Problem 1

$$H = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{r} = H_{kin} + H_{pot} \text{ for 1 electron}$$

$$\varphi_{\bar{r}}(r) = \left(\frac{1}{\bar{r}^3\pi}\right)^{1/2} e^{-r/\bar{r}}$$. We have:

$$E_{kin}(\bar{r}) = \langle \varphi_{\bar{r}} | H_{kin} | \varphi_{\bar{r}} \rangle = \frac{\hbar^2}{2m_e\bar{r}^2}$$

$$E_{pot}(\bar{r}) = \langle \varphi_{\bar{r}} H_{pot} | \varphi_{\bar{r}} \rangle = -\frac{Ze^2}{\bar{r}}$$

$$E = E_{kin} + E_{pot} = \frac{\hbar^2}{2m_e\bar{r}^2} - \frac{Ze^2}{\bar{r}}$$. Minimizing with respect to $\bar{r}$

$$\frac{dE}{d\bar{r}} = -\frac{\hbar^2}{m_e\bar{r}^3} + \frac{Ze^2}{\bar{r}^2} = 0 \Rightarrow \bar{r} = \frac{\hbar^2}{m_e e^2}\frac{1}{Z} = \frac{a_0}{Z}$$

$$\text{So } \varphi = \left(\frac{Z^3}{a_0^3\pi}\right)^{1/2} e^{-Zr/a_0}$$

With 2 electrons, assume

$$\Psi(r_1, r_2) = \varphi_{\bar{r}}(r_1)\varphi_{\bar{r}}(r_2)$$

and find $\bar{r}$ that minimizes the total energy.

$$E_{total} = E_1 + E_2 + E_{ee} = \frac{\hbar^2}{m_e\bar{r}^2} - \frac{2Ze^2}{\bar{r}} + E_{ee}$$

The interaction: $$H_{ee} = \frac{e^2}{|r_1 - r_2|}$$

$$E_{ee} = \langle \Psi | H_{ee} | \Psi \rangle$$

$$E_{ee} = \int d^3r_1\, d^3r_2 \; |\varphi_{\bar{r}}(r_1)|^2 |\varphi_{\bar{r}}(r_2)|^2 \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$$

Doing this integral gives

$$E_{ee} = \frac{5}{8} \frac{e^2}{\bar{r}} \quad , \text{ so}$$

$$E_{total}(\bar{r}) = \frac{\hbar^2}{m_e \bar{r}^2} - \frac{2Z e^2}{\bar{r}} + \frac{5}{8}\frac{e^2}{\bar{r}}$$

$$\frac{d E_{total}}{d\bar{r}} = 0 = -\frac{2\hbar^2}{m_e \bar{r}^3} + \frac{2Z e^2}{\bar{r}^2} - \frac{5}{8}\frac{e^2}{\bar{r}^2} \Rightarrow$$

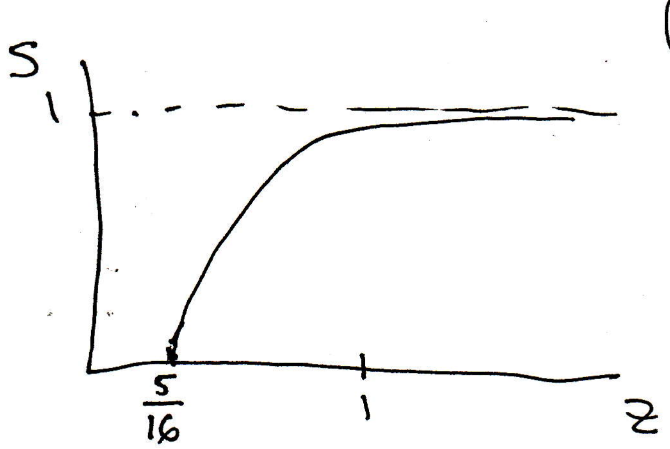
$$\Rightarrow \boxed{\bar{r} = a_0 / \left(Z - \frac{5}{16}\right)}$$

So $\bar{r} = \frac{a_0}{Z}$ for 1 electron, $\bar{r} = \frac{a_0}{Z - \frac{5}{16}}$ when there are 2 electrons

$\bar{r}$ is the most probable radius of the electron, i.e.
$P(r) = r^2 |\varphi_{\bar{r}}(r)|^2$ is maximum at $r = \bar{r}$.

The overlap

$$S \equiv \langle \varphi(r) \bar{\varphi}(r) \rangle = \frac{\left(1 - \frac{5}{16Z}\right)^{3/2}}{\left(1 - \frac{5}{32Z}\right)^3}$$



Problem 2

(a) $\sigma_1(\omega) = \dfrac{\sigma_0}{1+\omega^2\tau^2} \doteq \dfrac{ne^2}{m} \dfrac{\tau}{1+\omega^2\tau^2}$

$$\int_0^\infty d\omega\, \sigma_1(\omega) = \frac{ne^2}{m} \int_0^\infty d\omega \frac{\tau}{1+\omega^2\tau^2} = \frac{ne^2}{m} \int_0^\infty dx \frac{1}{1+x^2}$$

integrating in the complex plane, $\int_0^\infty dx \frac{1}{1+x^2} = \frac{\pi}{2} \Rightarrow$

$$\boxed{\int_0^\infty d\omega\, \sigma_1(\omega) = \frac{\pi n e^2}{2m}}$$

(b)

(i) $J(t) = \int_{-\infty}^{\infty} d\omega\, J(\omega) e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega\, \sigma(\omega) E(\omega) e^{-i\omega t} =$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \sigma(\omega) e^{-i\omega t} = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} d\omega\, \sigma_1(\omega) \cos\omega t + \int_{-\infty}^{\infty} d\omega\, \sigma_2(\omega) \sin\omega t \right\}$

because $\sigma_1(\omega) = \sigma_1(-\omega)$ and $\sigma_2(\omega) = -\sigma_2(-\omega)$

For $t<0$, $J(t) = 0$ by causality $\Rightarrow$ $\int_{-\infty}^{\infty} d\omega\, \sigma_1(\omega) \cos\omega t + \int_{-\infty}^{\infty} d\omega\, \sigma_2(\omega) \sin\omega t = 0$

for $t < 0$. Hence, for $t>0$, both integrals are same, hence

$$J(t>0) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, \sigma_1(\omega)\cos\omega t = \frac{2}{\pi}\int_0^\infty d\omega\, \sigma_1(\omega)\cos\omega t \quad (2)$$

(ii) The change in momentum of 1 particle is

$\Delta p = e\int dt\, E(t) = e \Rightarrow \Delta v = \frac{e}{m} \Rightarrow \Delta J = ne\Delta v = \frac{ne^2}{m}$

That is the current right after $t=0$, i.e. $J(t=0^+) = \frac{ne^2}{m}$, since there was no time for collisions, other forces to act, etc.

(iii) Comparing (ii) and Eq. (2), $\int_0^\infty d\omega\, \sigma_1(\omega) \underbrace{\cos(0^+)}_{1} = \frac{\pi n e^2}{2m}$

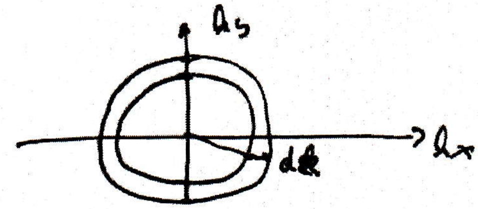
Problem 3

(a) With A = area of 2D system, $n = N/A$

$$\frac{A}{(2\pi)^2}\cdot 2\cdot \pi k_F^2 = N \Rightarrow n = \frac{2\pi}{(2\pi)^2} k_F^2 \Rightarrow$$

$$\Rightarrow \boxed{n = \frac{k_F^2}{2\pi}} \Rightarrow \boxed{k_F = (2\pi n)^{1/2}}$$

(b) $$\frac{A}{N} = \frac{1}{n} = \pi r_s^2 \Rightarrow \boxed{r_s = \frac{1}{(n\pi)^{1/2}}}$$

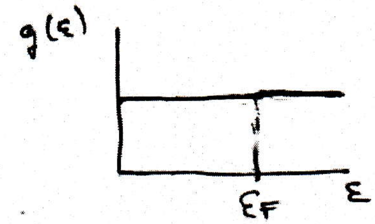


(c) Density of states:

$$g(\varepsilon)\,d\varepsilon = \frac{1}{\cancel{2}\pi\cancel{\pi}}\cdot \cancel{2}\cancel{\pi}\, k\,dk$$

$$g(\varepsilon) = \frac{1}{\pi}\, k\frac{dk}{d\varepsilon} = \frac{1}{2\pi}\frac{d}{d\varepsilon}k^2 \qquad \varepsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{dk^2}{d\varepsilon} = \frac{2m}{\hbar^2} \Rightarrow$$

$$\Rightarrow g(\varepsilon) = \frac{1}{2\pi}\cdot\frac{2m}{\hbar^2} \Rightarrow \boxed{g(\varepsilon) = \frac{m}{\pi\hbar^2}}$$



(d) Sommerfeld expansion:

$$n = \int_0^\mu d\varepsilon\, g(\varepsilon) + \sum_{n=1}^\infty \underbrace{\frac{d^{2n-1}}{d\varepsilon^{2n-1}} g(\varepsilon)\Big|_\mu}_{0 \text{ for } g(\varepsilon)=\text{constant}} (k_BT)^{2n} a_n$$

Since $n = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon) \Rightarrow \boxed{\mu = \varepsilon_F}$

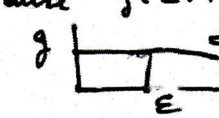
(e) $$n = \int d\varepsilon\, g(\varepsilon) f(\varepsilon) = \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon\, \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon\, \frac{e^{-\beta(\varepsilon-\mu)}}{e^{-\beta(\varepsilon-\mu)}+1} =$$

$$= \frac{m}{\pi\hbar^2}\, k_BT \ln\left(e^{-\beta(\varepsilon-\mu)}+1\right)\Big]_\infty^0 = \frac{m}{\pi\hbar^2}\, k_BT\ln\left(e^{\beta\mu}+1\right) = \frac{m}{\hbar^2\pi}\varepsilon_F \Rightarrow$$

For $T=0$; $$n = \frac{k_F^2}{2\pi} = \frac{1}{2\pi}\frac{\hbar^2 k_F^2}{2m}\frac{2m}{\hbar^2} = \frac{m}{\hbar^2\pi}\varepsilon_F \Rightarrow \varepsilon_F = k_BT\ln\left(e^{\beta\mu}+1\right)$$

$$\Rightarrow \boxed{\varepsilon_F = \mu + k_BT\ln\left(1+e^{-\mu/k_BT}\right)}$$

Sommerfeld expansion fails because $g(\varepsilon)$ is discontinuous at $\varepsilon=0$



$\Rightarrow$ Taylor expansion fails.

Problem 4

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \xrightarrow[\text{large } T]{} e^{-\beta(\varepsilon-\mu)}$$

$$n = \int d\varepsilon\, g(\varepsilon) f(\varepsilon) = \frac{3n}{2\varepsilon_F^{3/2}} \int d\varepsilon\, \varepsilon^{1/2} e^{-(\varepsilon-\mu)/kT} \Rightarrow$$

$$\Rightarrow 1 = \frac{3}{2\varepsilon_F^{3/2}} e^{\mu/k_BT} \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/k_BT}$$

let $\frac{\varepsilon}{k_BT} = x \Rightarrow d\varepsilon\, \varepsilon^{1/2} = (k_BT)^{3/2} x^{1/2} dx \Rightarrow$

$$\Rightarrow 1 = \frac{3}{2\varepsilon_F^{3/2}} e^{\mu/k_BT} (k_BT)^{3/2} \underbrace{\int_0^\infty dx\, x^{1/2} e^{-x}}_{I}$$

$$I = \int_0^\infty dx\, x^{1/2} e^{-x} = 2\int_0^\infty dy\, y^2 e^{-y^2} = \frac{\sqrt{\pi}}{2}$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}, \quad k_F = \frac{\left(\frac{9\pi}{4}\right)^{1/3}}{r_s}$$

Substituting,

$$\boxed{r_s = e^{-\mu/3k_BT}\, 3^{1/3}\, \pi^{1/6} \frac{\hbar}{(2mkT)^{1/2}}}$$

since $e^{-\mu/k_BT} >> 1 \Rightarrow$

$$r_s >> \left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}$$

The thermal de Broglie wavelength is

$\lambda_{th} = \frac{h}{p}$, where p is the momentum at temperature T

$$\frac{p^2}{2m} = \frac{3}{2}k_BT \Rightarrow p = (3mk_BT)^{1/2}$$

so $\lambda_{th} = \frac{h}{(3mk_BT)^{1/2}} = 2\pi\sqrt{\frac{2}{3}}\left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}$

so the condition above is $r_s >> \lambda_{th}$

i.e. interparticle average distance >> de Broglie wavelength, so wavefunctions don't overlap $\Rightarrow$ quantum statistics $\rightarrow$ classical statistics

(c) $a_0 = \frac{\hbar^2}{me^2} \Rightarrow \frac{r_s}{a_0} >> \left(\frac{\hbar^2}{2ma_0^2k_BT}\right)^{1/2}$

$$\frac{\hbar^2}{2ma_0^2k_B} = \frac{\cancel{\hbar^2}\cancel{m}e^2}{2\cancel{m}k_B\cancel{\hbar^2}a_0} = \frac{e^2}{2k_Ba_0} = \frac{14.4\times 11{,}600K}{2\times 0.529} = 1.6\times 10^5 K$$

$$\Rightarrow \boxed{\frac{r_s}{a_0} >> \left(\frac{10^5 K}{T}\right)^{1/2}}$$