Problem 1

$$g(\varepsilon) = \ln \frac{\varepsilon_0}{|\varepsilon|} \quad \text{for } -\varepsilon_0 < \varepsilon < \varepsilon_0$$

Pauli susceptibility:

$$\chi = \mu_B^2 \int d\varepsilon \, g(\varepsilon) \left(-\frac{df}{d\varepsilon}\right)$$

$$-\frac{df}{d\varepsilon} = \frac{1}{k_B T} \frac{e^{\beta\varepsilon}}{(e^{\beta\varepsilon}+1)^2}$$; changing variables to $\frac{\varepsilon}{k_B T} = x$

$$\chi = \mu_B^2 \int dx \, \ln\left(\frac{\varepsilon_0}{k_B T |x|}\right) \frac{e^x}{(e^x+1)^2} =$$ let $\varepsilon_0 \equiv k_B T_0$

$$= \mu_B^2 \Big[\underbrace{\ln\left(\frac{T_0}{T}\right)}_{\text{diverges as } T\to 0} \underbrace{\int dx \frac{e^x}{(e^x+1)^2}}_{\text{finite}} - \underbrace{\int dx \ln|x| \frac{e^x}{(e^x+1)^2}}_{\text{finite}} \Big]$$

So $\boxed{\chi \sim \ln \frac{T_0}{T}}$ diverges as $T \to 0$

Similarly specific heat:

$$u = \int d\varepsilon \, \varepsilon \ln \frac{\varepsilon_0}{|\varepsilon|} \frac{1}{e^{\beta\varepsilon}+1}$$; $\frac{\varepsilon}{k_B T} = x \Rightarrow$

$$u = \int dx \, (k_B T)^2 x \ln\left(\frac{T_0}{T|x|}\right) \frac{1}{e^x+1}$$

so at low temperatures the dominant term is

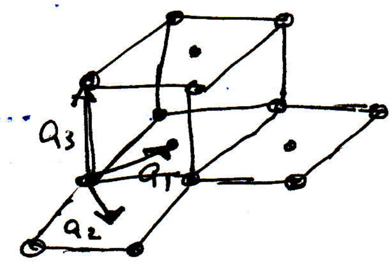
$$u \sim T^2 \ln T_0/T$$

$\Rightarrow \boxed{C_v \sim T \ln\left(\frac{T_0}{T}\right)}$ goes to 0 as $T \to 0$ but slower than $\sim T$

or, $\boxed{\gamma = \frac{C_v}{T} \text{ diverges as } T \to 0.}$

Problem 2: AM 4.1

(a) Base centered cubic



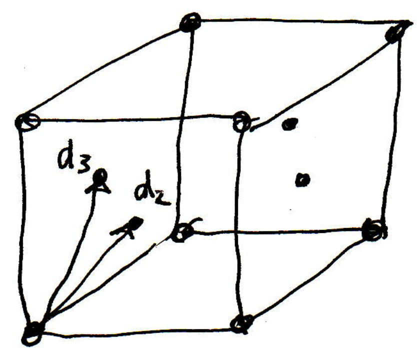
yes BL: simple tetragonal

$$\vec{a}_1 = \frac{a}{2}(\hat{x}+\hat{y})$$

$$\vec{a}_2 = \frac{a}{2}(\vec{x}-\hat{y})$$

$$\vec{a}_3 = a\hat{z}$$

(b) Side-centered cubic



not a Bravais lattice

BL is simple cubic

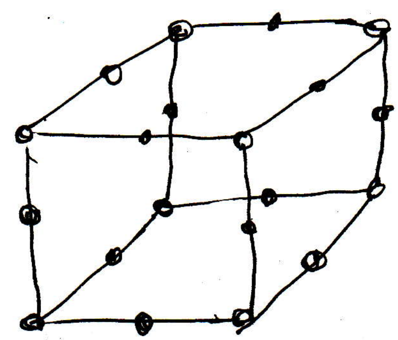
basis:

$$\vec{d}_1 = 0$$

$$\vec{d}_2 = \frac{a}{2}(\hat{x}+\hat{z})$$

$$\vec{d}_3 = \frac{a}{2}(\hat{y}+\hat{z})$$

(c) Edge-centered cubic



not a BL

BL is simple cubic

basis:

$$\vec{d}_1 = 0$$

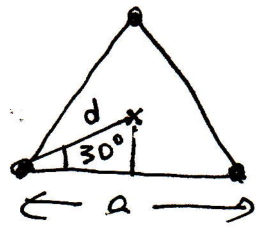
$$\vec{d}_2 = \frac{a}{2}\hat{x}$$

$$\vec{d}_3 = \frac{a}{2}\hat{y}$$

$$\vec{d}_4 = \frac{a}{2}\hat{z}$$

Problem 3 AM 4.5

(a)



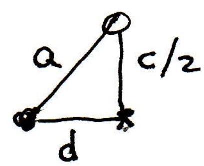
point x is equidistant from the vertices of triangle.

distance to a vertex = d

$$\cos 30° = \frac{a}{2d} = \frac{\sqrt{3}}{2} \Rightarrow d = \frac{a}{\sqrt{3}}$$

The point above x in the z direction is at distance $\frac{c}{2}$ from x

and distance a from the vertices:



$$\left(\frac{c}{2}\right)^2 + d^2 = a^2 \Rightarrow \frac{c^2}{4} + \frac{a^2}{3} = a^2 \Rightarrow \frac{c^2}{4} = \frac{2}{3} a^2 \Rightarrow \boxed{c = \sqrt{\frac{8}{3}}\, a}$$

(b) For hexagonal lattice

$$\vec{a}_1 = a\hat{x}, \quad \vec{a}_2 = \frac{a}{2}\hat{x} + \frac{\sqrt{3}}{2} a\hat{y}, \quad \vec{a}_3 = c\hat{z}$$

for close packed

volume of unit cell: $v = (\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3 = \frac{\sqrt{3}}{2} a^2 c = \sqrt{2}\, a^3$

there are 2 atoms in unit cell $\Rightarrow n = \frac{2}{\sqrt{2}\, a^3}$

For bcc, w/ spacing $a' = 4.23\,Å$, has 2 atoms in unit cell $\Rightarrow$

$\Rightarrow n = \frac{2}{a'^3}$; n is same for both $\Rightarrow \frac{2}{\sqrt{2}\, a^3} = \frac{2}{a'^3} \Rightarrow$

$$\Rightarrow \boxed{a = \frac{1}{6^{1/2}} a' = 3.77\,Å}$$

Problem 4 : AM 4.6

Let d be nearest neighbor distance, then radius of sphere R is $R = \frac{d}{2}$. Let n_e be # of atoms in conventional cubic unit cell. The packing fraction is

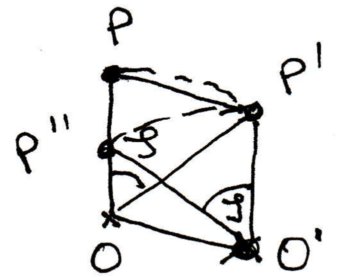
$$P = \frac{4}{3} \frac{\pi R^3 n_e}{a^3} = \frac{1}{6} \frac{\pi d^3}{a^3} n_e$$

	d	n_e	P
fcc	$a/\sqrt{2}$	4	$\frac{\pi}{3\sqrt{2}} = 0.740$
bcc	$\frac{\sqrt{3}}{2} a$	2	$\frac{\sqrt{3}}{8}\pi = 0.680$
sc	a	1	$\frac{\pi}{6} = 0.524$
diamond	$\frac{\sqrt{3}}{4}$	8	$\frac{\sqrt{3}\,\pi}{16} = 0.340$

Problem 5 :

Consider a rotation axis O. We can assume there exists a lattice plane perpendicular to O if $n \geq 3$ because: simply find a plane $\perp$ to O that contains a lattice point that is not on the axis, by rotating n times we generate n non-colinear points $\Rightarrow$ a plane.

Consider then the (n-fold) axis O perpendicular to the paper, and a (lattice) point P that is closer to O than any other point on the paper.



assume a = distance OP. Let $\varphi = 2\pi/n$

1) By rotation around O with angle φ, P goes into P', another lattice point.
2) translate O by (lattice) vector PP' to get point O'. O' is also an n-fold rotation axis.
3) A rotation around O' by angle $-\varphi$ will bring P' to a point P''
4) Use geometry to calculate the distance PP''. You should find $PP'' = 2a(1-\cos\varphi)$
5) Because we assumed P is closest to O than any other point, we must have either: (i) $PP'' = a$, $\Rightarrow \varphi = 60° \Rightarrow n = 6$ (here, $P'' = O$)
(ii) $PP'' \geq 2a$, $\Rightarrow$ $n = 4$ or $n = 3$

Finally, $n = 2$ is clearly possible. So the above shows that $n = 5$ and $n \geq 7$ is not.