(X_i, t)] T(X_j, X_k, X_l; t) + \cdots . \quad (7)$$

The second term is summed over pairs, the third over pairs of pairs, and the fourth over triplets. In the present treatment the series will be terminated after the second term, or up to second order in g. In the lowest order, the solution of Eq. (4) is such that the s-body function can be expressed in terms of one-body functions. In the next order, the s-body function can be expressed in terms of one- and two-body functions. We substitute,

$$f_s = \prod_{i=1}^{s} f(X_i, t) + \sum_{p} [\prod f^{(0)}(X_i, t)] \cdot P(X_j, X_k; t) + \cdots \quad (8)$$

into Eq. (5), and retain only terms of order less than g^2. Here P is of order g and $f = f^0 + f^1$, where f^1 is of order g. The result is

$$\sum_{i=1}^{s} (\prod f(X, t)) \left\{ \left[\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} - \frac{e}{m} \left(\mathbf{E}_M(\mathbf{x}_i, t) + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}_i} \right] f(X_i, t) - \frac{ne}{m} \int \mathbf{E}(\mathbf{x}_i, \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{v}_i} P(X_i, X; t)\, dX \right\}$$
$$+ \frac{1}{2} \sum_{i,j=1}^{s}{}' (\prod f^{(0)}(X, t)) \left\{ \frac{\partial}{\partial t} P(X_i, X_j; t) + \left[\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} P - \frac{e}{m} \left(\mathbf{E}_M{}^{(0)}(\mathbf{x}_i, t) + \frac{1}{c} \mathbf{v}_i \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}_i} P - \frac{ne}{m} \frac{\partial f^{(0)}}{\partial \mathbf{v}_i} \cdot \int \mathbf{E}(\mathbf{x}_i, \mathbf{x}) \cdot P(X_j, X; t)\, dX - \frac{e}{m} f^{(0)}(X_j, t) \mathbf{E}(\mathbf{x}_i, \mathbf{x}_j) \cdot \frac{\partial}{\partial \mathbf{v}_i} f^{(0)}(X_i, t) \right] + \begin{bmatrix} i \to j \\ j \to i \end{bmatrix} \right\} = 0, \quad (9)$$

where

$$\mathbf{E}(\mathbf{x}, \mathbf{x}') = e \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|},$$

and

$$\mathbf{E}_M(\mathbf{x}, t) = n \int \mathbf{E}(\mathbf{x}, \mathbf{x}') f(X', t)\, dX'.$$

The cases $s = 1$ and $s = 2$ give equations that define $f(X, t)$ and $P(X, X'; t)$. Because of the interchangeability of particle coordinates, it is clear that Eq. (8) satisfies Eq. (4) for $s > 2$. Because the only restriction on $f^0(X, t)$ is that it satisfy Eq. (6), the equation for $P(X, X'; t)$ is not very tractable. Some progress has been made in obtaining physically meaningful solutions for $\mathbf{B} = 0$.[4,5]

If one particle is singled out, the reduction of the Liouville equation can be accomplished in a similar way. There will be two kinds of s-body distribution functions according to whether or not the test particle is included. The two coupled chains of equations have been solved up to second order in g. To this order, the solutions for both kinds of s-body functions may be expressed in terms of one- and two-body functions only; i.e., closure is achieved as in the case where D has full symmetry. The results will be stated in terms of the equations that define the one- and two-body functions. In addition, ions of finite mass have been included.

The zero-order one-body functions are solutions of the equation

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left(\mathbf{E}_M{}^{(0)} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} \right\} f^{(0)}(X, t) = 0, \quad (10)$$

where

$$\mathbf{E}_M{}^{(0)} = n \sum_j \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}') f_j{}^{(0)}(X', t)\, dX';$$

$\mathbf{E}_j$ $(\mathbf{x}, \mathbf{x}')$ is the Coulomb electric field at $\mathbf{x}$ from a particle of species j at $\mathbf{x}'$; $j = 1$ means electrons, and $j = 2$ means ions; q_i is the charge of a particle; m_i is its mass, and $f_i{}^{(0)}(X, t)$, means the zero-order one-body distribution for a field particle.

We shall assume that the field particles are initially in thermal equilibrium, so

$$f_i{}^{(0)}(X, t) = \frac{1}{(2\pi v_i{}^2)^{\frac{3}{2}}} \exp(-v^2/2v_i{}^2), \quad (11)$$
$$\mathbf{E}_M{}^{(0)} = 0,$$

and $m_i v_i{}^2 = \Theta$ defines the thermal velocities. The zero-order one-body distribution for the test particle is designated by $w_i{}^{(0)}(X, t)$. It also satisfies Eq. (10). The general solution is

$$w_i{}^{(0)}(X, t) = \int \Omega(X_0)\, \delta[X - X_0(t)]\, dX_0, \quad (12)$$

where X_0 means the initial position and velocity coordinates and $\Omega(X_0)$ is an arbitrary function normalized such that $(1/V) \int \Omega(X_0) dX_0 = 1$. The $X_0(t)$ are the time-dependent orbits in a magnetic field. If we assume the magnetic field in the z direction,

$$\mathbf{x}_0(t) = \mathbf{e}_x[x_0 + (v_{x0}/\omega_i) \sin \omega_i t + (v_{y0}/\omega_i)(1 - \cos \omega_i t)] + \mathbf{e}_y[y_0 + (v_{y0}/\omega_i) \sin \omega_i t - (v_{x0}/\omega_i)(1 - \cos \omega_i t)] + \mathbf{e}_z[z_0 + v_{z0} t],$$

and $\mathbf{v}_0(t) = d\mathbf{x}_0(t)/dt$.

There are two kinds of two-body correlation functions. For the field particles, $\bar{P}_{ij}$ $(X, X'; t)$ satisfies the equation,

$$\frac{D}{Dt}\bar{P}_{ij}(X, X'; t) + \frac{nq_i}{m_i}\frac{\partial f_i^{(0)}}{\partial \mathbf{v}} \cdot \sum_l \int \mathbf{E}_l(\mathbf{x}, \mathbf{x}'')\bar{P}_{jl}(X', X'')\, dX'' + \frac{nq_j}{m_j}\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \cdot \sum_l \int \mathbf{E}_l(\mathbf{x}', \mathbf{x}'')\bar{P}_{il}(X, X'')\, dX'' + \frac{q_i}{m_i} f_j^{(0)}(v')\mathbf{E}_j(\mathbf{x}, \mathbf{x}')\cdot\frac{\partial}{\partial \mathbf{v}} f_i^{(0)} + \frac{q_j}{m_j} f_i^{(0)}(v)\mathbf{E}_i(\mathbf{x}', \mathbf{x})\cdot\frac{\partial}{\partial \mathbf{v}'} f_j^{(0)} = 0, \tag{13}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'} + \omega_i \mathbf{v}\times\mathbf{e}_z\cdot\frac{\partial}{\partial \mathbf{v}} + \omega_j \mathbf{v}'\times\mathbf{e}_z\cdot\frac{\partial}{\partial \mathbf{v}'},$$

and $\omega_i = q_i B/m_i c$ is the cyclotron frequency.

If the field particles are initially in thermal equilibrium, the solution of Eq. (13) is

$$\bar{P}_{ij}(X, X'; t) = -\frac{q_i}{\Theta} f_i^{(0)}(v) f_j^{(0)}(v')\Phi_j(\mathbf{x}, \mathbf{x}'), \tag{14}$$

where

$$\Phi_j(\mathbf{x}, \mathbf{x}') = \frac{q_j}{|\mathbf{x} - \mathbf{x}'|}\exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{L_D}\right),$$
$$1/L_D^2 = \sum_i 1/L_i^2,$$
$$1/L_i^2 = 4\pi n q_i^2/m_i v_i^2.$$

If the particle of species i is the test particle, the equation for P_{ij} $(X, X'; t)$ is

$$\frac{D}{Dt}P_{ij} + \frac{nq_j}{m_j}\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \cdot \sum_l \int \mathbf{E}_l(\mathbf{x}', \mathbf{x}'')P_{il}(X, X''; t)\, dX'' = \frac{q_i}{m_i} f_j^{(0)}(v')\frac{\partial \Phi_j(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{x}}\cdot\frac{\partial w_i^{(0)}}{\partial \mathbf{v}}. \tag{15}$$

To solve this equation, it is convenient to introduce the Green's function $\mathbf{G}_{ij}$ $(X, X'; t)$, defined by

$$P_{ij}(X, X'; t) = \int \Omega(X_0)\mathbf{G}_{ij}(X, X'; t) \cdot\frac{\partial}{\partial \mathbf{v}}\delta[X - X_0(t)]\, dX_0. \tag{16}$$

Equation (15) thus can be transformed to a set of coupled equations for $\mathbf{G}_{ij}$,

$$\frac{D}{Dt}\mathbf{G}_{ij} + \omega_i(\mathbf{e}_z\times\mathbf{G}_{ij}) = \frac{q_i}{m_i} f_j^{(0)}(v')\frac{\partial}{\partial \mathbf{x}}\Phi_j(\mathbf{x}, \mathbf{x}') + \frac{q_j}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{x}'}\right)\mathbf{A}_i(X, \mathbf{x}'; t),$$

and

$$\nabla'^2\mathbf{A}_i(X, \mathbf{x}'; t) = -4\pi n \sum_l q_l \int \mathbf{G}_{il}(X, X'; t)\, d\mathbf{v}'. \tag{17}$$

The one-body function up to order g^2 for field particles may be expressed as

$$f_j(X', t) = f_j^{(0)}(v') + \frac{1}{V}\int w_i^{(0)}(X, t)\,\delta f_{ij}(X, X'; t)\, dX, \tag{18}$$

where δf_{ij} $(X, X'; t)$ is also of the nature of a Green's function. It means the change in the distribution function at X', t due to a test particle at X. It satisfies the set of coupled equations:

$$\frac{D\,\delta f_{ij}}{Dt} = \frac{q_j}{m_j}\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'}\cdot\frac{\partial \Phi_i}{\partial \mathbf{x}'}(X, x'; t), \tag{19a}$$

and

$$\nabla'^2\Phi_i(X, \mathbf{x}'; t) = -4\pi q_i\,\delta(\mathbf{x}' - \mathbf{x}) - 4\pi n \sum_l q_l \int \delta f_{il}(X, X'; t)\, d\mathbf{v}'. \tag{19b}$$

Equations (17) and (19) are similar in type. They can be solved in a straightforward fashion by taking Fourier and Laplace transforms. It is only necessary to specify initial conditions. We shall assume that $\mathbf{G}_{ij}$ $(X, X'; 0) = \delta f_{ij}$ $(X, X'; 0) = 0$, which means that initially the test particle is not correlated with any other particle and does not possess a shield cloud.

The equation for the test-particle one-body function to order g^2 is of the Fokker-Planck type,

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_i}{m_i c}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}}\right)w_i + \frac{1}{m_i}\frac{\partial}{\partial \mathbf{v}}(\mathbf{F}_i' w_i^{(0)}) + \frac{1}{m_i}\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}(\mathbf{T}_i w_i^{(0)}) = 0. \tag{20}$$

The frictional drag force $\mathbf{F}_i'$ and the fluctuation tensor $\mathbf{T}_i$ are completely determined by the Green's functions,

$$\mathbf{F}_i' = \mathbf{F}_i - \frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{T}_i, \tag{21}$$

$$\mathbf{F}_i = nq_i \sum_j \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}')\ \delta f_{ij}(X, X'; t)\ dX', \tag{22}$$

$$\mathbf{T}_i = nq_i \sum_j \int \mathbf{E}_j(\mathbf{x}, \mathbf{x}')\mathbf{G}_{ij}(X, X'; t)\ dX'. \tag{23}$$

The starting point of this calculation is the Liouville equation. It possesses the property of time-reversal invariance. The expansion procedure preserves this property. Equation (20) differs from the usual Fokker-Planck equation because $\mathbf{F}_i'$ and $\mathbf{T}_i$ are functions of time, and the equation has the property of time-reversal invariance. If instead of the correct values of $\mathbf{F}_i$ and $\mathbf{T}_i$ the asymptotic values as $t \to \infty$ are substituted, Eq. (20) will be a conventional Fokker-Planck equation. The time reversal invariance will then be destroyed.

III. TEST-PARTICLE CALCULATIONS FOR $B = 0$

To solve Eqs. (19), Fourier and Laplace transforms are introduced. For example,

$$\Phi_i(X, x'; t) = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, (F\Phi_i)_{\mathbf{k},t} \exp\left[i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x})\right],$$

and

$$(LF\Phi_i)_{\mathbf{k},p} = \int_0^\infty dt\, (F\Phi_i)_{\mathbf{k},t} \exp(-pt).$$

The first of Eqs. (19) can be integrated along the unperturbed orbits which are linear trajectories for $\mathbf{B} = 0$:

$$\delta f_{ij}(X, X'; t) = \frac{q_j}{m_j}\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \cdot \int_{\tau=0}^{t} d\tau\, \frac{\partial \Phi_i}{\partial \mathbf{x}'}(\mathbf{x} - \mathbf{v}\tau, \mathbf{v}; \mathbf{x}' - \mathbf{v}'\tau, t - \tau).$$

After Fourier transforms have been taken, this equation becomes

$$(F\ \delta f_{ij})_{\mathbf{k},t} = \frac{q_j}{m_j} i\mathbf{k}\cdot\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \int_{\tau=0}^{t} d\tau\, (F\Phi_i)_{\mathbf{k},t-\tau} \cdot \exp\left[i\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')\tau\right].$$

The Laplace transform of this equation can be taken by making use of the "Faltung theorem"

$$L \int_{\tau=0}^{t} d\tau\, A(\tau)B(t - \tau)\, d\tau = (LA)(LB),$$

$$(LF\ \delta f_{ij})_{\mathbf{k},p} = \frac{q_j}{m_j} i\mathbf{k}\cdot\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} \frac{(LF\Phi_i)_{\mathbf{k},p}}{[p + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v})]}. \tag{19a}$$

Similarly, the second of Eqs. (19) becomes

$$k^2(LF\Phi_i)_{\mathbf{k},p} = \frac{4\pi q_i}{p} + 4\pi n \sum_l \int (LF\ \delta f_{il})_{\mathbf{k},p}\, d\mathbf{v}'. \tag{19b}$$

We may now substitute Eq. (19a) into Eq. (19b) and solve for $(LF\Phi_i)$. The result is

$$(LF\Phi_i)_{\mathbf{k},p} = \frac{4\pi q_i}{p[k^2 + (1/L_D^2)W]}, \tag{24}$$

where

$$W = 4\pi n L_D^2 \sum_j \frac{q_j}{m_j} \int \frac{d\mathbf{v}'}{[p + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v})]} i\mathbf{k} \cdot \frac{\partial f_j^{(0)}}{\partial \mathbf{v}'} = -\tfrac{1}{2} \sum_j \int_0^\infty dt\, \exp\{-[p - i(\mathbf{k}\cdot\mathbf{v})]t\} \cdot \frac{d}{dt} \exp\left[-\frac{(kv_jt)^2}{2}\right].$$

Then $(LF\delta f_{ij})$ can be obtained by substituting this expression into Eq. (19a),

$$(LF\ \delta f_{ij})_{\mathbf{k},p} = \frac{(-1)^{i+j+1}}{2np} \cdot \frac{i(\mathbf{k}\cdot\mathbf{v}')f_j^{(0)}(v')}{[p + i\mathbf{k}\cdot(\mathbf{v}' - \mathbf{v})][(kL_D)^2 + W]}. \tag{25}$$

The denominator of Eq. (25) has poles at $p = 0$, $p = i\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')$, and at the roots of $(kL_D)^2 + W = 0$. The $p = 0$ pole yields the asymptotic solution. The pole $p = i\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')$ produces a term that describes the removal of depleted charge from the vicinity of the test particle to infinity. Field particles move away from the test charge with their unperturbed velocities relative to the test charge. Ultimately, the field particle distribution achieves a permanent charge depletion around the test charge.

The equation $(kL_D)^2 + W(\mathbf{k}, p) = 0$ has an infinite set of roots that have been discussed by Landau.[7] All but one root produce terms that damp out in a negligible time. The last of the "Mohicans" is for

$$p = i[\mathbf{k}\cdot\mathbf{v} \pm \omega(k)] - \epsilon_L(k), \tag{26}$$

where

$$\omega(k) \cong \omega_p[1 + 3(kL_D)^2 + \cdots],$$

$$\epsilon_L(k) = \frac{\pi^{\frac{1}{2}}}{8} \frac{\omega_p}{(kL_D)^3} \exp\left[\frac{-1}{4(kL_D)^2}\right].$$

The term from this pole damps rapidly for $kL_D > 1$ and very slowly for $kL_D \ll 1$. If the Laplace and Fourier inversions of Eq. (25) are carried out, a term obtains that decays like $1/(\omega_p t)^{\frac{3}{2}}$. The cause of this decay is the dispersion of $\omega(k)$. In the rest of this paper we shall consider only the asymptotic solutions as $t \to \infty$.

Equations (17) are of the same type as Eqs. (19)

[7] L. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

and may be solved in the same way. The solution is

$$(LF\mathbf{G}_{ij})_{\mathbf{k},p} = \frac{(-1)^{i+j+1}}{2np} \cdot \frac{i\mathbf{k}v_i^2 f_i^{(0)}(v')}{[(kL_D)^2+1][p+i\mathbf{k}\cdot(\mathbf{v}'-\mathbf{v})]} \cdot \left\{1 - \frac{\sum_l(\mathbf{k}\cdot\mathbf{v}'/kv_l)U[(p-i\mathbf{k}\cdot\mathbf{v})/ikv_l]}{2[(kL_D)^2+W]}\right\}, \qquad (27)$$

where

$$U(x) = i\int_0^\infty \exp(-t^2/2)\exp(-ixt)\,dt.$$

The asymptotic solutions as $t\to\infty$, $\delta f_{ij}(X, X')$, and $\mathbf{G}_{ij}(X, X')$ must satisfy a detailed balance condition. This condition becomes apparent if we consider a Maxwell distribution for the test particle, i.e., $\Omega(X_0) = f_i^{(0)}(v_0)$ and $w_i^{(0)}(X, t) = f_i^{(0)}(v) = w_i(X, t)$. In this case the two-body function,

$$w_i(X)\left[f_j^{(0)}(X') + \frac{1}{V}\int w_i^{(0)}(X)\,\delta f_{ij}(X, X')\,dX\right] + \int \Omega(X_0)\mathbf{G}_{ij}(X, X')\cdot\frac{\partial}{\partial\mathbf{v}}\,\delta[X - X_0(t)]\,dX_0,$$

must be the thermal equilibrium two-body function $f_i^{(0)}(v)f_j^{(0)}(v') + \bar{P}_{ij}(X, X')$. The detailed balance condition is therefore

$$f_i^{(0)}(v)\left[\delta f_{ij}(X, X') - \frac{\mathbf{v}}{v_i^2}\cdot\mathbf{G}_{ij}(X, X')\right] = \bar{P}_{ij}(X, X'), \qquad (28)$$

$$\lim_{p\to 0}\left\{p\left[(LF\;\delta f_{ij})_{\mathbf{k},p} - \frac{\mathbf{v}}{v_i^2}\cdot(LF\mathbf{G}_{ij})_{\mathbf{k},p}\right]\right\} = \frac{(-1)^{i+j+1}}{2n}\,\frac{f_j^{(0)}(v')}{[(kL_D)^2+1]}.$$

That Eqs. (25) and (27) satisfy this condition can be verified by direct substitution. Equation (28)

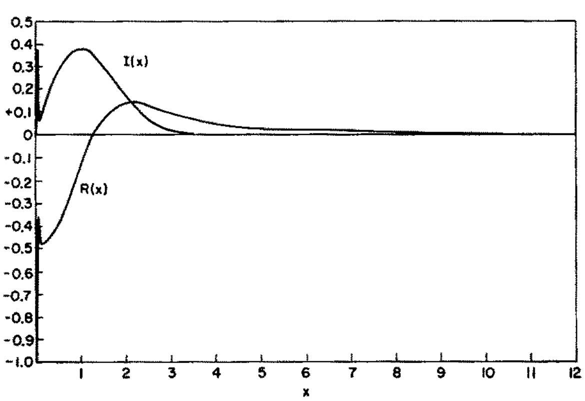

FIG. 1. The W function $W(x) = -R(x) + iI(x)$.

implies that

$$\mathbf{F}_i - (\mathbf{T}_i\cdot\mathbf{v}/v_i^2) = 0. \qquad (29)$$

If $\mathbf{F}_i$ and $\mathbf{T}_i$ are expressed in terms of a unit vector $\mathbf{e}_1$ parallel to $\mathbf{v}$ and $\mathbf{e}_2$, $\mathbf{e}_3$ perpendicular to $\mathbf{v}$,

$$\mathbf{F}_i = F_\parallel \mathbf{e}_1,$$
$$\mathbf{T}_i = T_\parallel \mathbf{e}_1\mathbf{e}_1 + T_\perp(\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3), \qquad (30)$$

and

$$\mathbf{F}_i' = F_\parallel' \mathbf{e}_1,$$

where

$$F_\parallel' = F_\parallel - \frac{\partial T_\parallel}{\partial v} - 2\left(\frac{T_\parallel - T_\perp}{v}\right).$$

There are only two independent quantities to compute, $F_\parallel$ and $T_\perp$, since $T_\parallel = (v_i^2 F_\parallel/v)$.

The frictional drag from the fluid response of the plasma to the test particle is obtained from Eqs. (22) and (25):

$$F_\parallel = \frac{4\pi i q^2}{(2\pi)^2}\int_0^\infty k\,dk\int_{-1}^1 \frac{\mu\,d\mu W(\mu v/v_1)}{(kL_D)^2 + W(\mu v/v_1)}. \qquad (31)$$

The function

$$W(x) = 1 + \frac{x}{2}U(-x) + \frac{x}{2}\left(\frac{m_2}{m_1}\right)^{\frac{1}{2}}U\left[-x\left(\frac{m_2}{m_1}\right)^{\frac{1}{2}}\right] = -R(x) + iI(x)$$

is plotted in Fig. 1. The k integral in Eq. (31) is divergent. That the present procedure breaks down at short distances is apparent from the case $\Omega(X_0) = \delta(X_0)$, i.e., a zero-velocity test charge. In this case,

$$\lim_{t\to\infty} f_j(X, t) = f_j^{(0)}(v)\left[1 - \frac{q_i q_j}{m_j v_j^2}\,\frac{\exp(-|\mathbf{x}|/L_D)}{|\mathbf{x}|}\right]. \qquad (32)$$

If $|\mathbf{x}| < q^2/\Theta$, f_j will not be positive when $q_i = q_j$. We should expect the present treatment to be appropriate for collective effects. For distances less than the interparticle spacing $r_0 = (3/4\pi n)^{\frac{1}{3}}$ there can be no collective effects, and the binary collision method should be applicable.[1] It is also necessary to employ a short distance cutoff with this method. For $q_i = -q_j$, a quantum-mechanical treatment is necessary to avoid divergence. The short distance cutoff employed in the classical collision treatment is q^2/Θ.[1] Instead of using the present treatment down to distances of the order r_0 and the binary collision method from r_0 to q^2/Θ, we shall simply use the present treatment with the cutoff $k_{\max} \cong \Theta/q^2$, which gives the same result.

Since $|R(x)| \leq 1$ and $|I(x)| < 1$, the denominator in Eq. (31) can be expanded in the domain $kL_D > 1$. This leads to a collision-type contribution,

$$F_{\parallel} = \frac{\Theta F_0}{2} \ln D \sum_i \frac{1}{m_i} \frac{\partial}{\partial v}\left(\frac{1}{v} \operatorname{erf} \frac{v}{\sqrt{2} v_i}\right)$$

$$\cong -\frac{2}{3(\pi)^{\frac{1}{2}}} F_0 \ln D \sum_i (v/v_i) \quad \text{for} \quad v \ll v_i,$$

$$\cong -\frac{F_0}{2} \ln D \sum_i (v_i/v)^2 \quad \text{for} \quad v \gg v_i, \tag{33}$$

where $F_0 = q^2/L_D^2$ and $D = k_{\max} L_D = 4\pi g^{-1}$.

For the domain $kL_D < 1$, it is convenient to write Eq. (31) in the form

$$F_{\parallel} = -1/v \int S(k, \mathbf{\Omega})\, dk\, d\mathbf{\Omega},$$

where

$$S(k, \mathbf{\Omega}) = \frac{v(kL_D)^2}{2\pi^2} \cdot \frac{k^3 \mu I(\mu v/v_1)}{[(kL_D)^2 - R(\mu v/v_1)]^2 + [I(\mu v/v_1)]^2},$$

and $d\mathbf{\Omega} = 2\pi d\mu$. The integrand is of a resonant character for $v \gg v_1$. In this case $R(\mu v/v_1) \cong (v_1/\sqrt{2}\mu v)^2 > 0$ and $|I(\mu v/v_1)| \ll 1$. The resonance corresponds to the emission of plasma waves. For a given k, the emission takes place at the angles $\mu = \cos\theta = \pm(v_1/\sqrt{2} v k L_D)$. After the $\mathbf{\Omega}$ integration is carried out, the result is

$$\bar{S}(k, v) = \frac{1}{4\pi} \int S(k, \mathbf{\Omega})\, d\mathbf{\Omega} = (q^2 \omega_p^2 / 4\pi k v)$$

for $v/v_1 > [1/\sqrt{2} k L_D]$ and negligible otherwise. The drag force due to plasma-wave emission is

$$F_{\parallel} = -\frac{4\pi}{v} \int_{k \sim (v_1/\sqrt{2} v L_D)}^{k \sim 1/L_D} \bar{S}(k, v)\, dk = -\frac{F_0}{2}\left(\frac{v_1}{v}\right)^2 \ln \frac{v}{v_1}. \tag{34}$$

The rate of emission of energy from the test particle is $F_{\parallel} v$, or $4\pi \bar{S}(k, v)\, dk$ from wave vector magnitudes between k and $k + dk$. The total emission per unit volume and wave vector magnitude for a plasma with a Maxwell distribution of particles is

$$Q(k) = 4\pi n \sum_i \int_{v > (v_1/\sqrt{2} k L_D)} \bar{S}(k, v) f_i^{(0)}(v)\, d^3\mathbf{v} = A(k)[\Theta k^2/2\pi^2], \tag{35}$$

where $A(k) = 2\epsilon_L(k)$ is the energy absorption constant and $\epsilon_L(k)$ is the Landau damping given by Eq. (26). Equation (35) is simply a statement of Kirkchoff's law of radiation with the classical Rayleigh-Jeans distribution $\Theta k^2/2\pi^2$.

By the same techniques, $T_\perp$ may be computed. The collision contribution from the domain $kL_D > 1$ is

$$T_\perp = -\frac{F_0}{4} \frac{v_i^2}{v} \ln D \sum_i \left(1 + \frac{\partial}{\partial v} \frac{v_i^2}{v}\right) \operatorname{erf} \frac{v}{\sqrt{2} v_i}. \tag{36}$$

The plasma-wave contribution from the domain $kL_D < 1$ is

$$T_\perp = \frac{-T_{\parallel}}{2} - \left(\frac{F_0 v_i^2}{4v}\right)\left[1 - \frac{1}{2}\left(\frac{v_1}{v}\right)^2 \cdots\right], \tag{37}$$

where $T_\perp$ and $T_{\parallel} = (F_{\parallel} v_i^2/v)$ are caused by field particle individuality. By use of Eqs. (30), $F_{\parallel}'$ can now be computed. The collision contribution is

$$F_{\parallel}' = \frac{\Theta F_0}{2} \ln D \cdot \sum_i \left(\frac{1}{m_i} + \frac{1}{m_i}\right) \frac{\partial}{\partial v}\left(\frac{1}{v} \operatorname{erf} \frac{v}{\sqrt{2} v_i}\right). \tag{38}$$

When this is compared with Eq. (25), it is apparent that the consequence of the individuality of the field particles is to replace the field particle mass $1/m_i$ by $1/m_{eff} = (1/m_i) + (1/m_i)$; i.e., recoil of the test particle is introduced.

The plasma-wave part of the total drag force is, for $v \gg v_1$,

$$F_{\parallel}' = -\frac{F_0}{2}\left(\frac{v_1}{v}\right)^2 \ln \frac{v}{v_1} - \frac{F_0}{2}\left(\frac{v_i}{v}\right)^2 + F_0 \frac{v_i^2 v_1^2}{v^4}. \tag{39}$$

The latter two terms are small corrections to the fluid response. The first term may be interpreted as due to spontaneous emission of plasma waves, the second as stimulated emission, and the third as absorption.

IV. EFFECT OF A CONSTANT EXTERNAL MAGNETIC FIELD

The procedure for solving Eqs. (17) and (19) is similar to the case where $\mathbf{B} = 0$. It is, however, more complicated because the unperturbed orbits are spirals about the magnetic field lines instead of straight lines. The details of this calculation will be omitted here.[8] In this section, the calculation of $\mathbf{F}_i$ and $\mathbf{T}_i$ will be considered, making use of the asymptotic results for δf_{ij} (X, X') and $\mathbf{G}_{ij}$ (X, X'). The

[8] M. N. Rosenbluth and Norman Rostoker, "Kinetic equations for a plasma, Part III," General Atomic Internal Report GAMD-663, 1959.

$\mathbf{F}_i$ and $\mathbf{T}_i$ are functions of $\mathbf{v}$, the test particle velocity. It is convenient to employ cylindrical coordinates $(v_\perp, \beta, v_z)$ for $\mathbf{v}$, since the test particle spirals about the z direction. The angular variable β is defined so that $v_x = -v_\perp \sin\beta$, $v_y = v_\perp \cos\beta$. All vectors and dyadics will be expressed in terms of the unit vectors, $(\mathbf{e}_\rho, \mathbf{e}_\beta, \mathbf{e}_z)$, where

$$\mathbf{e}_x = \mathbf{e}_\rho \cos\beta - \mathbf{e}_\beta \sin\beta,$$

$$\mathbf{e}_y = \mathbf{e}_\rho \sin\beta + \mathbf{e}_\beta \cos\beta.$$

In terms of these unit vectors,

$$\mathbf{v} = v_\perp \mathbf{e}_\beta + v_z \mathbf{e}_z, \tag{40}$$

$$\mathbf{F}_i = F_\rho \mathbf{e}_\rho + F_\beta \mathbf{e}_\beta + F_z \mathbf{e}_z$$

$$\mathbf{T}_i = T_{\rho\rho}\mathbf{e}_\rho\mathbf{e}_\rho + T_{\beta\beta}\mathbf{e}_\beta\mathbf{e}_\beta + T_{zz}\mathbf{e}_z\mathbf{e}_z + T_{\rho\beta}(\mathbf{e}_\rho\mathbf{e}_\beta - \mathbf{e}_\beta\mathbf{e}_\rho) + T_{\beta z}(\mathbf{e}_\beta\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_\beta) + T_{z\rho}(\mathbf{e}_z\mathbf{e}_\rho - \mathbf{e}_\rho\mathbf{e}_z).$$

The components F_ρ, $F_\beta \cdots T_{z\rho}$ are functions of $v_\perp$ and v_z, but not of β. They are as follows:

$$\mathbf{F}_i = 2\left(\frac{q_i L_D}{2\pi}\right)^2 \int d^3\mathbf{k} \cdot \sum_{m=-\infty}^{\infty} \frac{\boldsymbol{\gamma}_i(\mathbf{k}, m)}{[(kL_D)^2 + W(\mathbf{k}, m)]}, \tag{41}$$

$$\mathbf{T}_i = v_i^2\left(\frac{q_i L_D}{2\pi}\right)^2 \sum_j \int \frac{d^3\mathbf{k}}{[(kL_D)^2 + 1]} \cdot \sum_{m=-\infty}^{\infty} \frac{N_j(\mathbf{k}, m)\,\boldsymbol{\Gamma}_i(\mathbf{k}, m)}{[(kL_D)^2 + W(\mathbf{k}, m)]}, \tag{42}$$

$$N_j(\mathbf{k}, m) = i\int_0^\infty dt\, \exp\,[i(k_z v_z - m\omega_i)t] \cdot \exp\,[-\xi_j(t)], \tag{43}$$

$$W(\mathbf{k}, m) = -\frac{1}{2}\sum_j \int_0^\infty dt\, \exp\,[i(k_z v_z - m\omega_i)t]\,\frac{d}{dt} \cdot \exp\,[-\xi_j(t)] = 1 + \tfrac{1}{2}(k_z v_z - m\omega_i)\sum_j N_j(\mathbf{k}, m), \tag{44}$$

$$\xi_j(t) = \tfrac{1}{2}(k_z v_j t)^2 + (k_\perp \bar{a}_j)^2(1 - \cos\omega_j t).$$

Cylindrical coordinates $(k_\perp, \alpha, k_z)$ are employed for $\mathbf{k}$; $\bar{a}_j = v_j/\omega_j$ is the mean Larmor radius for field particles of species j.

The components of the vector $\boldsymbol{\gamma}_i$ and the tensor $\boldsymbol{\Gamma}_i$ contain Bessel functions of argument $r = k_\perp a_i$, where $a_i = v_\perp/\omega_i$ is the Larmor radius of the test particle. These components are

$$\left.\begin{aligned}
\gamma_\rho &= k_\perp J_m(r) J_m'(r),\\
\gamma_\beta &= i(m/a_i)[J_m(r)]^2,\\
\gamma_z &= -ik_z[J_m(r)]^2,\\
\Gamma_{\rho\rho} &= ik_\perp^2[J_m'(r)]^2,\\
\Gamma_{\beta\beta} &= i[(m/a_i)J_m(r)]^2,\\
\Gamma_{zz} &= i[k_z J_m(r)]^2,\\
\Gamma_{\rho\beta} &= k_\perp(m/a_i)J_m(r)J_m'(r),\\
\Gamma_{\beta z} &= -ik_z(m/a_i)[J_m(r)]^2,\\
\Gamma_{z\rho} &= k_\perp k_z J_m(r) J_m'(r),
\end{aligned}\right\} \tag{45}$$

where J_m is the Bessel function of the first kind of order m and $J_m'(r) = dJ_m(r)/dr$. Equations (41) to (45) were obtained by solving Eqs. (17) and (19) exactly.

Of the nine components given by Eq. (45), only six are independent; $\mathbf{F}_i$ and $\mathbf{T}_i$ satisfy the detailed balance relationship of Eq. (29), which gives three equations connecting the components. The dyadic $\mathbf{T}_i$ contains a symmetric and an antisymmetric part. Only the symmetric part need be retained in the term $(1/m_i)(\partial/\partial\mathbf{v})\cdot(\partial/\partial\mathbf{v})\cdot(\mathbf{T}_i w_i^{(0)})$ of Eq. (20). The total frictional drag force is given by Eq. (21). The components of $\mathbf{F}_i'$ are

$$\left.\begin{aligned}
F_\rho' &= F_\rho + \frac{\partial T_{\rho\beta}}{\partial v_\perp} - \frac{\partial T_{z\rho}}{\partial v_z},\\
F_\beta' &= F_\beta - \frac{\partial T_{\beta\beta}}{\partial v_\perp} - \frac{\partial T_{\beta z}}{\partial v_z} - \frac{(T_{\beta\beta} - T_{\rho\rho})}{v_\perp},\\
F_z' &= F_z - \frac{\partial T_{\beta z}}{\partial v_\perp} - \frac{\partial T_{zz}}{\partial v_z} - \frac{T_{\beta z}}{v_\perp}.
\end{aligned}\right\} \tag{46}$$

The components $T_{\rho\beta}$ and $T_{z\rho}$, play only a part in the determination of the radial force F_ρ'. The radial force is in the same direction as the Lorentz force. It is smaller than the Lorentz force by a factor g so that if the present expansion procedure converges rapidly, the radial force and the antisymmetric part of $\mathbf{T}_i$ may be neglected.

To make any further progress in the calculation of $\mathbf{F}_i$ and $\mathbf{T}_i$, it is necessary to carry out the integrations of Eqs. (41) and (42) approximately. The method consists of dividing the $\mathbf{k}$ space into regions where asymptotic forms of N_j $(\mathbf{k}, m)$ and $W(\mathbf{k}, m)$ are applicable. If the results are insensitive to how the division is made, they are considered to be acceptable. The asymptotic forms of N_j $(\mathbf{k}, m)$ are as follows:

1. If $|k_z| \gg 1/\bar{a}_j$, the term $\exp\,[-\frac{1}{2}(k_z v_j t)^2]$ annihilates the integrand in a time $t \ll 1/\omega_j$. We may

therefore expand $\cos \omega_i t = 1 - \frac{1}{2}(\omega_i t)^2 \cdots$ and retain only the first two terms. The asymptotic form is

$$N_i(\mathbf{k}, m) \cong \frac{1}{kv_i} U\left(\frac{m\omega_i - k_z v_z}{kv_i}\right), \tag{47}$$

and

$$W(\mathbf{k}, m) \cong W\left(\frac{k_z v_z - m\omega_i}{kv_1}\right),$$

where $U(x)$ and $W(x)$ are defined in Eqs. (27) and (31).

2. If $|k_z| \ll 1/\bar{a}_i$ and $k_\perp \ll 1/\bar{a}_i$,

$$\exp\left[-(k_\perp \bar{a}_i)^2(1 - \cos \omega_i t)\right] \cong 1$$

and

$$N_i(\mathbf{k}, m) \cong \frac{1}{|k_z|\, v_i} U\left(\frac{m\omega_i - k_z v_z}{|k_z|\, v_i}\right). \tag{48}$$

3. If $|k_z| \ll 1/\bar{a}_i$ and $k_\perp \gg 1/\bar{a}_i$, $\exp[-\xi_i(t)] \ll 1$ except in the neighborhood of $\omega_i t = 2n\pi$. In the neighborhood of these points we may set $t = 2n\pi/\omega_i + \tau$ and $\cos \omega_i t \cong [1 - \frac{1}{2}(\omega_i \tau)^2]$. The result is

$$N_i(\mathbf{k}, m) \cong \frac{1}{kv_i}\left\{U\left(\frac{m\omega_i - k_z v_z}{kv_i}\right) + 2i\left[U_I\left(\frac{m\omega_i - k_z v_z}{kv_i}\right)\right] \sum_{n=1}^{\infty} \exp\left[-\frac{(k_z \bar{a}_i 2\pi n)^2}{2}\right] \cdot \exp\left[i2\pi n\left(\frac{k_z v_z - m\omega_i}{\omega_i}\right)\right]\right\}, \tag{49}$$

where $U_I(x) = (\pi/2)^{\frac{1}{2}} \exp(-x^2/2)$.

The second term is of a resonant character. When $k_z v_z - m\omega_i = s\omega_i$, where $s = 0, \pm 1, \pm 2$, etc., the sum will be

$$\sum_{n=1}^{\infty} \exp\left[-\frac{(k_z \bar{a}_i 2\pi n)^2}{2}\right] \cong \frac{1}{2(2\pi)^{\frac{1}{2}}\, |k_z|\, \bar{a}_i} \gg 1. \tag{50}$$

If $v_z \gg v_i$, the resonances will be sharp and the spacing narrow. The width of a resonance will be $\delta(k_z \bar{a}_i) \sim (k_z \bar{a}_i)\,(v_i/v_z)$, and the spacing $\Delta(k_z \bar{a}_i) \sim (v_i/v_z)$. The fraction of k_z space occupied by resonances will be of order $k_z \bar{a}_i \ll 1$. The resonances may be neglected in this case. If $v_z \ll v_i$, the resonances will be broad and the spacing large. Assuming $\omega_1 \neq \omega_2$ times an integer, the only case in which the resonant term is significant is when $i = j$ and $v_z \ll v_j$. In this case the sum is given by Eq. (50).

The subdivision of $\mathbf{k}$ space is illustrated in Fig. 2. Except in the resonance regions indicated by hatching, $|W(\mathbf{k}, m)| \gtrsim 1$. If $|W(k, m)| < 1$ and $kL_D > 1$, the denominator in Eqs. (41) and (42)

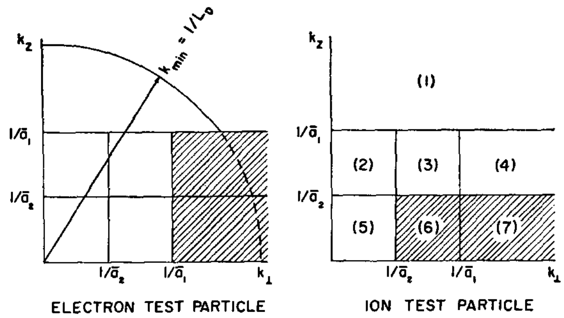

FIG. 2. Subdivision of $\mathbf{k}$ space.

may be expanded as follows:

$$\frac{1}{(kL_D)^2 + W(\mathbf{k}, m)} \cong \frac{1}{(kL_D)^2}\left[1 - \frac{W(k, m)}{(kL_D)^2} \cdots\right]. \tag{51}$$

The contributions from all domains where this expansion is permissible are designated collision type, otherwise they are resonance type.

Consider a typical integral such as

$$F_z = -4\pi\left(\frac{q_i L_D}{2\pi}\right)^2 \int k_\perp\, dk_\perp \int dk_z\, ik_z \cdot \sum_{m=-\infty}^{\infty} \frac{J_m^2(k_\perp a_i)}{[(kL_D)^2 + W(\mathbf{k}, m)]}. \tag{52}$$

For the collision contribution, the domain of integration is bounded on the outside by a sphere of radius $k_{\max} = (k_\perp^2 + k_z^2)^{\frac{1}{2}}$ and on the inside by a sphere of radius L_D if $L_D < \bar{a}_1$. The resonance region indicated in Fig. 2 must be omitted. This, however, involves a small angular region so that it may be included in the domain of integration provided the asymptotic form of Eq. (47) is employed.

After expanding the denominator in Eq. (52), the sum can be expressed as

$$\sum_{m=-\infty}^{\infty} \frac{J_m^2(k_\perp a_i)}{[(kL_D)^2 + W(\mathbf{k}, m)]} \cong \frac{1}{(kL_D)^2} \cdot \left\{1 + \frac{1}{2(kL_D)^2} \sum_{i,m} \int_0^\infty dt\, J_m^2(k_\perp a_i) \cdot \exp[i(k_z v_z - m\omega_i)t]\left[\frac{d}{dt} \exp\left(-\frac{(kv_i t)^2}{2}\right)\right]\right\}.$$

The first term can be omitted because it is even in k_z. In the second term,

$$\sum_m J_m^2(k_\perp a_i) \exp(-im\omega_i t) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp\{ik_\perp a_i[\cos(\alpha - \omega_i t) - \cos \alpha]\} \cong \frac{1}{2\pi} \int_0^{2\pi} d\alpha \exp(ik_\perp v_\perp t \sin \alpha),$$

TABLE I. Maximum impact parameter for simple collision terms.

Electron test particle	Ion test particle	Region of applicability
L_D	L_D	$L_D < \bar{a}_1$
$\bar{a}_1$	L_D	$\bar{a}_1 < L_D < \bar{a}_2$
$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_2 < L_D$

making use of the fact that $\exp[-\frac{1}{2}(kv_it)^2]$ annihilates the integrand before ω_it departs appreciably from zero. Thus Eq. (52) is reduced to

$$F_z \cong (q_i/2\pi L_D)^2 \int d^3\mathbf{k}\, \frac{ik_z}{k^4} \cdot \sum_i (\mathbf{k}\cdot\mathbf{v}/kv_i) U(-\mathbf{k}\cdot\mathbf{v}/kv_i)$$

$$= -4\pi(v_z/v)(q_i/2\pi L_D)^2 \int_{k_{\min}}^{k_{\max}} \frac{dk}{k} \cdot \int_0^1 d\mu\, \mu^2 \sum_i (v/v_i) U_I(-\mu v/v_i)$$

$$= \frac{F_0}{2}\frac{v_z}{v} \ln D \sum_i \frac{\partial}{\partial v}\left(\frac{v_i^2}{v} \operatorname{erf} \frac{v}{\sqrt{2}v_i}\right). \quad (52a)$$

If $L_D < \bar{a}_1$, $D = k_{\max}L_D$. If $L_D \not< \bar{a}_1$, the calculation is the same except that the domain of integration is defined by the limits of applicability of the asymptotic form given by Eq. (47). Thus, $D = k_{\max}b_{\max}$, where $b_{\max} = 1/k_{\min}$, is given in Table I.

Similarly, the collision contribution to all other components may be determined. The results can be expressed most conveniently in a coordinate system for which $\mathbf{T}_i$ is diagonal, neglecting the antisymmetric part. The unit vectors $\mathbf{e}_\beta$ and $\mathbf{e}_z$ are replaced by $\mathbf{e}_v$ and $\mathbf{e}_\sigma$, where

$$\mathbf{e}_v = \mathbf{v}/v = (v_\perp/v)\mathbf{e}_\beta + (v_z/v)\mathbf{e}_z,$$
$$\mathbf{e}_\sigma = (\mathbf{v}\times\mathbf{B}\times\mathbf{v})/vv_\perp B = -(v_z/v)\mathbf{e}_\beta + (v_\perp/v)\mathbf{e}_z. \quad (53)$$

In terms of $\mathbf{e}_\rho$, $\mathbf{e}_v$, $\mathbf{e}_\sigma$,

$$\mathbf{F}_i \cong F_\parallel \mathbf{e}_v,$$
$$\mathbf{T}_i \cong T_\parallel \mathbf{e}_v\mathbf{e}_v + T_\perp(\mathbf{e}_\rho\mathbf{e}_\rho + \mathbf{e}_\sigma\mathbf{e}_\sigma). \quad (54)$$

The equations for $F_\parallel$, $T_\parallel$, and $T_\perp$ are the same as in the zero-magnetic-field case except for the definition of D.

The radial force has been neglected. The result for F_ρ is

$$F_\rho \cong -(F_0v_\perp\omega_i/\pi v^2k_{\min}) \cdot \{1 - \tfrac{1}{2} \sum_i [(v/v_i) + (v_i/v)] U_R(v/v_i)\}$$

$$\cong (F_0v_\perp\omega_i/\pi v^2k_{\min}) \sum_i (v_i/v)^2 \quad \text{for} \quad v \gg v_i$$

$$\cong (F_0v_\perp\omega_i/3\pi v^2k_{\min}) \sum_i (v/v_i)^2 \quad \text{for} \quad v \ll v_i. \quad (55)$$

Because $\mathbf{e}_\rho$ has been defined so that it is always directed from the guiding center of the particle towards the particle, $\mathbf{F}_\rho$ is always antiparallel to the Lorentz force. The magnitude of the Lorentz force is $|F_L| = m_iv_\perp\omega_i$ so that

$$|F_\rho|/|F_L| \cong (2/k_{\max}k_{\min}L_D^2)(v_i/v)^2 \sum_i (v_i/v)^2 \quad \text{for} \quad v \gg v_i$$

$$\cong (2/k_{\max}k_{\min}L_D^2) \sum_i (v_i/v_i)^2 \quad \text{for} \quad v \ll v_i.$$

Since $(1/k_{\max}k_{\min}L_D^2) \ll 1/k_{\max}L_D = g/4\pi$, and $g \ll 1$ is the basis of the entire calculation, F_ρ may be neglected. Similarly, it can be demonstrated that the components $T_{\rho\beta}$ and $T_{z\rho}$ are negligible.

The radial force has a simple physical interpretation. The polarized cloud around the test charge gives the test charge an effective mass. It can be estimated, by considering the energy of the test charge, which is $\frac{1}{2}m_iv^2 + q_i(\Phi - \Phi_0)$, where $\Phi - \Phi_0$ is the change in potential produced by the test charge at the location of the test charge. If the collision contribution is determined in the same sense that F_ρ was calculated, we may use Eq. (24) for $\Phi - \Phi_0$:

$$\Phi - \Phi_0 \cong \frac{4\pi q_i}{(2\pi)^3} L_D^2 \cdot \int d^3\mathbf{k}\left[\frac{1}{(kL_D)^2 + W} - \frac{1}{(kL_D)^2}\right]. \quad (56)$$

For a slow test charge, $\Phi - \Phi_0$ is proportional to v^2 so that the energy is $\frac{1}{2}\bar{m}_iv^2$, where

$$\bar{m}_i = m_i + \frac{1}{3\pi}\frac{q_i^2}{k_{\min}L_D^2} \sum_i (1/v_i^2).$$

The additional mass $\delta m_i = (q_i^2/3\pi k_{\min} L_D^2) \sum_i (1/v_i^2)$ may be regarded as an inertial mass due to the polarized cloud. If it follows the motion of the test charge, there must be a centripetal force acting on it of magnitude

$$F_\rho = \delta m_iv_\perp^2/a_i = (F_0v_\perp\omega_i/3\pi v^2k_{\min}) \sum_i (v/v_i)^2.$$

This force must be produced by a radial asymmetry of polarization, and therefore an equal force must act outward on the test charge. The result agrees with Eq. (55).

There are additional long-wavelength collision contributions when $L_D > \bar{a}_1$, as well as resonance contributions. For the remaining contributions we shall consider only the limiting cases

$$v_\perp = 0 \qquad v_z/v_i \ll 1 \qquad v_z/v_i \gg 1,$$
$$v_z = 0 \qquad v_\perp/v_i \ll 1 \qquad v_\perp/v_i \gg 1,$$

and calculate only $F_\parallel$. These calculations are of the nature of a survey where the objective is simply to see if there are any large effects due to the magnetic field. Description of these calculations will be limited to the more interesting terms. A complete description may be found elsewhere.[8] The results are listed in Tables II, III, and IV. For each term, the region of **k** space from which it came is indicated by a number that refers to Fig. 2.

The collision terms can be added together and expressed in the form

$$F_\parallel^{(i)} = -\frac{2}{3\pi^{1/2}} F_0 \sum_i (\ln k_{\max} b_{ij})(v/v_i) \qquad \text{for} \quad v \ll v_i$$
$$= -\tfrac{1}{2} F_0 \sum_j (\ln k_{\max} b_{ij})(v_j/v)^2 \qquad \text{for} \quad v \gg v_i. \qquad (57)$$

TABLE II. Force on an electron or ion test particle, $-F_\parallel/F_0$, when $v_\perp = 0$.

	$L_D \ll \bar{a}_1$	Region[a]	$L_D \gg \bar{a}_2$	Region[a]
$v_z \ll v_2$	$[2/3(\pi)^{1/2}](\ln k_m L_D) \sum_j (v_z/v_j)$	1	$[2/3(\pi)^{1/2}](\ln k_m \bar{a}_1) \sum_j (v_z/v_j)$	1
			$+ [1/2(2\pi)^{1/2}] \sum_j (\ln L_D/\bar{a}_j)(v_z/v_j)$	2, 3, 5, 6
			$+ [1/3(\pi)^{1/2}](\ln m_2/m_1)(v_z/v_2)$	2, 3, 6
$v_2 \ll v_z \ll v_1$	$[2/3(\pi)^{1/2}](\ln k_m L_D)(v_z/v_1)$	1	$[2/3(\pi)^{1/2}][\ln k_m \bar{a}_1)(v_z/v_1)$	1
	$+ \frac{1}{2}(\ln k_m L_D)(v_2/v_z)^2$	1	$+ \frac{1}{2}(\ln k_m \bar{a}_1)(v_2/v_z)^2$	1
			$+ [1/2(2\pi)^{1/2}](\ln L_D/\bar{a}_1)(v_z/v_1)$	5
			$+ \frac{1}{4}(\ln m_2/m_1)(v_2/v_z)^2$	2, 3, 6
$v_z \gg v_1$	$\frac{1}{2}(\ln k_m L_D) \sum_j (v_j/v_z)^2$	1	$\frac{1}{2}(\ln k_m \bar{a}_1) \sum_j (v_j/v_z)^2$	1
	$+ \frac{1}{2}(\ln \bar{a}_1/L_D)(v_1/v_z)^2$	1–7	$+ \frac{1}{4}(\ln m_2/m_1)(v_2/v_z)^2$	2, 3, 6

[a] See Fig. 2.

TABLE III. Force on an electron test particle, $-F_\parallel/F_0$, when $v_z = 0$.

	$L_D \ll \bar{a}_1$	Region[a]	$L_D \gg \bar{a}_2$	Region[a]
$v_\perp \ll v_2$	$(2/3\pi^{1/2})(\ln k_m L_D) \sum_j (v_\perp/v_j)$	1	$(2/3\pi^{1/2})(\ln k_m \bar{a}_1) \sum_j (v_\perp/v_j)$	1
			$+ (1/2\pi)(\ln L_D/\bar{a}_1)(v_\perp/v_1)$	4, 7
$v_2 \ll v_\perp \ll v_1$	$(2/3\pi^{1/2})(\ln k_m L_D)(v_\perp/v_1)$	1	$(2/3\pi^{1/2})(\ln k_m \bar{a}_1)(v_\perp/v_1)$	1
	$+ \frac{1}{2}(\ln k_m L_D)(v_2/v_\perp)^2$	1	$+ \frac{1}{2}(\ln k_m \bar{a}_1)(v_2/v_\perp)^2$	1
			$+ (1/3\pi^{1/2})(\ln L_D/\bar{a}_2)(v_2/v_\perp)^2$	5
			$+ (1/3\pi^{1/2})(\ln m_2/m_1)(v_2/v_\perp)^2$	2, 3, 6
			$+ (1/2\pi)(\ln L_D/\bar{a}_1)(v_\perp/v_1)$	4, 7
$v_\perp \gg v_1$	$\frac{1}{2}(\ln k_m L_D) \sum_j (v_j/v_\perp)^2$	1	$\frac{1}{2}(\ln k_m \bar{a}_1) \sum_j (v_j/v_\perp)^2$	1
	$+ (2/3\pi)(\ln L_D v_\perp/\bar{a}_1 v_1)(v_1/v_\perp)^2$	1–7	$+ (1/3\pi^{1/2}) \sum_j (\ln L_D/\bar{a}_j)(v_j/v_\perp)^2$	2, 3, 5, 6
			$+ (1/3\pi^{1/2})(\ln m_2/m_1)(v_2/v_\perp)^2$	2, 3, 6
			$+ (2/3\pi^2)(\ln L_D/\bar{a}_1)(v_1/v_\perp)^2$	4, 7

[a] See Fig. 2.

TABLE IV. Force on an ion test particle, $-F_\parallel/F_0$, when $v_z = 0$.

	$L_D \ll \bar{a}_1$	Region[a]	$L_D \gg \bar{a}_2$	Region[a]
$v_\perp \ll v_2$	$(2/3\pi^{1/2})(\ln k_m L_D) \sum_i (v_\perp/v_i)$	1	$(2/3\pi^{1/2})(\ln k_m \bar{a}_2) \sum_j (v_\perp/v_j)$	1–4
			$+ [1/2(2\pi)^{1/2}][\ln (m_2/m_1)^{1/2}]$	
			$(\ln L_D/\bar{a}_2)(v_\perp/v_1)$	5
			$+ (1/2\pi)(\ln L_D/\bar{a}_2)(v_\perp/v_2)$	6, 7
$v_2 \ll v_\perp \ll v_1$	$(2/3\pi^{1/2})(\ln k_m L_D)(v_\perp/v_1)$	1	$(2/3\pi^{1/2})(\ln k_m \bar{a}_2)(v_\perp/v_1)$	1–4
	$+ \frac{1}{2}(\ln k_m L_D)(v_2/v_\perp)^2$	1	$+ \frac{1}{2}(\ln k_m \bar{a}_2)(v_2/v_\perp)^2$	1–4
			$+ [1/2(2\pi)^{1/2}](\ln v_1/v_\perp)(\ln L_D/\bar{a}_2)(v_\perp/v_1)$	5
			$+ (2/3\pi^2)(\ln L_D/\bar{a}_2)(v_2/v_\perp)^2$	6, 7
$v_\perp \gg v_1$	$\frac{1}{2}(\ln k_m L_D) \sum_j (v_j/v_\perp)^2$	1	$\frac{1}{2}(\ln k_m \bar{a}_2) \sum_j (v_j/v_\perp)^2$	1–4
	$+ (2/3\pi)(\ln L_D v_\perp/\bar{a}_1 v_1)(v_1/v_\perp)^2$	1–7	$+ (1/3\pi^{1/2})(\ln L_D/\bar{a}_2) \sum_j (v_j/v_\perp)^2$	5
			$+ (2/3\pi^2)(\ln L_D/\bar{a}_2)(v_2/v_\perp)^2$	6, 7

[a] See Fig. 2.

When $L_D \ll \bar{a}_i$, $b_{ij} = L_D$. When $L_D \gg \bar{a}_i$ and $v_\perp = 0$,

$$b_{ij} \cong (L_D \bar{a}_j)^{\frac{1}{2}} \quad \text{if} \quad v_z \ll v_j,$$
$$b_{ij} = \bar{a}_j \quad \text{if} \quad v_z \gg v_j.$$

When $L_D \gg \bar{a}_j$ and $v_z = 0$,

$$b_{ij} \cong \begin{bmatrix} (\bar{a}_1 L_D)^{\frac{1}{2}}, & \bar{a}_1 \\ \bar{a}_2, & (\bar{a}_2 L_D)^{\frac{1}{2}} \end{bmatrix} \quad \text{if} \quad v_\perp \ll v_j,$$

$$b_{ij} \cong \begin{bmatrix} (\bar{a}_1 L_D)^{\frac{1}{2}}, & (\bar{a}_1 \bar{a}_2 L_D)^{\frac{1}{3}} \\ (\bar{a}_2{}^2 L_D)^{\frac{1}{3}}, & (\bar{a}_2 L_D)^{\frac{1}{2}} \end{bmatrix} \quad \text{if} \quad v_\perp \gg v_j.$$

In general, the maximum impact parameter, is something between the Debye length and a Larmor radius. The shield cloud of the test particle has a complex form so that it is not easy to see in an intuitive way what the particular value of b_{ij} should be. Equation (57) accounts for all collision terms except in the case of a slow ion moving perpendicular to the field. The terms

$$-F_\parallel / F_0 = \frac{1}{2(2\pi)^{\frac{1}{2}}} [\ln (m_2/m_1)^{\frac{1}{2}}] \cdot (\ln L_D/\bar{a}_2)(v_\perp/v_1) \quad \text{for} \quad v_\perp < v_2 \tag{58}$$

and

$$-F_\parallel / F_0 = \frac{1}{2(2\pi)^{\frac{1}{2}}} (\ln v_1/v_\perp)(\ln L_D/\bar{a}_2)(v_\perp/v_1) \quad \text{for} \quad v_2 \ll v_\perp \ll v_1 \tag{59}$$

are of a qualitatively different form so that the calculation will be described. These terms come from region 5, where $|k_z| \ll 1/\bar{a}_2$, $k_\perp \ll 1/\bar{a}_2$. Making use of Eqs. (41), (48), and (51) gives

$$F_\parallel = -(F_0/2\pi a_2) \sum_j \int_{(\text{region } 5)} k_\perp \, dk_\perp \int \frac{dk_z}{k^4} \cdot \sum_{m=-\infty}^{\infty} \frac{m^2 \omega_2}{|k_z| \, v_j} U_I\left(\frac{m\omega_2}{|k_z| \, v_j}\right) J_m{}^2(k_\perp a_2).$$

The function U_I has the property that

$$U_I \frac{m\omega_2}{|k_z| \, v_j} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{m\omega_2}{|k_z| \, \bar{a}_j \omega_j}\right)^2$$
$$\cong \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \quad \text{if} \quad |m| \frac{\omega_2}{\omega_j} < \sqrt{2} \, |k_z| \, \bar{a}_j$$
$$\cong 0 \quad \text{if} \quad |m| \frac{\omega_2}{\omega_j} > \sqrt{2} \, |k_z| \, \bar{a}_j.$$

Therefore if $v_\perp \ll v_2$ and $k_\perp a_2 \ll 1$,

$$S = \sum_m m^2 U_I \frac{m\omega_2}{|k_z| \, v_j} J_m{}^2(k_\perp a_2) \cong (\pi/2)^{\frac{1}{2}} (k_\perp a_2)^2/2$$

if $|k_z| \, \bar{a}_j > \omega_2/\sqrt{2}\omega_j$, and $S \cong 0$ if $|k_z| \, \bar{a}_j < \omega_2/\sqrt{2}\omega_j$. If $v_1 \gg v_\perp \gg v_2$ and $k_\perp a_2 \gg 1$, we can employ the asymptotic form of the Bessel functions $J_m{}^2 \, (k_\perp a_2) \cong (1/\pi k_\perp a_2)$ [1 + rapidly oscillating term]. Therefore

$$S \cong \frac{1}{(2\pi)^{\frac{1}{2}} k_\perp a_2} \sum_{-M}^{M} m^2.$$

If $\sqrt{2} \, |k_z| \, \bar{a}_j < (\omega_2/\omega_j) \, k_\perp a_2$,

$$M \cong \sqrt{2}\omega_j \, |k_z| \, \bar{a}_j/\omega_2.$$

The sum is terminated by the cutoff of the U_I function and

$$S \cong \frac{4}{3(\pi)^{\frac{1}{2}} k_\perp a_2} \left(|k_z| \, \bar{a}_j \frac{\omega_j}{\omega_2}\right)^3.$$

If $\sqrt{2} \, |k_z| \, \bar{a}_j > (\omega_2/\omega_j) k_\perp a_2$,

$$S \cong \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_m m^2 J_m{}^2(k_\perp a_2) = (\pi/2)^{\frac{1}{2}} (k_\perp a_2)^2/2.$$

When $v_\perp \ll v_2$, the ion-electron force dominates. The ion-ion force is negligible, because $S \cong 0$ for $|k_z| < 1/\sqrt{2}\bar{a}_2$, which includes the entire range of integration:

$$F_\parallel \cong -(F_0/2\pi a_2) \int k_\perp \, dk_\perp \cdot \int \frac{dk_z}{k^4} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(k_\perp a_2)^2}{2} \frac{\omega_2}{|k_z| \, v_1}$$
$$= -\frac{F_0}{2(2\pi)^{\frac{1}{2}}} \frac{v_\perp}{v_1} \int_{1/L_D}^{1/\bar{a}_2} \frac{dk}{k} \int_{\mu \sim (m_1/m_2)^{\frac{1}{2}}}^{1} d\mu \, \frac{(1-\mu^2)}{\mu}.$$

Thus Eq. (58) is obtained. If $v_2 \ll v_\perp \ll v_1$, the only change is that the lower limit of the μ integration is replaced by $\mu \cong v_\perp/v_1$.

The forces caused by plasma-wave emission are significantly modified by the magnetic field. If $L_D \gg \bar{a}_2$, they disappear entirely. If $L_D \ll \bar{a}_1$ and $v_\perp = 0$, the modification is the replacement of $\frac{1}{2} (\ln v_z/v_1) \, (v_1/v_z)^2$ by $(\frac{1}{2}) \, (\ln \bar{a}_1/L_D) \, (v_1/v_z)^2$. The reason is that in the presence of a magnetic field, the plasma-wave Cerenkov effect takes place only for wavelengths $1/\bar{a}_1 < k < 1/L_D$ instead of $0 < k < 1/L_D$. When there is no magnetic field, the leading term of the dispersion relation for plasma waves is $\omega(k) = \omega_p$. The phase velocity is ω_p/k. Plasma waves are emitted by the test charge mainly at the angle $\cos\theta = \omega_p/kv = (v_1/\sqrt{2} \, v_z k L_D)$. This leads to the usual Cerenkov shock front. In the presence of a magnetic field, the leading term of the dispersion relation is $\omega(k) = \omega_p$ for $1/\bar{a}_1 < k < 1/L_D$, and $\omega(k) = \omega_p \, |k_z|/k = \omega_p$, $\cos\theta$[9] for

[9] I. Bernstein, Phys. Rev. 109, 10 (1958).

$0 < k < 1/\bar{a}_1$. The resonance now takes place for $\cos\theta = \omega_p \cdot \cos\theta/kv$ or $\omega_p = kv$, independent of angle. This plasma-wave resonance of the second kind produces a negligible drag on a particle moving in the direction of the field. The effect is to replace the lower limit of integration in Eq. (34) by $k = 1/\bar{a}_1$.

For a particle moving perpendicular to the field, the plasma-wave drag

$$F_{\parallel} = -(2F_0/3\pi)\,(\ln L_D v_\perp/\bar{a}_1 v_1)(v_1/v_\perp)^2, \qquad (60)$$

is entirely due to plasma-wave resonance of the second kind. The calculation of this term will be described.

Consider the case of an ion test particle for which

$$F_{\parallel} = (F_0 L_D^4/\pi a_2)\int_{(kL_D<1)} k_\perp\, dk_\perp \int dk_z \cdot \sum_{m=-\infty}^{\infty} \frac{mJ_m^2(k_\perp a_2)W_I(\mathbf{k}, m)}{[(kL_D)^2 + W_R]^2 + W_I^2},$$

where $W(k, m) = W_R + iW_I$. In the domain $k < 1/\bar{a}_2$,

$$W_I \cong -\frac{1}{2}\frac{m}{|k_z|\,\bar{a}_2}\left(\frac{m_1}{m_2}\right)^{\frac{1}{2}} U_I\left[\frac{m}{|k_z|\,\bar{a}_2}\left(\frac{m_1}{m_2}\right)^{\frac{1}{2}}\right] - \tfrac{1}{2}\frac{m}{|k_z|\,\bar{a}_2}\, U_I\left(\frac{m}{|k_z|\,\bar{a}_2}\right).$$

If $|k_z| \ll (|m|/\bar{a}_2)\,(m_1/m_2)^{\frac{1}{2}}$, then $W_I \ll 1$, and the contribution from the mth term will be significant if $(kL_D)^2 + W_R = 0$, where

$$W_R = -\tfrac{1}{2}\left(\frac{k_z \bar{a}_2}{m}\right)^2 \frac{m_2}{m_1}.$$

Therefore, only terms for which $|k_z|\bar{a}_2\,(m_2/m_1)^{\frac{1}{2}} \ll |m| < (\bar{a}_2/L_D)\,(m_2/2m_1)^{\frac{1}{2}}$ can contribute. There will be a significant number of terms only if $L_D \ll \bar{a}_2$. If $k_z = k\mu$, the resonant condition is

$$(kL_D)^2 + W_R = k^2 L_D^2\left[1 - \tfrac{1}{2}\frac{m_2}{m_1}\left(\frac{\bar{a}_2}{L_D}\right)^2\left(\frac{\mu}{m}\right)^2\right] = 0,$$

or

$$|\mu| = \mu_0 = \left(\frac{2m_1}{m_2}\right)^{\frac{1}{2}}\frac{L_D}{\bar{a}_2}\,|m| = |m|\,\omega_2/\omega_p.$$

Resonances can take place for harmonics of the ion cyclotron frequency up to $|m| = \omega_p/\omega_2$. The contribution from each resonance is proportional to $J_m^2\,(k_\perp a_2)$, which becomes negligible for $|m| \gg k_\perp a_2$. Therefore, a significant result will be obtained only for a fast test particle such that $k_\perp a_2 > \omega_p/\omega_2$:

$$F_{\parallel} = (2F_0/\pi a_2)\int_0^{1/\bar{a}_2} k^2\, dk \int_0^1 d\mu \cdot \sum_m \frac{mJ_m^2[ka_2(1-\mu^2)^{\frac{1}{2}}]W_I}{\{k^4 L_D^4[1-(\mu/\mu_0)^2]^2 + W_I^2\}}.$$

The μ integration can be carried out by making use of the resonant character of the denominator:

$$F_{\parallel} \cong -2\left(\frac{2m_1}{m_2}\right)^{\frac{1}{2}}\frac{F_0 L_D^3}{a_2\bar{a}_2}\int_0^{1/\bar{a}_2} dk \sum_m m^2 J_m^2(ka_2).$$

If $a_2/\bar{a}_2 > (m_2/2m_1)^{\frac{1}{2}}\,(\bar{a}_2/L_D)$, then for

$$\{1/a_2[(m_2/2m_1)^{\frac{1}{2}}(\bar{a}_2/L_D)]\} < k < 1/\bar{a}_2,$$

$$\sum_m m^2 J_m^2(ka_2) \cong \frac{2}{\pi k a_2}\sum_{m=0}^{m\sim(m_2/2m_1)^{\frac{1}{2}}(\bar{a}_2/L_D)} m^2 = \frac{2}{3\pi k a_2}\,[(m_2/2m_1)^{\frac{1}{2}}(\bar{a}_2/L_D)]^3.$$

Therefore,

$$F_{\parallel} = -(2F_0/3\pi)(v_1/v_\perp)^2 \ln\,[a_2 L_D (m_2)^{\frac{1}{2}}/\bar{a}_2 a_2 (2m_1)^{\frac{1}{2}}].$$

This contribution, is from the domain $0 < k < 1/\bar{a}_2$. Similarly, the contribution from the domain $1/\bar{a}_2 < k < 1/\bar{a}_1$ is

$$F_{\parallel} = -(2F_0/3\pi)(v_1/v_\perp)^2 \ln\,(\bar{a}_2/\bar{a}_1).$$

By combining these two, we obtain the result of Eq. (60). In the domain $1/\bar{a}_1 < k < 1/L_D$,

$$W_R = -\frac{1}{2}\left(\frac{k\bar{a}_2}{m}\right)^2\frac{m_2}{m_1},$$

so that

$$(kL_D)^2 + W_R = k^2\left[L_D^2 - \tfrac{1}{2}\left(\frac{\bar{a}_2}{m}\right)^2\frac{m_2}{m_1}\right]$$

can vanish only if $|m| = [m_2/2m_1]^{\frac{1}{2}}\,(\bar{a}_2/L_D)$ happens to be an integer. The contribution from this domain is therefore negligible. The result for an electron test particle is the same as for an ion.

The results obtained with a magnetic field depend only logarithmically on the way in which **k** space was divided. $F_{\parallel}(\lambda)$ is a slowly varying function of $\lambda = L_D/\bar{a}$. If either of these conditions were violated, the calculation procedure would be of doubtful validity.

In a thermonuclear machine where the magnetic field is produced by currents within the plasma, $(L_D/\bar{a}_1)^2 = B^2/4\pi n m_1 c^2$, where $B^2/4\pi \cong n\Theta$. Therefore $(L_D/\bar{a}_1)^2 \cong \Theta/m_1c^2 < 1$ for $\Theta < 510$ kev. In a mirror machine where B is produced by external coils, it is possible to have $L_D > \bar{a}_2$ for sufficiently low density. This case may also arise in astrophysical problems. The less extreme case, $\bar{a}_1 < L_D < \bar{a}_2$, is more likely. We have not listed results for this case, but they can easily be obtained by applying the same methods.

If the cluster expansion is continued (i.e., $f_s = f_s^{(0)} + gf_s^{(1)} + g^2 f_s^{(2)} \cdots$), it is probable that $f_s^{(2)}$ will contain the factor ln g. This speculation was made some time ago by Bogoliubov[10] for a thermal equilibrium cluster expansion and appears in the recent results of Abe.[11] The expansion in terms of g is not a power series. However, for the kind of plasma that is of current interest for thermonuclear machines, g is very small. For example, if $\Theta = 100$ ev and $n = 10^{16}$ cm^{-3}, $g \sim 10^{-5}$ so that a calculation to order g^2 should give satisfactory quantitative results.

[10] N. Bogoliubov, J. Phys. (U.S.S.R.) 10, 257 (1946).
[11] R. Abe, Progr. Theoret. Phys. (Kyoto) 21, 475 (1959).

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