PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW SOLUTIONS #3 : STOCHASTIC CALCULUS

(1) Evaluate, for general α, the averages of the following stochastic integrals:

$$\int_0^t dW(s)\, W(s)\, s \quad , \quad \int_0^t dW(s)\, W^3(s)\, e^{-\lambda s} \quad , \quad \int_0^t dW(s)\, W^{2k+1}(s) \quad .$$

Solution:

We evaluate the general stochastic integral,

$$\begin{aligned} I_k[\phi] &= \int_0^t dW(s)\, W^{2k+1}(s)\, \phi(s) \\ &= \sum_{j=0}^{N-1} \Big[(1-\alpha)\, W_j^{2k+1}\, \phi_j + \alpha\, W_{j+1}^{2k+1}\, \phi_{j+1}\Big]\big(W_{j+1} - W_j\big) \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} \big\langle I_k[\phi]\big\rangle &= \alpha \sum_{j=0}^{N-1} \phi_{j+1}\Big(\big\langle W_{j+1}^{2k+2}\big\rangle - \big\langle W_j\, W_{j+1}^{2k+1}\big\rangle\Big) \\ &= \frac{(2k+2)!}{2^{k+1}(k+1)!} \cdot \alpha \int_0^t ds\; s^k\, \phi(s) \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \Big\langle \int_0^t dW(s)\, W(s)\, s \Big\rangle &= \tfrac{1}{2}\, \alpha t^2 \\ \Big\langle \int_0^t dW(s)\, W^3(s)\, e^{-\lambda s} \Big\rangle &= 3\alpha \int_0^t ds\; s\, e^{-\lambda s} = \frac{1}{\lambda^2}\big(1 - e^{-\lambda t}\big) + t\, e^{-\lambda t} \\ \Big\langle \int_0^t dW(s)\, W^{2k+1}(s) \Big\rangle &= \frac{(2k+2)!}{2^{k+1}(k+1)!} \cdot \frac{\alpha\, t^{k+1}}{k+1} \quad . \end{aligned}$$

Note that in the limit $\lambda \to 0$ the term in curly brackets on the RHS of the second integral yields $(\lambda t)^4/24 + \mathcal{O}(\lambda^5)$ after a cancellation of the first four terms in the Taylor expansion of $e^{\lambda t}$, hence the integral becomes $\frac{3}{4}\alpha\, t^4$ in this limit, which is correct.

(2) Derive Eqn. 3.105 of the lecture notes.

Solution:

Starting from

$$
\begin{aligned}
du &= -\beta\, u\, dt + \beta\, dW(t) \\
\frac{dz}{dt} &= i\nu\, z + i\lambda\, u(t)\, z\ ,
\end{aligned}
$$

we obtained the solution

$$
z(t) = z(0)\, \exp\left\{ i\nu t + \frac{i\lambda}{\beta}\, u(0)\, \big(1 - e^{-\beta t}\big) + i\lambda \int\limits_0^t dW(s)\, \Big(1 - e^{-\beta(t-s)}\Big) \right\} .
$$

We wish to compute the quantity $Y(s) = \lim_{t\to\infty} \big\langle z(t+s)\, z^*(t) \big\rangle$. We therefore have

$$
\big\langle z(t+s)\, z^*(t) \big\rangle = |z(0)|^2\, e^{i\nu s}\, \exp\left\{ \frac{i\lambda}{\beta}\, u(0)\, e^{-\beta t} \big(e^{-\beta s} - 1\big) \right\} \times
$$

$$
\exp\left\{ -\frac{\lambda^2}{2} \left\langle \left(\int\limits_0^t dW(\sigma)\, \Big[1 - e^{-\beta(t-\sigma)}\Big] - \int\limits_0^{t+s} dW(\sigma)\, \Big[1 - e^{-\beta(t+s-\sigma)}\Big] \right)^2 \right\rangle \right\}
$$

We now invoke the result of Eqn. 3.27,

$$
\left\langle \int\limits_0^t dW(s)\, F(s) \int\limits_0^{t'} dW(s')\, G(s') \right\rangle = \int\limits_0^{\tilde t} ds\, F(s)\, G(s)\ ,
$$

where $\tilde t = \min(t, t')$, to obtain

$$
\begin{aligned}
\left\langle \left(\int\limits_0^t dW(\sigma)\, \Big[1 - e^{-\beta(t-\sigma)}\Big] - \int\limits_0^{t+s} dW(\sigma)\, \Big[1 - e^{-\beta(t+s-\sigma)}\Big] \right)^2 \right\rangle &= t - \frac{2}{\beta}\big(1 - e^{-\beta t}\big) + \frac{1}{2\beta}\big(1 - e^{-2\beta t}\big) \\
&\quad + (t+s) - \frac{2}{\beta}\big(1 - e^{-\beta(t+s)}\big) + \frac{1}{2\beta}\big(1 - e^{-2\beta(t+s)}\big) \\
&\quad - 2t + \frac{2}{\beta}\big(1 - e^{-\beta t}\big) + \frac{2}{\beta}\big(1 - e^{-\beta t}\big)\, e^{-\beta s} - \frac{2}{2\beta}\big(1 - e^{-2\beta t}\big)\, e^{-\beta s} \\
&\underset{t\to\infty}{=} s - \frac{1}{\beta}\big(1 - e^{-\beta s}\big) \quad ,
\end{aligned}
$$

where we have assumed $s > 0$. For $s < 0$, it is clear that we must replace s with $|s|$. The final result is

$$
Y(s) = \lim_{t\to\infty} \big\langle z(t+s)\, z^*(t) \big\rangle = |z(0)|^2\, \exp\left\{ i\nu s - \tfrac{1}{2}\lambda^2\, |s| + \frac{\lambda^2}{2\beta}\, \Big(1 - e^{-\beta |s|}\Big) \right\} .
$$

(3) For the colored noise example in §3.5.3 of the notes, compute numerically $\hat{Y}(\omega)$ and plot your results as a function of $\omega - \nu$. Set $\lambda \equiv 1$ and plot your results for a representative set of different values of the parameter β.

Solution:

We may derive an expansion for $\hat{Y}(\omega)$ as follows. First, for convenience we set $|z(0)|^2 = 1$. Then we have

$$\begin{aligned} Y(s) &= \exp\left\{ i\nu s - \tfrac{1}{2}\lambda^2\,|s| + \frac{\lambda^2}{2\beta}\left(1 - e^{-\beta|s|}\right)\right\} \\ &= e^{i\nu s} e^{-\lambda^2|s|/2}\, e^{\lambda^2/2\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(-\frac{\lambda^2}{2\beta}\bigg)^n e^{-n\beta|s|} \end{aligned}$$

Taking the Fourier transform, we have

$$\hat{Y}(\omega) = e^{\lambda^2/2\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(-\frac{\lambda^2}{2\beta}\bigg)^n \frac{2\,(n\beta + \frac{1}{2}\lambda^2)}{(\omega-\nu)^2 + (n\beta + \frac{1}{2}\lambda^2)^2} \ .$$

Define the parameter $\varepsilon \equiv \lambda^2/2\beta$, and define rescaled frequencies $\bar{\omega} \equiv \omega/\beta$ and $\bar{\nu} \equiv \nu/\beta$. Then $\hat{Y}(\omega) = \beta^{-1}\hat{\mathcal{Y}}_\varepsilon(\delta)$, where $\delta = \bar{\omega} - \bar{\nu}$ and

$$\begin{aligned} \hat{\mathcal{Y}}_\varepsilon(\delta) &= 2\exp(\varepsilon) \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n}{n!}\, \frac{n+\varepsilon}{\delta^2 + (n+\varepsilon)^2} \\ &= 2\int_0^\infty \! d\tau\, \cos(\delta\tau)\, \exp\Big\{-\varepsilon\left(e^{-\tau} - 1 + \tau\right)\Big\} \quad . \end{aligned}$$

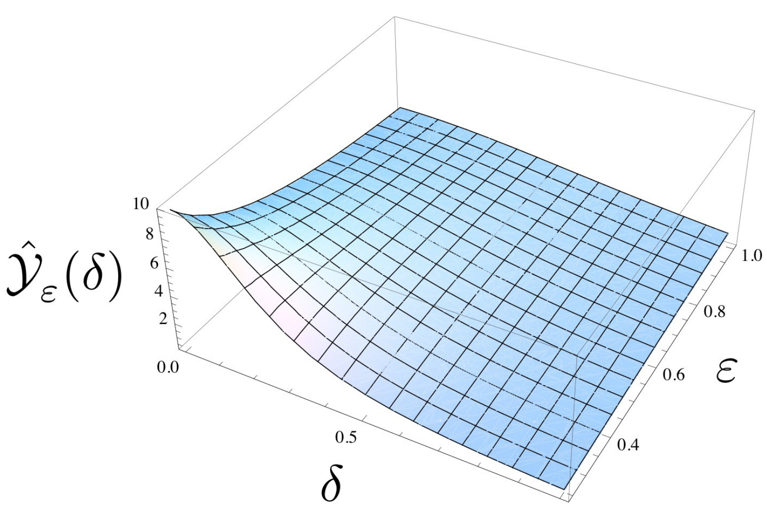

Figure 1: The integral $\hat{\mathcal{Y}}_\varepsilon(\delta)$ from problem (3).

Note that $f(\tau) = e^{-\tau} - 1 + \tau$ is nonnegative and monotonically increasing for $\tau \geq 0$, with $f(0) = 0$. For $\varepsilon \ll 1$, we can expand $f(\tau) = \frac{1}{2}\tau^2 + \mathcal{O}(\tau^3)$ and obtain

$$\hat{\mathcal{Y}}_\varepsilon(\delta) \simeq \sqrt{\frac{2\pi}{\varepsilon}}\, e^{-\delta^2/2\varepsilon} \qquad (\varepsilon \to 0) \ .$$

We evaluate numerically via `Mathematica`, *viz.*

```
Y[x_, a_] := NIntegrate[2 Cos[x * y] Exp[-a (Exp[-y] - 1 + y )], {y, 0, Infinity}]
Plot3D[Y[x, a], {x, 0, 1}, {a, 0.25, 1}, PlotRange → Full]
```

The resulting plot is shown in Fig. 1.

(4) Consider the following stochastic differential equation,

$$dx = -\beta x\, dt + \sqrt{2\beta(a^2 - x^2)}\, dW(t) \ ,$$

where $x \in [-a, a]$.

(i) Find the corresponding Fokker-Planck equation.

(ii) Find the normalized steady state probability $\mathcal{P}(x)$.

(iii) Find and solve for the eigenfunctions $P_n(x)$ and $Q_n(x)$. *Hint: learn a bit about Chebyshev polynomials.*

(iv) Find an expression for $\langle x^3(t)\, x^3(0)\rangle$, assuming $x_0 \equiv x(0)$ is distributed according to $\mathcal{P}(x_0)$.

Solution:

(a) From §3.3.4 of the notes, assuming the stochastic differential equation is in the Itô form (parameter α=0),

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(fP) + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}(g^2 P) \quad ,$$

with $f(x) = -\beta x$ and $g(x) = \sqrt{2\beta(a^2 - x^2)}$. Thus,

$$\frac{\partial P}{\partial t} = \beta\frac{\partial}{\partial x}(xP) + \beta\,\frac{\partial^2}{\partial x^2}\big[(a^2 - x^2)\, P\big] \quad .$$

At the boundaries $x = \pm a$ the diffusion constant vanishes, and the drift is into the interval, hence *the boundaries are reflecting*.

(b) We set the LHS of the FPE to zero to find the steady state solution. Assuming no currents at the boundaries, we have $P(x, t \to \infty) = \mathcal{P}(x)$, where the equilibrium distribution $\mathcal{P}(x)$ satisfies the first order equation

$$0 = x\,\mathcal{P} + \frac{d}{dx}\big[(a^2 - x^2)\,\mathcal{P}\big] \quad .$$

This may be rewritten as

$$\frac{d}{dx}\ln\big[(a^2-x^2)\,\mathcal{P}\big] = -\frac{x}{a^2-x^2} = \frac{d}{dx}\,\tfrac{1}{2}\ln(a^2-x^2) \quad ,$$

and therefore

$$\mathcal{P}(x) = \frac{1}{\pi}\,\frac{1}{\sqrt{a^2-x^2}} \quad ,$$

which is normalized with $\int\limits_{-a}^{a} dx\,\mathcal{P}(x) = 1$.

(c) The eigenfunctions $P_n(x)$ satisfy $\mathcal{L}\,P_n(x) = -\lambda_n P_n(x)$, with $Q_n(x) = P_n(x)/\mathcal{P}(x)$ satisfying $\mathcal{L}^\dagger Q_n = -\lambda_n Q_n$. It is useful to measure distances in units of a and times in units of β^{-1}. Then the FPE is $\partial_t P = \mathcal{L}\,P$, where our Fokker-Planck operator is

$$\mathcal{L} = \frac{d}{dx}\,x + \frac{d^2}{dx^2}\,(1-x^2) \quad .$$

The eigenfunctions $Q_n(x)$ satisfy $\mathcal{L}^\dagger Q_n = -\lambda_n\,Q_n$. Thus,

$$(1-x)^2\,\frac{d^2Q_n}{dx^2} - x\,\frac{dQ_n}{dx} = -\lambda_n Q_n \,.$$

This is Chebyshev's equation. The solution are the Chebyshev polynomials $T_n(x)$, and the eigenvalues are $\lambda_n = n^2$. The eigenfunctions $P_n(x)$ are given by $P_n(x) = \mathcal{P}(x)\,Q_n(x)$, with $\mathcal{P}(x) = \pi^{-1}(1-x^2)^{-1/2}$.

A good place to learn about Chebyshev polynomials is Wikipedia. The Chebyshev polynomials of the first kind are an orthonormal family of functions $\big\{T_n(x)\big\}$ on the interval $x\in[-1,1]$, satisfying the recurrence relation

$$T_0(x) = 1 \qquad , \qquad T_1(x) = x \qquad , \qquad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x) \quad .$$

They satisfy the differential equation

$$(1-x^2)\frac{d^2T_n}{dx^2} - x\,\frac{dT_n}{dx} + n^2\,T_n = 0 \quad .$$

There are several generating functions for the $\big\{T_n(x)\big\}$:

$$\begin{aligned}
\frac{1-tx}{1-2tx+t^2} &= \sum_{n=0}^\infty t^n\,T_n(x) \\
e^{tx}\cos\Big(t\sqrt{1-x^2}\Big) &= \sum_{n=0}^\infty \frac{t^n}{n!}\,T_n(x) \\
-\tfrac{1}{2}\ln\big(1-2tx+t^2\big) &= \sum_{n=1}^\infty \frac{t^n}{n}\,T_n(x) \quad .
\end{aligned}$$

The orthogonality relation is

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\,T_m(x)\,T_n(x)=\begin{cases}0 & \text{if } m\neq n\\ 1 & \text{if } m=n=0\\ \frac{1}{2} & \text{if } m=n\neq 0\end{cases}\quad .$$

The first few $T_n(x)$ are

$$\begin{aligned}
&T_0(x)=1 && T_6(x)=32x^6-48x^4+18x^2-1\\
&T_1(x)=x && T_7(x)=64x^7-112x^5+56x^3-7x\\
&T_2(x)=2x^2-1 && T_8(x)=128x^8-256x^6+160x^4-32x^2+1\\
&T_3(x)=4x^3-3x && T_9(x)=256x^9-576x^7+432x^5-120x^3+9x\\
&T_4(x)=8x^4-8x^2+1 && T_{10}(x)=512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1\\
&T_5(x)=16x^5-20x^3+5x && T_{11}(x)=1024x^{11}-2816x^9+2816x^7-1232x^5+220x^3-11x\quad .
\end{aligned}$$

The general solution of the Fokker-Planck equation is then

$$P(x,t)=\sum_{n=0}^{\infty}A_n\,\mathcal{P}(x)\,T_n(x)\,e^{-n^2t}\quad .$$

The coefficients A_n are obtained from initial data $P(x,0)$, *viz.*

$$A_0=\int\limits_{-1}^{1}dx\,P(x,0)\qquad,\qquad A_{n>0}=2\int\limits_{-1}^{1}dx\,P(x,0)\,T_n(x)\quad .$$

(d) From the conclusion of §4.2.4 of the notes, we have that

$$P(x,t\,|\,x_0,0)=\sum_n Q_n(x_0)\,P_n(x)\,e^{-\lambda_n t}\quad ,$$

where $P_0(x)=\mathcal{P}(x)$ and $P_{n>0}(x)=\sqrt{2}\,T_n(x)\,\mathcal{P}(x)$. Thus, assuming x_0 is distributed according to $\mathcal{P}(x_0)$,

$$\begin{aligned}
\big\langle x^3(t)\,x^3(0)\big\rangle&=\int\limits_{-1}^{1}dx_0\,\mathcal{P}(x_0)\,x_0^3\int\limits_{-1}^{1}dx\,P(x,t\,|\,x_0,0)\\
&=\sum_n\big|\big\langle\,x^3\,|\,P_n\,\big\rangle\big|^2\,e^{-n^2t}\quad ,
\end{aligned}$$

where

$$\big\langle\,x^3\,|\,P_n\,\big\rangle=\sqrt{2}\int\limits_{-1}^{1}dx\,\mathcal{P}(x)\,x^3\,T_n(x)=\tfrac{1}{\sqrt{2}}\Big(\tfrac{1}{4}\,\delta_{n,3}+\tfrac{3}{4}\,\delta_{n,1}\Big)\quad ,$$

since $x^3 = \frac{1}{4}\,T_3(x) + \frac{3}{4}\,T_1(x)$. Thus,

$$\big\langle x^3(t)\,x^3(0)\big\rangle = \tfrac{1}{32}\,e^{-3t} + \tfrac{9}{32}\,e^{-t} \quad .$$

Note that $\big\langle x^6(0)\big\rangle = \frac{5}{16}$, which agrees with the calculation

$$\begin{aligned}\big\langle x^6(0)\big\rangle &= \int\limits_{-1}^{1} dx_0\,\mathcal{P}(x_0)\,x_0^6 \\ &= \frac{1}{\pi}\int\limits_0^{\pi} d\theta\ \cos^6\!\theta = \frac{1}{2^6}\binom{6}{3} = \tfrac{5}{16} \quad .\end{aligned}$$