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4

Nonlinear wave–particle interaction

流れにうかぶうたかたはかつきえかつむすびて久しくとどまりたるためしなし
Vortices on the flow either annihilate or emerge and have never stayed long.

(Kamo no Chomei, Hojoki)

4.1 Prologue and overview

In this chapter, and those which follow, we introduce and discuss *plasma turbulence theory*. A working theory of plasma turbulence is critical to our understanding of the saturation mechanisms and levels for plasma instabilities, and their associated turbulent transport. Quasi-linear theory alone is not sufficient for these purposes, since fluctuations and turbulence can saturate by coupling to other modes, and ultimately to dissipation, as well as by relaxing the mean distribution function. The energy flow in plasma turbulence is shown schematically in Figure 4.1. In nearly all cases of interest, linear instabilities, driven by externally pumped free energy reservoirs, grow and interact to produce a state of fluctuations and turbulence. As an aside, we note that while the word *turbulence* is used freely here, we emphasize that the state in question is frequently one of spatiotemporal chaos, wave turbulence or weak turbulence, all of which bear little resemblance to the familiar paradigm of high Reynolds number fluid turbulence, with its characteristically broad inertial range. Indeed, it is frequently not even possible to identify an inertial range in plasma turbulence, since sources and sinks are themselves distributed over a wide range of scales. Fundamentally there are two channels by which turbulence can evolve to a saturated state. These are by:

(i) quasi-linear relaxation of the distribution function or profile gradient associated with the free energy reservoir. In this channel $\langle f \rangle$, evolves toward a state where $\gamma_k \to 0$, for all modes k;
(ii) nonlinear interaction of unstable modes with other modes and ultimately with damped modes. In this channel, which resembles the well-known cascade in fluid turbulence,

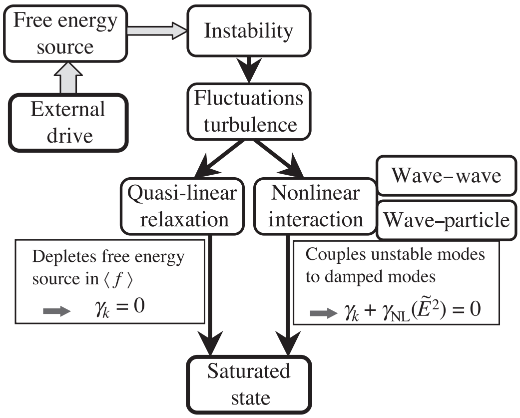

Fig. 4.1. Energy flow in plasma turbulence.

γ_k remains finite ($\gamma_k \neq 0$) but $\gamma_k^{L} + \gamma_k^{NL}\left(|E|^2\right) \rightarrow 0$, at each k, so that the *sum* of linear growth or damping and nonlinear transfer allows a finite free energy source and growth rate to be sustained against collisional or Landau damping.

Taken together, routes (i) and (ii) define the pathway from instability to a saturated state of plasma turbulence. Several caveats should be added here. First, the saturated state need not be absolutely stationary, but rather can be cyclic or in bursts, as long as the net intensity does not increase on average in the observed time. Second, the mechanism of nonlinear transfer can work either by wave–wave coupling or by the nonlinear scattering of waves on particles. These two mechanisms subdivide the "nonlinear interaction" channel into two sub-categories referred to, respectively, as "*nonlinear wave–wave interaction*" and "*nonlinear wave–particle interaction*". These two sub-categories for energy transfer are the subjects of this chapter (wave–particle) and the next (wave–wave), and together define the topic of wave turbulence in plasma. The ideas and material of these chapters constitute the essential foundations of the subject of plasma turbulence theory. Finally, we should add that in nearly all cases of practical interest, quasi-linear relaxation co-exists with a variety of nonlinear interaction processes (wave–wave, wave–particle, etc.). Only in rare cases does a single process or transfer channel dominate all the others.

The quasi-linear theory and its structure have already been discussed. The major elements of plasma turbulence theory which we must address are the theory of nonlinear wave–particle and wave–wave interactions. The general structure of plasma turbulence theory is shown schematically in Figure 4.2. Both for particles (via the evolution of f_k) or waves (via the evolution of N, the wave population density), a paradigmatic goal is to derive and understand the physics of the turbulent collision operators C_k^{P} and C_k^{W}. In practice, these two operators are usually strongly coupled. Examples of C_k^{P} include:

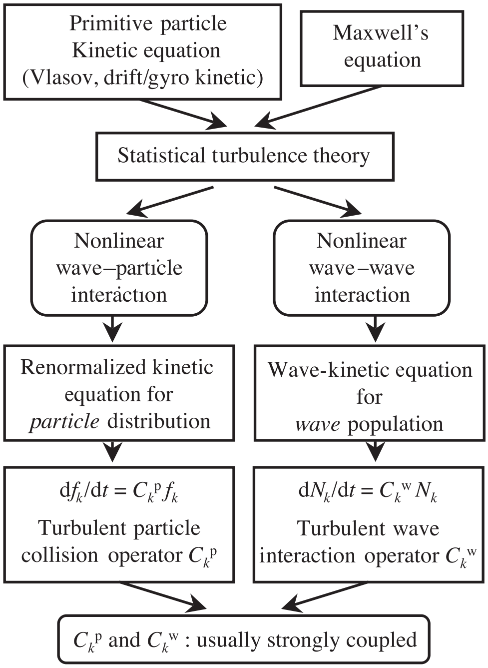

Fig. 4.2. General structure of plasma turbulence theory.

(i) the quasi-linear operator for $\langle f \rangle$ relaxation, discussed in Chapter 3;
(ii) the Balescu–Lenard collision integral, describing the relaxation of $\langle f \rangle$ in a stable plasma near equilibrium, discussed in Chapter 2;
(iii) the Vlasov propagator renormalization,

$$\frac{q}{m}\tilde{E}\frac{\partial \tilde{f}}{\partial v} \rightarrow -\frac{\partial}{\partial v}D_{k,w}\frac{\partial f_{k,\omega}}{\partial v},$$

where $D_{k,\omega} = D\left\{\left|E_{k',\omega'}\right|^2, \tau^{\rm ac}_{k',\omega'}\right\}$. Here the operator $-\partial/\partial v\, D_{k,\omega}\, \partial/\partial v$ may be viewed as similar to an "eddy viscosity" for Vlasov turbulence, and describes particle scattering by a spectrum of fluctuating electrostatic modes;
(iv) the scattering operator for nonlinear Landau damping $C_k^{\rm P} f_k \sim N f_k$, which is due to the class of interactions described schematically in Figure 4.3(a). In this case, a nonlinearly generated *beat* or *virtual* mode resonates with particles with their velocity equal to its phase velocity. Mechanisms (iii) and (iv) are discussed in this chapter.

Nonlinear wave–wave interaction processes result from resonant coupling, which is schematically depicted in Figure 4.3(b). The wave–wave collision operator has the generic form $C_k^W N \sim NN$, so the spectral evolution equation takes the form,

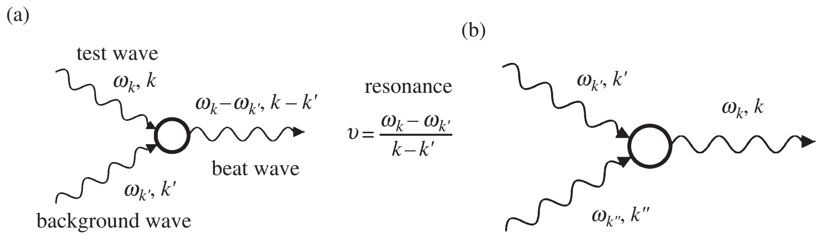

Fig. 4.3. Nonlinear wave–wave interaction process. (a) Nonlinear Landau resonance interaction with beat waves. (b) Nonlinear wave–wave coupling of three resonant modes, where $\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'' = 0$ and $\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'} + \omega_{\boldsymbol{k}''} = 0$.

$$\frac{\partial |E_{\boldsymbol{k}}|^2}{\partial t} - \gamma_{\boldsymbol{k}} |E_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k}'} C_1 \left(\boldsymbol{k}, \boldsymbol{k}'\right) |E_{\boldsymbol{k}'}|^2 |E_{\boldsymbol{k}}|^2 = \sum_{\substack{\boldsymbol{k}', \boldsymbol{k}'' \\ \boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{k}}} C_2 \left(\boldsymbol{k}', \boldsymbol{k}''\right) |E_{\boldsymbol{k}'}|^2 |E_{\boldsymbol{k}''}|^2.$$

Here, the wave kinetics is akin to that of a *birth and death process*. Usually, the incoherent emission term on the right-hand side (so named because it is *not* proportional to $|E_k|^2$) corresponds to *birth*, while the coherent mode coupling term i.e. the third term on the left-hand side (so named because it is proportional to, and coherent with, $|E_{\boldsymbol{k}}|^2$) corresponds to *death* or nonlinear damping. The competition between these two defines the process of nonlinear wave energy transfer, i.e. nonlinear cascade. Generally, $\sum_k$ (incoherent) $=$ $\sum_k$ (coherent), confirming that energy is conserved in the couplings. For weak turbulence, the three-wave resonance function $R_{k,k',k''}$ has negligible width, so Re $R_{k,k',k''} = \pi\delta\left(\omega_k - \omega_{k'} - \omega_{k''}\right)$, while for strong turbulence, $R_{k,k',k''}$ is broadened, and has the form $R_{k,k',k''} = i \big/ \left\{ \left(\omega_k - \omega_{k'} - \omega_{k''}\right) + i \left(\Delta\omega_k + \Delta\omega_{k'} + \Delta\omega_{k''}\right) \right\}$. The width of $R_{k,k',k''}$ is due to the effects of nonlinear scrambling on the coherence of the three interacting modes. The subject of nonlinear wave–wave interaction is discussed at length in Chapters 4 and 5.

4.2 Resonance broadening theory

4.2.1 Approach via resonance broadening theory

We begin our discussion of nonlinear wave–particle interaction by presenting the theory of resonance broadening (Dupree, 1966). Recall that quasi-linear theory answers the question, "How does $\langle f \rangle$ evolve in the presence of a spectrum of waves, given that the particle orbits are stochastic?". Continuing in that vein, resonance broadening theory answers the question, "How does the plasma distribution

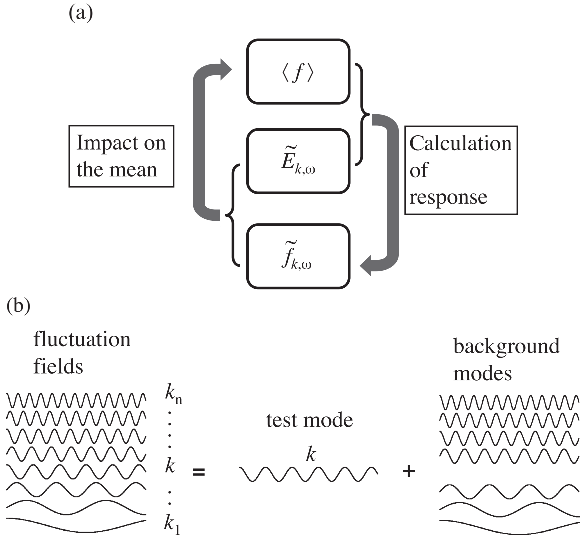

Fig. 4.4. Calculation of the plasma response for given electric field perturbation, and calculation of the evolution of the mean (a). Coupling with test wave and background spectrum is given in (b), depicting the test wave approximation.

function f respond to a test wave $E_{k,\omega}$ at (k, ω), given an existing spectrum of background waves?". This situation is depicted in Figure 4.4.

4.2.1.1 Basic assumptions

It cannot be over-emphasized that *resonance broadening theory rests upon two fundamental assumptions*. First the particle orbits are assumed to be stochastic, so excursions from unperturbed orbits may be treated as a diffusion process. Thus, resonance broadening theory (RBT) is valid only in regions of phase space where the islands around phase space resonances overlap. Resonance broadening theory also tacitly assumes the convergence of the second moment of the scattering step pdf (probability density function) i.e.

$$\langle (\Delta v)^2 \, P \rangle < \infty,$$

for velocity step Δv with pdf P. This allows the scattering to be treated as a diffusion process, via the central limit theorem. Second, the "test wave" approximation is assumed to be valid. The test wave approximation, which appears in various forms in virtually *every* statistical theory of turbulence, envisions the ensemble of interacting modes to be sufficiently large and statistically homogeneous so that *any* one mode may be removed from the ensemble and treated as a *test* wave, without altering the physics of the ensemble of remaining modes. The test wave hypothesis is thus not applicable to problems involving coherent mode coupling, nor to problems involving a few large amplitude or coherent modes interacting with a stochastic bath. In practice, validity of the test wave approximation almost always requires that the number of waves in the ensemble be large, and that the spectral auto-correlation time be short.

4.2.1.2 Ensemble and path integral

The essential idea of resonance broadening theory resembles that of the Weiner–Feynman path integral, in that the formal solution of the Vlasov equation, which can be written as an integration over the time history of the exact ("perturbed") orbit, is replaced by an average over a statistical ensemble of excursions from the linear or, "unperturbed", orbit. This concept is shown schematically in Figure 4.5. To implement this, it is useful to recall that the Vlasov response f_k to an electric field fluctuation E_k may be written as an integration over orbits. So starting from,

$$\frac{\mathrm{d}f_k}{\mathrm{d}t} = -\frac{q}{m}E_k\frac{\partial\langle f\rangle}{\partial v}, \tag{4.1a}$$

when $\mathrm{d}/\mathrm{d}t$ is determined by the characteristic equations (of the Vlasov equation),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m}E, \tag{4.1b}$$

which are also the equations of particle motion, we can write,

$$f_{k,\omega} = -\frac{q}{m}\mathrm{e}^{-ikx}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{ik\tau}u(-\tau)\left[\mathrm{e}^{ikx}E_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}\right], \tag{4.2a}$$

where $u(-\tau)$ is the formal, exact orbit propagator, which has the property that,

$$u(-\tau)\mathrm{e}^{ikx} = \mathrm{e}^{ikx(-\tau)}. \tag{4.2b}$$

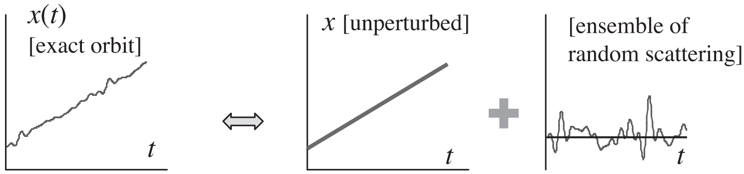

Fig. 4.5. Schematic illustration for decomposition of the particle orbit in resonance broadening theory.

Here, $x(-\tau)$ is the full, perturbed orbit. We can again formally decompose $x(-\tau)$ into the unperturbed piece and a fluctuation around it, as,

$$x(-\tau) = x_0(-\tau) + \delta x(-\tau). \tag{4.2c}$$

Along the unperturbed orbit $x_0(-\tau)$, in this case, $x_0(-\tau) = x - v\tau$. This gives,

$$f_{k,\omega} = -\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{i(\omega-kv)\tau} \mathrm{e}^{ik\delta x(-\tau)} \frac{q}{m} E_{k,\omega} \frac{\partial \langle f \rangle}{\partial v}. \tag{4.3}$$

Note that here we have assumed that the time scale for orbit scattering τ_s is short compared to the time scale upon which $\langle f \rangle$ varies (i.e. $\tau_\mathrm{s} < \tau_\mathrm{relax}$, where τ_relax is the quasi-linear relaxation time in Table 3.1), so that $\langle f \rangle$ may be treated as constant.

4.2.1.3 Introduction of approximation

So far all the calculations have been purely formal. We now come to the critical substantive step of resonance broadening theory, which is to *approximate* $f_{k,\omega}$ by its average over a statistical ensemble of orbit perturbations, i.e. to take $f_{k,\omega} \to \langle f_{k,\omega} \rangle_\mathrm{OE}$, where

$$\langle f_{k,\omega} \rangle_\mathrm{OE} = -\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{i(\omega-kv)\tau} \left\langle \mathrm{e}^{ik\delta x(-\tau)} \right\rangle_\mathrm{OE} \frac{q}{m} E_{k,\omega} \frac{\partial \langle f \rangle}{\partial v}. \tag{4.4}$$

Here the bracket $\langle \quad \rangle_\mathrm{OE}$ signifies an average over an ensemble of orbits. Note that by employing this ansatz, the orbit perturbation factor appearing in the response time history, i.e. $\exp[ik\delta x(-\tau)]$, which we don't know, is replaced by its ensemble average, which we *can* calculate, by exploiting an assumption concerning the pdf of δx. The approximation, which is used in quasi-linear theory, corresponds to,

$$\left\langle \mathrm{e}^{i\omega\tau - ikx(\tau)} \right\rangle_{QL} \simeq \exp \langle i\omega\tau - ikx(\tau) \rangle = \exp(i\omega\tau - ikv\tau).$$

The mean field theory is employed in evaluating the quasi-linear response.

To calculate $\langle \exp[ik\delta x(-\tau)] \rangle_\mathrm{OE}$ in RBT, we first note that,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = v_0 + \delta v, \tag{4.5a}$$

so

$$\delta x(-\tau) = -\int_0^\tau \mathrm{d}\tau'\, \delta v\left(-\tau'\right), \tag{4.5b}$$

and we find,

$$\langle \exp[ik\delta x(-\tau)] \rangle_{\text{OE}} = \left\langle \exp\left[-ik \int_0^\tau \delta v(-\tau')\mathrm{d}\tau' \right] \right\rangle_{\text{OE}}. \tag{4.5c}$$

Thus the problem has now been reduced to calculating the expectation value in Eq.(4.5c). Now, excursions in velocity from the unperturbed orbit are produced by the fluctuating electric fields of the turbulent wave ensemble, i.e. $\mathrm{d}\delta v/\mathrm{d}t = q\tilde{E}/m$. Consistent with the test wave hypothesis of a statistically homogeneous ensemble of weakly correlated fluctuation, we *assume* a Gaussian pdf of $\tilde{E}$, so that δv behaves diffusively, i.e.

$$\text{pdf}[\delta v] = \frac{1}{\sqrt{\pi D\tau}} \exp\left[-\delta v^2/D\tau \right], \tag{4.6a}$$

and the expectation value of $A(\delta v)$ is just,

$$\langle A \rangle_{\text{OE}} = \int \frac{\mathrm{d}\,\delta v}{\sqrt{\pi D\tau}} \exp\left[-\delta v^2/D\tau \right] A. \tag{4.6b}$$

Here D is the velocity diffusion coefficient which, like the quasi-linear D, characterizes stochastic scattering of particles by the wave ensemble. In practice, D has the same structure as does the quasi-linear diffusion coefficient. It cannot be over emphasized that the Gaussian statistics of $\tilde{E}$ and the diffusive pdf of δv are *input by assumption*, only. While Gaussian statistics, etc., are often characteristic of nonlinear systems with large numbers of interacting degrees of freedom, there is no a-priori guarantee this will be the case. One well-known example of a dramatic departure from Gaussian behaviour is the Kuramoto transition, in which the phases of an ensemble of N (for $N \gg 1$) strongly coupled oscillators synchronize for coupling parameters above some critical strength (Kuramoto, 1984). Another is the plethora of findings of super-diffusive or sub-diffusive scalings in various studies of the transport of test particles in a turbulent flow. Such non-diffusive behaviours (Yoshizawa *et al.*, 2004), which demand more advanced methods, like SOC models (Bak *et al.*, 1987; Dendy and Helander, 1997; Carreras *et al.*, 1998) and fractional kinetic (Podlubny, 1998; del Castillo-Negrete *et al.*, 2004; Zaslavsky, 2005; Sanchez, 2005), are frequently, but not always associated with the presence of structures in the flow. In spite of these caveats, the Gaussian diffusive assumption is a logical starting point. Furthermore, it is quite plausible that resonance broadening theory can be generalized to treat the fractional kinetics of orbit perturbations, and

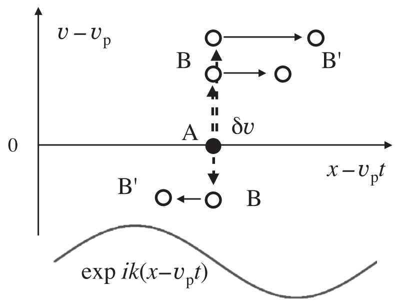

Fig. 4.6. Accelerated decorrelation of resonant particles via diffusion in the velocity space. The deviation of the particle velocity from the phase velocity of the wave enhances the rate of phase change, $\omega - kv$.

so encompass non-diffusive particle motion. Proceeding then, the orbit averaged response factor is given by,

$$\left\langle \exp\left[-ik\int_0^{\tau} \mathrm{d}\tau'\,\delta v\left(-\tau'\right)\right]\right\rangle_{\mathrm{OE}} = \exp\left[-\frac{k^2 D\tau^3}{6}\right]. \tag{4.7a}$$

The scaling $\langle\delta x^2\rangle \sim D\tau^3$ is a consequence of the fact that velocity, not position, is directly scattered by electric field fluctuations, so linear streaming can couple to the random walk in velocity to enhance decorrelation. This is shown schematically in Figure 4.6.

4.2.1.4 *Response function and decorrelation rate*

Using Eq.(4.6a), the RBT approximation to the response function is then just,

$$f_{k,\omega} = -\int_0^{\infty} \mathrm{d}\tau\ \exp\left[i(\omega - kv)\tau - \frac{k^2 D\tau^3}{6}\right]\frac{q}{m}E_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}. \tag{4.7b}$$

If we define the wave–particle correlation time τ_{c} according to the width in time of the kernel of this response, we have,

$$\frac{1}{\tau_{\mathrm{c}}} = \left(\frac{k^2 D}{6}\right)^{1/3}, \tag{4.7c}$$

so that,

$$f_{k,\omega} = -\int_0^{\infty} \mathrm{d}\tau\ \exp\left[i(\omega - kv)\tau - \frac{\tau^3}{\tau_{\mathrm{c}}^3}\right]\frac{q}{m}E_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}. \tag{4.7d}$$

Of course, the origin of the name "resonance broadening theory" is now clear, since the effect of scattering by the turbulent spectrum of background waves is to broaden the linear wave–particle resonance, from a delta function of zero width in linear theory to a function of finite width proportional to $1/\tau_c$. In this regard, it is often useful to approximate the result of Eq.(4.7d) by a Lorentzian of width $1/\tau_c$, so,

$$f_{k,\omega} = -\frac{i}{(\omega - kv + i/\tau_c)}\frac{q}{m}E_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}. \tag{4.7e}$$

The principal result of RBT is the identification of $1/\tau_c$ as given by Eq.(4.7c), as the wave–particle decorrelation rate. This is the rate (inverse time) at which a resonant particle scatters a distance of one wavelength ($\lambda = 2\pi/k$) relative to the test wave, and so defines the individual coherence time of a resonant particle with a specific test wave. Of course, τ_c corresponds to τ_s, the particle scattering time referred to earlier but not defined. Since $1/\tau_c$ defines the width of the resonance in time, it also determines a width in velocity, i.e.

$$\frac{1/\tau_c}{(\omega - kv)^2 + 1/\tau_c^2} = \frac{1/\tau_c}{\left\{(\omega/k - v)^2 + 1/k^2\tau_c^2\right\}\mathrm{k}^2},$$

so

$$\Delta v_T = \frac{1}{k\tau_c} = \left(\frac{D}{6k}\right)^{1/3} \tag{4.8}$$

is the width in velocity of the broadened resonance. Together, $\Delta x \sim k^{-1}$ and Δv_T define the fundamental scales of an element or *chunk* of turbulent phase space fluid. This fluid element is the analogue for Vlasov turbulence of eddy paradigm, familiar from ordinary fluid turbulence. Note that in contrast to the eddy, with spatial scale independent of amplitude, the velocity scale of a turbulent Vlasov fluid element varies with turbulence intensity via its dependence upon D. In this regard then, $1/\tau_{ck}$ may be viewed as the analogue of the eddy turn-over or decay rate $\Delta\omega_k \sim k\tilde{v}_k$.

To better understand the dependencies and scalings of the RBT parameters τ_c, Δv, etc, and to place these new scales in the context of the fundamental time scales which we encountered in Chapter 3, it is instructive to re-visit the calculation of the quasi-linear diffusion coefficient D_{QL}, now employing the response f_k calculation via RBT. Thus,

$$D = \mathrm{Re}\left[\frac{q^2}{m^2}\sum_k |E_k|^2 \frac{i}{\omega - kv + i/\tau_{ck}}\right]; \tag{4.9a}$$

so employing the Lorentzian spectrum as before,

$$D = \mathrm{Re}\left[\frac{q^2}{m^2}\sum_k \frac{\left|\tilde{E}\right|^2 / \Delta k}{1 + \{(k - k_0)/\Delta k\}^2}\frac{i}{\omega - kv + i/\tau_{ck}}\right], \tag{4.9b}$$

and performing the spectral summation by contour integration gives,

$$D = \frac{q^2}{m^2}\left|\tilde{E}\right|^2 \mathrm{Re}\left\{i/\left(\omega_{k_0} - k_0 v + i\,|\Delta k|\left|v_{\mathrm{gr}} - v\right| + i/\tau_{ck}\right)\right\}. \tag{4.9c}$$

Taking $v = \omega_{k_0}/k_0$ at resonance, we see that D reduces to its quasi-linear antecedent ($D \to D_{\mathrm{QL}}$) for $\left|\Delta k\left(v_{\mathrm{gr}} - v_{\mathrm{ph}}\right)_k\right| > 1/\tau_{ck}$. Thus quasi-linear diffusion is recovered for $\tau_{\mathrm{ac}} < \tau_{\mathrm{c}}$.

4.2.2 Application to various decorrelation processes

4.2.2.1 Scattering in action variable

The structure of the above calculation and the resulting super-diffusive decorrelation are straightforward consequences of the fact that the action variable is scattered (i.e. particle velocity $v \to v + \delta v$), while the response decorrelation is measured by the excursion of the associated angle variable (i.e. $x = \int v \mathrm{d}\tau \to x + \delta x$). Moreover drag, a key component in Brownian dynamics, is absent. Thus, we have,

$$\phi \sim \int \omega(J)\,\mathrm{d}\tau, \tag{4.10a}$$

so the scattering induced excursion is,

$$\delta\phi \sim \int \frac{\partial\omega}{\partial J}\delta J\,\mathrm{d}\tau, \tag{4.10b}$$

and

$$\langle\delta\phi\rangle \sim \left(\frac{\partial\omega}{\partial J}\right)^2 D_J \tau^3, \tag{4.10c}$$

so the mean square excursion of the angle variable grows in time $\sim \tau^3$, not $\sim \tau$.

Two particularly important examples of this type of decorrelation process are concerned with:

(a) the decorrelation of an electron by radial scattering in a sheared magnetic field (Fig. 4.7);
(b) the decorrelation of a particle or fluid element by radial scattering in a sheared flow, (Fig. 4.9).

4.2.2.2 Decorrelation in a sheared magnetic field

Regarding electron scattering in the configuration of Figure 4.7, consideration of streaming in the poloidal direction gives,

$$r\frac{\mathrm{d}\theta}{\mathrm{d}t} = v_{\parallel}\frac{B_{\theta}}{B_{\mathrm{T}}} \tag{4.11a}$$

(θ: poloidal angle), so

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v_{\parallel}}{Rq\,(r)}. \tag{4.11b}$$

With radial scattering δr, the change in the poloidal angle follows

$$\begin{aligned}\delta\theta &\sim \int \frac{v_{\parallel}}{Rq\,(r+\delta r)}\mathrm{d}\tau\\ &\sim \frac{-v_{\parallel}}{Rq^{2}}q'\delta r\,\mathrm{d}\tau,\end{aligned} \tag{4.11c}$$

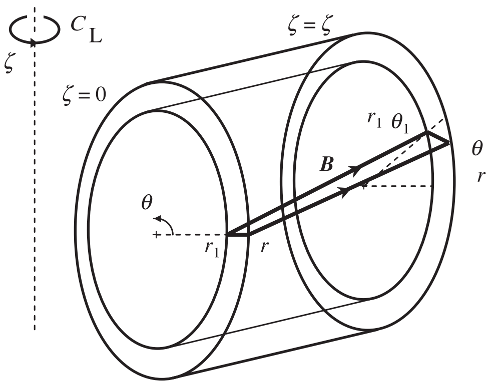

Fig. 4.7. Illustration of the sheared magnetic field in the toroidal plasma. When the pitch of magnetic field is different from one magnetic surface to the other surface, magnetic field has a shear. In this example, the pitch is weaker if the minor radius r increases. Two starting points are at the same poloidal angle at C (the toroidal angle $\zeta = 0$), but $\theta < \theta_1$ holds at C′ following the magnetic field lines.

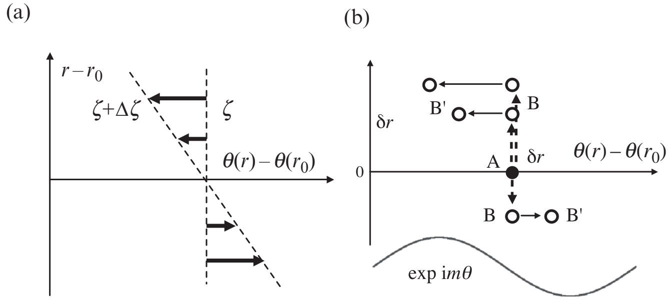

Fig. 4.8. In sheared magnetic field configuration, the poloidal angle of two neighbouring magnetic field lines deviates when the magnetic field lines are followed (a). In this circumstance, accelerated decorrelation (against the wave propagating in the poloidal direction) occurs via diffusion in the velocity space (b).

as is illustrated in Figure 4.8. So, we have (Hirshman and Molvig, 1979),

$$\left\langle \delta\theta^2 \right\rangle \sim \frac{v_\parallel^2}{L_s^2}\frac{D}{r^2}\tau^3. \tag{4.11d}$$

Here $L_s^{-1} = rq'/Rq^2$ is the magnetic shear length and D is the radial diffusion coefficient. As in the 1D velocity scattering case, decorrelation occurs via the synergy of radial scattering with parallel streaming. A useful measure of decorrelation is the time at which $k_\theta^2 r^2 \left\langle \delta\theta^2 \right\rangle \sim 1$. This then defines the decorrelation time τ_c, where,

$$\frac{1}{\tau_c} \sim \left(\frac{k_\theta^2 v_\parallel^2}{L_s^2} D \right)^{1/3}. \tag{4.11e}$$

In practice, the result of Eq.(4.11e) is a very rapid rate for electron scattering, and one which frequently exceeds the wave frequency for drift waves. Note too, that since $1/\tau_c \sim D^{1/3}$, this process is less sensitive to fluctuation levels, etc. than the familiar purely diffusive decorrelation rate $1/\tau_c \sim k_\perp^2 D$. Of course, this is a simple consequence of the underlying hybrid structure of the decorrelation process.

4.2.2.3 Decorrelation in sheared mean flow

Similarly, if one considers the motion of a particle or fluid element which undergoes radial scattering in a sheared flow (Fig. 4.9) (Biglari *et al.*, 1990; Itoh and Itoh, 1990; Shaing *et al.*, 1990; Zhang and Mahajan, 1992), we have,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = V_y(x), \tag{4.12a}$$

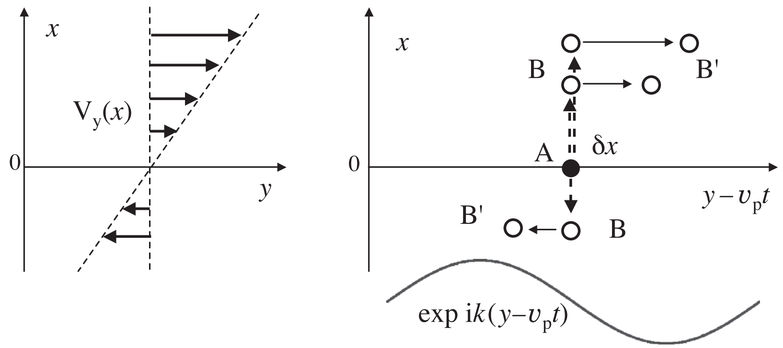

Fig. 4.9. The case of sheared mean flow. Accelerated decorrelation occurs via diffusion in the x-direction. The deviation of the plasma elements in the x-direction enhances the rate of phase change, $\omega - kV$.

so

$$y \sim \int d\tau \, V_y \left(x + \delta x\right), \tag{4.12b}$$

and

$$\delta y \sim \int d\tau \left(\frac{\partial V_y}{\partial x}\right) \delta x. \tag{4.12c}$$

The variance in the y-direction evolves as,

$$\left\langle \delta y^2 \right\rangle \sim \left(\frac{\partial V_y}{\partial x}\right)^2 D_x \tau^3, \tag{4.12d}$$

and the decorrelation rate follows as,

$$\frac{1}{\tau_c} \sim \left(\left(\frac{\partial V_y}{\partial x}\right)^2 k_y^2 D_x\right)^{1/3}. \tag{4.12e}$$

Again, note that the hybrid character of the process implies reduced sensitivity to D_x and the fluctuation levels which drive it (Biglari *et al.*, 1990). The interplay of sheared streaming and radial scattering which yields the hybrid decorrelation rate given in Eq.(4.12e) is relevant to the phenomena of suppression of turbulence and transport by a sheared flow (Hahm and Burrell, 1995; Itoh and Itoh, 1996; Terry, 2000). Combined with the theory of electric field bifurcation (Itoh and Itoh, 1988), this turbulence suppression mechanism plays a central role in understanding the phenomenon of confinement improvement (such as the H-mode

(Wagner *et al.*, 1982)). Extension to meso scale radial electric field (zonal flows) has also been performed (Diamond *et al.*, 2005b). These issues are explained in Volume 2.

4.2.3 Influence of resonance broadening on mean evolution

The appearance of resonance broadening by orbit scattering has several interesting implications for diffusion and the structure of the mean field theory for $\langle f \rangle$ evolution. The mean field velocity flux is given by,

$$\Gamma_v = \sum_{\boldsymbol{k},\omega} \frac{q}{m} E_{\substack{\boldsymbol{k} \\ -\omega}} f_{\substack{\boldsymbol{k} \\ \omega}}$$

$$= -D \frac{\partial \langle f \rangle}{\partial v}, \tag{4.13a}$$

where by substitution of Eq.(4.7e) for $f_{\substack{k \\ \omega}}$ into Eq.(4.13a), we find,

$$D = \frac{q^2}{m^2} \sum_{k,\omega} \left| E_{\substack{\boldsymbol{k} \\ \omega}} \right|^2 \frac{1/\tau_{c\boldsymbol{k}}}{(\omega - kv)^2 + 1/\tau_{c\boldsymbol{k}}^2}. \tag{4.13b}$$

Noting that $1/\tau_{c\boldsymbol{k}} = \left(k^2 D/6\right)^{1/3}$, we see that D is, in principle, defined as a function of itself in Eq.(4.13). Of course, $\tau_{c\boldsymbol{k}}$ constitutes the individual coherence time of distribution perturbations measured relative to a test wave $\boldsymbol{k}$. We also see that $\tau_{c\boldsymbol{k}}$ enters the resonance width, and acts to broaden the wave–particle resonance to a finite width.

$$\Delta v_{\mathrm{T}} = \frac{1}{k\tau_{\mathrm{c}}}, \tag{4.14a}$$

so the resonance function now has the broadened form,

$$\frac{1/\tau_{c\boldsymbol{k}}}{(\omega - kv)^2 + 1/\tau_{c\boldsymbol{k}}^2} \to \frac{\Delta v_{\mathrm{T}}/k}{(\omega/k - v)^2 + (1/k\tau_{c\boldsymbol{k}})^2}. \tag{4.14b}$$

Given that the main effect of orbit decorrelation is to broaden the linear, singular resonance function $\sim \delta\,(\omega - kv)$ to one of finite width, one might naturally ask, "How does D really charge," and "is the finite width of the wave–particle resonance at all significant, in the event that the width of the fluctuation spectrum is larger?" As in Chapter 3, we proceed by an ansatz, a simple form of the fluctuation spectrum which facilitates performing the spectrum integrations in order to identify the basic time scales. Assuming D is driven by a Lorentzian spectrum of modes, we have,

$$D = \mathrm{Re} \int \mathrm{d}k \, \frac{q^2}{m^2} \frac{\left|\tilde{E}_0\right|^2}{\left[1 + \left(\frac{k-k_0}{\Delta k}\right)^2\right]} \frac{i}{\omega - kv + \frac{i}{\tau_c}}$$

$$\sim \mathrm{Re} \left\{ \frac{q^2}{m^2} \left|\tilde{E}_0\right|^2 i \Big/ \left[\omega_{k_0} - k_0 v + i \left|\Delta k\right| \left|\frac{\mathrm{d}\omega}{\mathrm{d}k} - v\right| + \frac{i}{\tau_{ck}}\right]\right\}. \quad (4.15a)$$

Thus we immediately see that if,

$$\left|\Delta k \left(v_{\mathrm{gr}} - \frac{\omega}{k}\right)\right| > \frac{1}{\tau_c}, \quad (4.15b)$$

so that the spectral auto-correlation time $\tau_{\mathrm{ac}} < \tau_c$, the resonance broadening is irrelevant and $D \to D_{\mathrm{QL}}$. Combining Eqs.(4.15a) and (4.7c), Eq.(4.15b) is rewritten as,

$$\left(\frac{\Delta k}{k}\right)^2 \left|\frac{v_g}{v_p} - 1\right|^2 > \frac{e\phi}{m v_p^2}.$$

This condition is equivalent to the validity condition for the quasi-linear theory, Eq.(3.11) and Fig. 3.11.

Note that broad spectral width, alone, is *not* sufficient to ensure that D is quasi-linear. Dispersion must be sufficient to ensure that the fluctuation pattern seen by a resonant particle (one with $v = \omega/k$) is short lived in comparison with the correlation time. In this limit of $D \to D_{\mathrm{QL}}$, the particle–wave decorrelation rate scales as,

$$\frac{1}{\tau_{ck}} \sim \left(k^2 \left\langle \tilde{a}^2 \right\rangle \tau_{\mathrm{ac}}\right)^{1/3} \quad (4.16a)$$

and the resonance width scales as

$$\frac{1}{k\tau_{ck}} \sim \left(\left\langle \tilde{a}^2 \right\rangle \frac{\tau_{\mathrm{ac}}}{k}\right)^{1/3}. \quad (4.16b)$$

Here $\left\langle \tilde{a}^2 \right\rangle$ is the acceleration fluctuation spectrum and τ_{ac} is the spectral auto-correlation time.

Given the discussion above, the opposite limit of short correlation time naturally arouses one's curiosity. In this limit, for resonant particles $i/(\omega - kv + i/\tau_{ck}) \sim \tau_{ck}$. Hence, ignoring the k-dependence of τ_c, we have,

$$\frac{1}{\tau_c^3} = \frac{k^2 D}{6} = \frac{k^2}{6} \sum_k \frac{q^2}{m^2} |E_k|^2 \tau_c, \tag{4.17a}$$

so

$$\frac{1}{\tau_c^4} \sim \frac{q^2}{m^2} k^4 \left\langle \tilde{\phi}^2 \right\rangle, \tag{4.17b}$$

and

$$\frac{1}{\tau_c} \sim k \left(\frac{q^2}{m^2} \left\langle \tilde{\phi}^2 \right\rangle \right)^{1/4}, \tag{4.17c}$$

and

$$\Delta v_T \sim \left(\frac{q^2}{m^2} \left\langle \tilde{\phi}^2 \right\rangle \right)^{1/4} \sim \left(\frac{q}{m} \left\langle \tilde{\phi}^2 \right\rangle^{1/2} \right)^{1/2}. \tag{4.17d}$$

Not surprisingly, the results for the short-τ_c limit resemble those for a particle interaction with a single wave.

4.3 Renormalization in Vlasov turbulence I: Vlasov response function

4.3.1 Issues in renormalization in Vlasov turbulence

While intuitively appealing in many respects, resonance broadening theory is inherently unsatisfactory, for several reasons. These include, but are not limited to:

1. the theory is intrinsically one of the 'test wave' genre, yet treats E_k as fixed while f_k evolves nonlinearly in response to it – i.e. f_k and E_k are treated asymmetrically. This is especially dubious since the Vlasov nonlinearity consists of the product $E \frac{\partial f}{\partial v}$;
2. the resonance broadening theory does not conserve energy, and indeed does not address the issue of energetics. We will elaborate on this point further below, in our discussion of renormalization for drift wave turbulence;
3. resonance broadening theory treats the evolution of f as a Markov process, and neglects memory effects;
4. resonance broadening theory asserts Gaussian statistics a priori for particle orbit scattering statistics.

In view of these limitations, it is natural to explore other, more systematic, approaches to the problem of renormalization. The reader is forewarned, however, that *all* renormalization procedures involve some degree of uncertainty in the accuracy of the approximations they employ. *None* can be fully justified on a rigorous function. None can predict their own errors.

The aim here is to determine the response function relating $f_{k,\omega}$ to the electric field perturbation $E_{k,\omega}$. In formal terms, if one assigns each $E_{k,\omega}$, a multiplicative phase factor $\alpha_{k,\omega} = \mathrm{e}^{i\theta_{k,\omega}}$, where $\theta_{k,\omega}$ is the phase of the fluctuation at k, ω, then the Vlasov response function is simply $\delta f_{k,\omega}/\delta\alpha_{k,\omega}$. In other words, the aim here is to extract the portion of the Vlasov equation nonlinearity which is phase coherent with $\alpha_{k,\omega}$. Note that the phase-coherent portion can and will contain pieces proportional to both $f_{k,\omega}$ and $E_{k,\omega}$.

4.3.2 One-dimensional electron plasmas

Proceeding, consider a simple 1D electron plasma with ions responding via a given susceptibility $\chi_i\,(k,\omega)$. Then the Vlasov–Poisson system is just,

$$-i\,(\omega - kv)\,f_{k,\omega} + \frac{\partial}{\partial v}\sum_{k',\omega'}\frac{q}{m}E_{\substack{-k'\\-\omega'}}f_{\substack{k+k'\\\omega+\omega'}} = -\frac{q}{m}E_{\substack{k\\\omega}}\frac{\partial\langle f\rangle}{\partial v}, \tag{4.18a}$$

and

$$ikE_{\substack{k\\\omega}} = 4\pi n_0 q\int \mathrm{d}v\, f_{\substack{k\\\omega}} - 4\pi n_0 q\,\chi_i\,(k,\omega)\,\frac{q}{T}\phi_{k,\omega}. \tag{4.18b}$$

To obtain the response function for mode k, ω, i.e. $\delta f_{k,\omega}/\delta\phi_{k,\omega}$, we seek to isolate the part of the nonlinearity $(q/m)\,E\,\partial f/\partial v$ which is phase coherent with $E_{k,\omega}$. Here, 'phase coherent' means having the same phase as does $\phi_{k,\omega}$. Note that the philosophy here presumes the utility of a test wave approach, which 'tags' each mode by a phase $\alpha_{k,\omega}$, and assumes that one may examine the phase-coherent response of a given mode in the (dynamic) background of all modes, without altering the statistics and dynamics of the ensemble. Thus, the aim of this procedure is to systematically approximate the nonlinearity,

$$N_{k,\omega} = \frac{\partial}{\partial v}\sum_{k',\omega'}\frac{q}{m}E_{\substack{-k'\\\omega'}}f_{\substack{k+k'\\\omega+\omega'}} \tag{4.19a}$$

by a function of the form,

$$N_{k,\omega} = C_{\substack{f\\\boldsymbol{k},\omega}}\,f_{\boldsymbol{k},\omega} + C_{\substack{E\\k,\omega}}\frac{q}{m}E_{\substack{k\\\omega}}, \tag{4.19b}$$

where $C_{\substack{f\\k,\omega}}$ and $C_{\substack{E\\k,\omega}}$ are *phase independent* operator function of the fluctuation spectrum, and the response function itself. Note that the form of the renormalized $N_{k,\omega}$ in Eq.(4.19b) suggests that $C_{\substack{f\\k,\omega}}$ should reduce to the familiar diffusion operator from resonance broadening theory in certain limits.

4.3.2.1 Renormalization procedure

To answer the obvious question of how one obtains C_f, C_E, we proceed by a perturbative approach. To this end, it is useful to write the Vlasov equation as,

$$-i\left(\omega-kv\right)f_{\substack{k\\\omega}}+\frac{\partial}{\partial v}\sum_{k',\omega'}\left(\frac{q}{m}E_{\substack{-k'\\-\omega'}}f^{(2)}_{\substack{k+k'\\\omega+\omega'}}+f^{(2)}_{\substack{-k'\\-\omega'}}\frac{q}{m}E_{\substack{k+k'\\\omega+\omega'}}\right)=-\frac{q}{m}E_{\substack{k\\\omega}}\frac{\partial\left\langle f\right\rangle}{\partial v}. \tag{4.20a}$$

Here, the superscript (2) signifies that the quantity so labelled is driven by the *direct* beating of two modes or fluctuations. Thus,

$$f^{(2)}_{\substack{k+k'\\\omega+\omega'}}\sim E^{(1)}_{\substack{k'\\\omega'}}f^{(1)}_{\substack{k\\\omega}}\propto\exp i\left[\theta_{k',\omega'}+\theta_{k,\omega}\right], \tag{4.20b}$$

and similarly for $E^{(2)}_{\substack{k+k'\\\omega+\omega'}}$ (here θ is a phase of the mode.) This ensures that the resulting $N_{k,\omega}$ is phase coherent with $\phi_{k,\omega}$. The remaining step of relating $f^{(2)}_{\substack{k+k'\\\omega+\omega'}}$, $E^{(2)}_{\substack{k+k'\\\omega+\omega'}}$ to the amplitudes of primary modes is done by perturbation theory, i.e. for $f^{(2)}_{\substack{k+k'\\\omega+\omega'}}$ we write,

$$\left[-i\left(\omega+\omega'-\left(k+k'\right)v\right)+Cf_{\substack{k+k'\\\omega+\omega'}}\right]f^{(2)}_{\substack{k+k'\\\omega+\omega'}}=-\frac{q}{m}\frac{\partial f^{(1)}_{\substack{k'\\\omega'}}}{\partial v}E^{(1)}_{\substack{k\\\omega}}, \tag{4.21a}$$

so

$$f^{(2)}_{\substack{k+k'\\\omega+\omega'}}=L_{\substack{k+k'\\\omega+\omega'}}\left\{\left(\frac{q}{m}\right)^2E^{(1)}_{\substack{k'\\\omega'}}\frac{\partial}{\partial v}L_{\substack{k\\\omega}}\frac{\partial\langle f\rangle}{\partial v}E^{(1)}_{\substack{k\\\omega}}\right\}, \tag{4.21b}$$

where the propagator is,

$$L^{-1}_{\substack{k+k'\\\omega+\omega'}}=-i\left[\omega+\omega'-\left(k+k'\right)v+iC_{\substack{k+k'\\\omega+\omega'}}\right]. \tag{4.21c}$$

Here we have dropped $E^{(2)}$ contributions since we are concerned with nonlinear wave–particle interaction and the response function for f. Terms from $E^{(2)}$ involve only *moments* of f, and so are not directly relevant to wave–particle interaction. Given this simplification, the subscript 'f' on C_f has also been dropped. Then, substituting Eqs.(4.21b), (4.21c) into Eq.(4.20a) gives the renormalized nonlinearity,

$$N_{k,\omega} = -\frac{\partial}{\partial v}\sum_{k',\omega'}\frac{q^2}{m^2}\left|E_{\substack{k'\\ \omega'}}\right|^2 L_{\substack{k''\\ \omega''}}\frac{\partial}{\partial v}f_{\substack{k\\ \omega}} - \frac{\partial}{\partial v}\sum_{k',\omega'}E_{\substack{-k'\\ \omega'}}\frac{\partial f_{\substack{k\\ \omega}}}{\partial v}L_{\substack{k''\\ \omega''}}\frac{q}{m}E_{\substack{k\\ \omega}}, \quad (4.22a)$$

and the renormalized Vlasov equation thus follows as,

$$-i\left(\omega - kv\right) f_{k,\omega} - \frac{\partial}{\partial v}D_{k,\omega}\frac{\partial}{\partial v}f_{k,\omega} = -\frac{q}{m}E_{k,\omega}\left(\frac{\partial\langle f\rangle}{\partial v} + \frac{\partial}{\partial v}\bar{f}_{k,\omega}\right), \quad (4.22b)$$

where by correspondence with $N_{k,\omega}$,

$$D_{k,\omega} = \sum_{k',\omega'}\frac{q^2}{m^2}\left|E_{k',\omega'}\right|^2 L_{\substack{k''\\ \omega''}} \quad (4.22c)$$

$$\bar{f}_{\substack{k\\ \omega}} = \sum_{k',\omega'}\frac{q}{m}E_{\substack{-k'\\ -\omega'}}\frac{\partial f_{\substack{k'\\ \omega'}}}{\partial v}L_{k'',\omega''}. \quad (4.22d)$$

Cleary, the operator $-\partial/\partial v\, D_{k,\omega}\,\partial/\partial v$ constitutes a propagator dressing or "self-energy" correlation to the bare Vlasov propagator $L_{k,\omega} = i/(\omega - kv)$. We term this a 'self-energy' because it reflects the effect of interactions between the test mode and the ambient spectrum of background waves, just as the self-energy renormalization of the electron propagated in quantum electrodynamics accounts for the interaction of a bare electron with ambient photons induced by vacuum polarization. In a similar vein, $\bar{f}_{k,\omega}$ corresponds to a renormalization of the background or ambient distribution function, and so bears a resemblence to wave function renormalization, familiar from quantum electrodynamics.

4.3.2.2 *Non-Markovian property*

Although the propagator renormalization derived above has the structure of a diffusion operator, here the diffusion coefficient $D_{k,\omega}$ depends explicitly upon the wave number and frequency of the fluctuation. This important feature reflects the fundamentally *non-Markovian* character of the nonlinear interactions. Recall that a Markovian process, which may be described by a Fokker–Planck equation with a space-time-independent diffusion coefficient, is one with no memory. Thus a Markovian model for f evolution is one with the form,

$$f(t+\tau, v) = f(t, v) + \int \mathrm{d}(\Delta v)\, T(v, \Delta v, \tau)\, f(v - \Delta v, t). \qquad (4.23)$$

Here, $T(v, \Delta v, \tau)$ is the transition probability for a step $v - \Delta v \rightarrow v$ in time interval τ. Making a standard Fokker–Planck expansion for small Δv, and taking $\tau \sim \tau_{\mathrm{ac}}$ and $\Delta v \sim \Delta v_{\mathrm{T}}$ recovers the resonance broadening theory result, with $D = D_{\mathrm{QL}}$. Application of this model to the evolution of $f_{k,\omega}$ is sensible *only* if $\partial f_{k,\omega}/\partial t \ll f_{k,\omega}/\tau_{\mathrm{ac}}$, and $\left|k'\right| \gg |k|$, $|\omega|' \gg |\omega|$, so that the spectrum of ambient modes appears as a stochastic bath to the test mode in question. Of course, examination of $D_{k,\omega}$ in the $\tau_{\mathrm{ac}} < \tau_{\mathrm{c}}$ limit (so $L_{k'',\omega''}$ may be taken to be bare) reveals that the k, ω dependence of $D_{k,\omega}$, $\bar{f}_{k,\omega}$ is present previously because the test mode at (k, ω) has spatio-temporal scales comparable to, not slower than, the other modes. Indeed, since,

$$D_{k,\omega} = \sum_{k',\omega'} \frac{q^2}{m^2} \left| E_{\substack{k'\\ \omega'}} \right|^2 \mathrm{Re}\left\{ \frac{i}{\omega + \omega' - (k + k')\, v} \right\}, \qquad (4.24)$$

we see that $D_{k,\omega} \rightarrow D$ if $k \ll k'$, $\omega \ll \omega'$, which corresponds to the Markovian limit where the random 'kicking' by other modes is so fast that it appears as a sequence of random kicks. Hence, it is apparent that the non-Markovian structure of the theory is a consequence of the fact that the test mode scales are comparable to other scales in the spectrum. However, it should be noted that regardless of the ratio of test wave space-time scales to background scales, the Markovian approximation is always valid for *resonant* particles, for which $\omega = kv$, since in this case,

$$D_{k,\omega}^{(v)} = \sum_{k',\omega'} \frac{q^2}{m^2} \left| E_{\substack{k'\\ \omega'}} \right|^2 \mathrm{Re}\left\{ \frac{i}{\omega' - k'v} \right\} \rightarrow D_{\mathrm{QL}}. \qquad (4.25)$$

Thus, the *resonant* particle response *is* amenable to treatment by a Markovian theory.

4.3.2.3 Background distribution renormalization

The other new feature in the theory is the background distribution renormalization $\bar{f}_{k,\omega}$. The function $\bar{f}_{k,\omega}$ accounts for the renormalization or "dressing" of the background distribution function which is necessary for the renormalized response of $f_{k,\omega}$ to reduce to the weak turbulence theory expansion result for $f_{k,\omega}$ in the limit of $kv \ll \omega$ and small fluctuation levels. More generally, $\bar{f}_{k,\omega}$ preserves certain structural properties of $N_{k,\omega}$ which are crucial to energetics and its treatment by the renormalized theory. These features are most readily illustrated in the context of the drift wave dynamics, so it is to this problem we now turn.

4.4 Renormalization in Vlasov turbulence II: drift wave turbulence

4.4.1 Kinetic description of drift wave fluctuations

Recall from Chapter 3 that a simple model for low frequency plasma dynamics in a strongly magnetized plasma is the drift-kinetic equation,

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B_0} \nabla \phi \times z \cdot \nabla f + \frac{q}{m} E_z \frac{\partial f}{\partial v_z} = 0. \tag{4.26a}$$

The geometry of the plasma is illustrated in Figure 3.14. Indeed, the simplest possible model of drift wave dynamics consists of drift-kinetic ion dynamics, as described by Eq.(4.26a) and a 'nearly Boltzmann' electron response, along with quasi-neutrality, so,

$$\frac{n_{i\underset{\omega}{\boldsymbol{k}}}}{n_0} = \int \mathrm{d}v\, f_{\underset{\omega}{\boldsymbol{k}}}(v) = \frac{n_{e\underset{\omega}{\boldsymbol{k}}}}{n_0} = \left(1 - i\delta_{\underset{\omega}{\boldsymbol{k}}}\right) \frac{|e|\,\phi_{\boldsymbol{k},\omega}}{T}, \tag{4.26b}$$

where ϕ is the electrostatic potential, so that the electric field in the direction of the main magnetic field, E_z, is given by $-\partial\phi/\partial z$. This simple model can be reduced even further by ignoring the $(qE_z/m)\,\partial f/\partial v_z$ nonlinearity, since $k_z \ll k_\perp$. In that case, the drift-kinetic equation for f simplifies to,

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{cE}{B_0} \nabla f = \frac{c}{B_0} \frac{\partial \phi}{\partial y} \frac{\partial \langle f \rangle}{\partial r} - \frac{q}{m} F_z \frac{\partial \langle f \rangle}{\partial v_z}. \tag{4.26c}$$

Equation (4.26c) has the generic structure explained below,

$$\frac{\partial f}{\partial t} + \underset{①}{v_z \frac{\partial}{\partial z} f} + \underset{②}{\boldsymbol{v}_\perp \cdot \nabla_\perp f} = \underset{③}{\frac{\delta S}{\delta \phi} \phi \langle f \rangle}, \tag{4.27a}$$

where, in Eq.(4.27a), the meaning of the terms is as follows: ①: parallel streaming along $B_0 z$, ②: advection by fluid with $\nabla \cdot \boldsymbol{v} = 0 \to$ spatial scattering, ③: source-potential perturbation $\langle f \rangle = \langle f(r, v_z) \rangle$. Many variations on this generic form are possible obviously, one is to take $v_\perp \to 0$ (i.e. $B_0 \to \infty$), which recovers the structure of the linearized equation.

A second is to take $k_z \to 0$ and integrate over v, thus recovering,

$$\frac{\partial n}{\partial t} + \boldsymbol{v}_\perp \cdot \nabla_\perp n = \frac{\delta S'}{\delta \phi} \phi, \tag{4.27b}$$

which is similar to the equation for the evolution of a 2D fluid, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla\phi \times z \cdot \nabla\rho - \nu\nabla^2\rho = 0, \tag{4.27c}$$

where

$$\rho = \nabla^2\phi. \tag{4.27d}$$

This type of structure appears in the descriptions of 2D fluids, guiding centre plasmas, non-neutral plasmas, etc.

A third variation is found by retaining finite Larmor radius effects, so

$$f_{\boldsymbol{k},\omega} = J_0(k\rho)\, f^{\mathrm{gc}}_{\boldsymbol{k},\omega}. \tag{4.27e}$$

Here, $f^{\mathrm{gc}}_{\boldsymbol{k},\omega}$ refers to the guiding centre distribution function and $\rho = v_\perp/\omega_c$. The guiding centre distribution obeys the gyrokinetic equation,

$$\frac{\partial}{\partial t} f^{\mathrm{gc}} + v_z \frac{\partial f^{\mathrm{gc}}}{\partial z} + \langle v_\perp \rangle_\theta \cdot \nabla_\perp f^{\mathrm{gc}} = \left\langle \frac{\partial S}{\partial \phi} \right\rangle_\theta \phi \langle f \rangle . \tag{4.27f}$$

Here, the averages $\langle \quad \rangle_\theta$ refer to gyro-angle averages. It should be apparent, then, that the structure of Eq.(4.26c) is indeed of general interest and relevant to a wide range of problems.

4.4.2 Coherent nonlinear effect via resonance broadening theory

The simplest approach to the task of obtaining a renormalized response of $f_{k,\omega}$ to $\phi_{k,\omega}$ in Eq.(4.26c) is to employ resonance broadening theory, i.e.

$$\begin{aligned} f_{\substack{k\\ \omega}} &= \mathrm{e}^{ik_z z}\mathrm{e}^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp} \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{i\omega\tau} \langle u(-\tau)\rangle^* \mathrm{e}^{ik_z z}\mathrm{e}^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp} \frac{\delta S_{k,\omega}}{\delta\phi}\left(-\frac{|e|\,\phi_{\boldsymbol{k},\omega}}{T_i}\right)\langle f\rangle \\ &= \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{i(\omega - k_z v_z)\tau} \left\langle \mathrm{e}^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\delta} r_\perp(-\tau)}\right\rangle^* \frac{\delta S_{k,\omega}}{\delta\phi}\left(-\frac{|e|\,\phi_{\boldsymbol{k},\omega}}{T_i}\right)\langle f\rangle . \end{aligned} \tag{4.28}$$

Here $\delta\boldsymbol{r}$ is the excursion in r induced by random $E \times B$ scattering. Taking the statistical distribution of $\delta\boldsymbol{r}$ to be Gaussian, one then finds,

$$\begin{aligned} \left\langle \mathrm{e}^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\delta} r(-\tau)}\right\rangle &\cong \left\langle 1 + i\boldsymbol{k}_\perp \cdot \boldsymbol{\delta} r - \frac{(k_\perp \delta r)^2}{2} \right\rangle \\ &= \exp\left[-\left(\boldsymbol{k}_\perp \cdot \boldsymbol{D} \cdot \boldsymbol{k}_\perp\right)\tau\right] \\ &= \exp\left[-k_\perp^2 \boldsymbol{D}\tau\right] \end{aligned} \tag{4.29}$$

for isotropic turbulence. Contrary to Eq.(4.7a), a simple exponential decay is recovered. Note that the diffusion is *spatial* here. The precise form of the diffusion tensor $\boldsymbol{D}_\perp$ may be straightforwardly obtained by a quasi-linear calculation on the underlying drift-kinetic equation (i.e. Eq.(4.26c)), yielding,

$$\boldsymbol{k}_\perp \cdot \boldsymbol{D} \cdot \boldsymbol{k}_\perp = \sum_{\boldsymbol{k}'} \frac{c^2}{B_0^2} \left(\boldsymbol{k}_\perp \cdot \boldsymbol{k}'_\perp z\right)^2 \left|\phi_{\boldsymbol{k}'}\right|^2 \pi\delta\left(\omega - k_z v_z\right). \tag{4.30}$$

Thus, for turbulence which is isotropic in $\boldsymbol{k}$ we find the familiar 'classic' form of the renormalized response, as given by resonance broadening theory,

$$f_{k,\omega} = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{i\left(\omega - k_z v_z + i k_\perp^2 D\right)\tau} \frac{\delta S_{k,\omega}}{\delta\phi_{k,\omega}} \left(-\frac{|e|}{T_i}\phi_{k,\omega}\right)\langle f\rangle . \tag{4.31}$$

So, for drift-kinetic turbulence, we see that the decorrelation rate for turbulent scattering of a test particle from its unperturbed trajectory scattering is given by,

$$\frac{1}{\tau_{c\boldsymbol{k}}} = k^2 D_\perp . \tag{4.32}$$

This result is, in turn, the underpinning of the classic mixing length theory estimate for the saturated transport associated with drift wave instabilities, namely,

$$D_\perp = \gamma_{\boldsymbol{k}} \Big/ k_\perp^2 . \tag{4.33}$$

Note that the idea here is simply that the instability saturates when the rate at which a particle is scattered one perpendicular wave length (i.e. $1\big/\tau_{c\,\boldsymbol{k}}$) equals the growth rate $\gamma_{\boldsymbol{k}}$. Of course, coupling to magnetic or flow velocity shear can increase the decorrelation rate, as discussed earlier in this chapter.

4.4.3 Conservation revisited

Note, too, that the essence of resonance broadening theory is simply to replace the nonlinearity of the drift-kinetic equation (i.e. Eq.(4.26c)) by a diffusion operator, so (RHS = right-hand side):

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B}\nabla\phi \times z \cdot \nabla f = \mathrm{RHS} \tag{4.34a}$$

becomes

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \nabla_\perp \cdot \boldsymbol{D} \cdot \nabla_\perp f = \mathrm{RHS}. \tag{4.34b}$$

Thus, we see that in this application, the result of resonance broadening theory resembles that of a simple 'eddy viscosity' model as used in modelling in fluid turbulence. The apparent direct correspondence between resonance broadening theory and simple eddy viscosity methods then begs the question, "*Is application of resonance broadening theory to the drift-kinetic equation in the vein discussed above correct?*". Two simple observations are quite pertinent to answering this question. One is that since the $\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity is independent of velocity (except for the velocity dependence of f), we can integrate Eq.(4.9c) over velocity ($\int \mathrm{d}^3 v$) to obtain,

$$\frac{\partial n}{\partial t} + \frac{\partial v_z}{\partial z} - \frac{c}{B_0} \nabla \phi \times \boldsymbol{z} \cdot \nabla n = \int \mathrm{d}^3 v \, \mathrm{RHS}, \tag{4.35a}$$

which may then be straightforwardly re-written in the form,

$$\frac{\partial n}{\partial t} + \frac{\partial v_z}{\partial z} + \nabla_\perp \cdot \boldsymbol{J}_\perp = \int \mathrm{d}^3 v \, \mathrm{RHS}, \tag{4.35b}$$

where

$$\boldsymbol{J}_\perp = \boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} \, n \tag{4.35c}$$

is the perpendicular current carried by $\boldsymbol{E} \times \boldsymbol{B}$ advection of particles. Of course, such a current can not couple to the perpendicular electric field, because $J_\perp$ in Eq.(4.35c) is perpendicular to $E_\perp$ (either to do work or have work done upon it), so we require,

$$\left\langle \boldsymbol{E}_\perp^* \cdot \boldsymbol{J}_\perp \right\rangle = 0, \tag{4.36}$$

where the brackets signify a space-time average. In drift kinetics, the only heating possible is *parallel* heating, so $\langle \boldsymbol{E} \cdot \boldsymbol{J} \rangle = \left\langle E_\parallel J_\parallel \right\rangle$, as in the discussion of the energetics of quasi-linear theory for drift wave turbulence which was presented in Chapter 3. This is seen trivially in real space or in $\boldsymbol{k}$-space, since

$$\left\langle \boldsymbol{E}_\perp^* \cdot \boldsymbol{J}_\perp \right\rangle = \sum_{\boldsymbol{k},\omega} \sum_{\boldsymbol{k}',\omega'} \frac{c}{B_0} \left(\boldsymbol{k} \cdot \boldsymbol{k}', \boldsymbol{\omega}' \times \boldsymbol{z} \right) \phi_{\substack{-\boldsymbol{k} \\ -\omega}} \phi_{\substack{-\boldsymbol{k}' \\ -\omega'}} n_{\substack{\boldsymbol{k}+\boldsymbol{k}' \\ \omega+\omega'}}, \tag{4.37}$$

so that the interchange $(k, \omega) \leftrightarrow \left(k', \omega'\right)$ leaves all in Eq.(4.37) invariant, except for the coupling coefficient $\boldsymbol{k} \cdot \boldsymbol{k}' \times \boldsymbol{z}$, which is anti-symmetry, i.e. $\boldsymbol{k} \cdot \boldsymbol{k}' \times \boldsymbol{z} \to -\boldsymbol{k} \cdot \boldsymbol{k}' \times \boldsymbol{z}$ so $\left\langle \boldsymbol{E}^* \cdot \boldsymbol{J}_\perp \right\rangle = -\left\langle \boldsymbol{E}^* \cdot \boldsymbol{J}_\perp \right\rangle = 0$. The condition that $\left\langle \boldsymbol{E}^* \cdot \boldsymbol{J}_\perp \right\rangle = 0$ can be re-written as a condition on the nonlinear term $N_{k,\omega}$ since,

$$\left\langle \boldsymbol{E}_\perp^* \cdot \boldsymbol{J}_\perp \right\rangle = \sum_{\boldsymbol{k},\omega} \phi_{\substack{-\boldsymbol{k} \\ -\omega}} N_{\substack{\boldsymbol{k} \\ \omega}} = 0. \tag{4.38}$$

Any renormalization of the nonlinearity $N_{\boldsymbol{k},\omega}$ must satisfy the condition expressed by Eq.(4.38). From this discussion, we also see that the problem of renormalization in turbulence theory is one of 'representation', i.e. the aim of renormalization theory is to 'represent' the 'bare' nonlinearity by a simpler, more tractable operator which maintains its essential physical properties.

A second, somewhat related property of N is that it annihilates the adiabatic or Boltzmann response, to the lowest order in $1\big/k_\perp L_\perp$, where $L_\perp$ is the perpendicular scale length of $\langle f\rangle$ variation. In calculations related to drift wave turbulence, it is often useful to write the total fluctuating distribution function as the sum of the Boltzmann response (f_{B}) plus a non-Boltzmann correction. Thus for ions, we often write,

$$f_{\boldsymbol{k},\omega} = -\frac{|e|}{T_i}\phi_{\boldsymbol{k},\omega}\langle f\rangle + g_{\boldsymbol{k},\omega}. \tag{4.39}$$

Now it is obvious that $\boldsymbol{v}_{\boldsymbol{E}\times\mathrm{B}}\cdot\nabla f_B \rightarrow 0$ to the lowest order, since $\boldsymbol{E}\times\boldsymbol{B}\cdot\nabla\phi = 0$. Hence any representation of $N_{\boldsymbol{k},\omega}$ must preserve the property that $\lim_{f\rightarrow f_{\mathrm{B}}} N_{\boldsymbol{k},\omega}\rightarrow 0$, to the lowest order.

It is painfully clear that resonance broadening theory satisfies *neither* of the constraints discussed above. In particular, in resonance broadening theory, $N_{\boldsymbol{k},\omega} = k_\perp^2 D f_{\boldsymbol{k},\omega}$, and rather obviously,

$$\sum_{\boldsymbol{k},\omega}\phi_{\substack{-\boldsymbol{k}\\-\omega}} N_{\substack{-\boldsymbol{k}\\-\omega}} \rightarrow \sum_{\boldsymbol{k},\omega}\phi_{\substack{-\boldsymbol{k}\\-\omega}} k_\perp^2 D f_{\substack{\boldsymbol{k}\\\omega}} \neq 0, \tag{4.40a}$$

so that $\left\langle \boldsymbol{E}_\perp^*\cdot\boldsymbol{J}_\perp\right\rangle\neq 0$, as it should be. Also, in resonance broadening theory,

$$\lim_{f\rightarrow f_{\mathrm{B}}} N_{\substack{\boldsymbol{k}\\\omega}} = k^2 D\left(-\frac{|e|}{T}\phi_{\substack{\boldsymbol{k}\\\omega}}\langle f\rangle\right)\neq 0, \tag{4.40b}$$

so the Boltzmann response is *not* annihilated, either. Hence, resonance broadening theory fails both 'tests' of a successful renormalization. The reason for these shortcomings is the neglect of background renormalization, i.e. $\bar{f}_{\boldsymbol{k},\omega}$, by the resonance broadening approach.

4.4.4 Conservative formulations

This shortcoming can be rectified by the perturbative renormalization procedure presented above in the context of the 1D Vlasov plasma. We now turn to the application of this methodology to drift wave turbulence.

The primitive drift-kinetic equation is, in $\boldsymbol{k}$, ω-space,

$$-i\omega\left(\omega - k_z v_z\right) f_{\substack{\boldsymbol{k}\\ \omega}} + \frac{c}{B_0}\sum_{\boldsymbol{k}'}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times\boldsymbol{z}\right) f_{\substack{-\boldsymbol{k}'\\ -\omega'}} f_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = \frac{\delta S}{\delta\phi}\left(-\frac{|e|}{T_i}\phi_{\boldsymbol{k},\omega}\right)\langle f\rangle\,, \tag{4.41a}$$

so that the portion of the nonlinearity which is phase coherent with $\alpha_{\boldsymbol{k},\omega}$, where $\alpha_{\boldsymbol{k},\omega} = \exp i\theta_{k\omega}$ and θ is a phase, may be written as,

$$N_{\substack{\boldsymbol{k}\\ \omega}} \cong \frac{c}{B_0}\sum_{\boldsymbol{k}',\omega}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times\boldsymbol{z}\right)\left(\phi^{(1)}_{\substack{-\boldsymbol{k}'\\ -\omega'}} f^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} - f^{(1)}_{\substack{-\boldsymbol{k}'\\ -\omega'}}\phi^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}\right). \tag{4.41b}$$

As before, here we are interested in nonlinear wave–particle interaction, so we ignore $\phi^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}$ hereafter. The quantity $f^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}$ is given by,

$$\begin{aligned}&-i\left\{\left(\omega+\omega'\right) - \left(k_z + k_z'\right)\right\} f^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} + C_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} f^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}\\ &= \frac{c}{B_0}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times\boldsymbol{z}\right)\left(\phi_{\substack{\boldsymbol{k}'\\ \omega'}} f_{\substack{\boldsymbol{k}\\ \omega}} - f_{\substack{\boldsymbol{k}'\\ \omega'}}\phi_{\substack{\boldsymbol{k}\\ \omega}}\right),\end{aligned} \tag{4.42a}$$

so

$$f^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = L_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}\frac{c}{B_0}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times\boldsymbol{z}\right)\left(\phi_{\substack{\boldsymbol{k}'\\ \omega'}} f_{\substack{\boldsymbol{k}\\ \omega}} - f_{\substack{\boldsymbol{k}'\\ \omega'}}\phi_{\substack{\boldsymbol{k}\\ \omega}}\right), \tag{4.42b}$$

where

$$L^{-1}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = -i\left\{\left(\omega+\omega'\right) - \left(k_z + k_z'\right)v_z + iC_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}\right\} \tag{4.42c}$$

is the beat wave propagator. Thus, we see that the renormalized nonlinearity has the form,

$$N_{\boldsymbol{k},\omega} = d_{\boldsymbol{k},\omega} f_{\boldsymbol{k},\omega} - \bar{f}_{\boldsymbol{k},\omega}\phi_{\boldsymbol{k},\omega}, \tag{4.43a}$$

where

$$d_{\boldsymbol{k},\omega} = \sum_{\boldsymbol{k}',\omega'}\frac{c^2}{B_0^2}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times\boldsymbol{z}\right)^2\left|\phi_{\substack{\boldsymbol{k}'\\ \omega'}}\right|^2 L_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} \cong k_\perp^2 D_{\boldsymbol{k},\omega} \tag{4.43b}$$

for isotropic turbulence, and

$$\bar{f}_{\boldsymbol{k},\omega} = \frac{c^2}{B_0^2} \sum_{\boldsymbol{k}',\omega'} \left(\boldsymbol{k} \cdot \boldsymbol{k}' \times z\right)^2 \phi_{\substack{-\boldsymbol{k}'\\-\omega'}} \phi_{\substack{\boldsymbol{k}'\boldsymbol{k}+\boldsymbol{k}'\\ \omega'\omega+\omega'}}. \tag{4.43c}$$

Here, $d_{\boldsymbol{k},\omega}$ is the rate of test particle scattering and $\bar{f}_{\boldsymbol{k},\omega}$ is the background distribution renormalization. The term $d_{\boldsymbol{k},\omega}$ is referred to as the 'test particle scattering rate', since it is different from the actual particle flux, which is necessarily regulated to the non-adiabatic electron response $\delta_{\boldsymbol{k},\omega}|e|\phi_{\boldsymbol{k},\omega}\big/T_{\mathrm{e}}$. Retaining $d_{\boldsymbol{k},\omega}$, $\bar{f}_{\boldsymbol{k},\omega}$, we thus see that the renormalized distribution response satisfies the equation

$$-i\left(\omega - k_z v_z\right) f_{\substack{\boldsymbol{k}\\ \omega}} + d_{\substack{\boldsymbol{k}\\ \omega}} f_{\substack{\boldsymbol{k}\\ \omega}} = \left(\frac{\mathrm{d}S}{\mathrm{d}\phi}\langle f\rangle + \bar{f}_{\substack{\boldsymbol{k}\\ \omega}}\right)\phi_{\substack{\boldsymbol{k}\\ \omega}}. \tag{4.44}$$

As in 1D, both resonance broadening and background distribution renormalization are non-Markovian. Finally, note that yet another way to argue for the existence of $\bar{f}_{\boldsymbol{k},\omega}$ is that both test particles and background particles (somewhat akin to field particles in collision theory) are scattered by the ensemble of fluctuations.

Given the motivation, we first check that the renormalized $N_{\boldsymbol{k},\omega}$ as given by Eq.(4.43a), satisfies the two properties. First, using Eq.(4.43) we can easily show that,

$$\sum_{\boldsymbol{k},\omega} \phi_{\substack{-\boldsymbol{k}\\-\omega}} N_{\substack{\boldsymbol{k}\\ \omega}} = \sum_{\boldsymbol{k},\omega}\sum_{\boldsymbol{k}',\omega'} \frac{c^2}{B_0^2}\left(\boldsymbol{k}\cdot\boldsymbol{k}'\times z\right)^2 \left\{\left|\phi_{\substack{\boldsymbol{k}'\\ \omega'}}\right|^2 \left(\phi_{\substack{-\boldsymbol{k}\\-\omega}} f_{\substack{\boldsymbol{k}\\ \omega}}\right) - \left|\phi_{\substack{\boldsymbol{k}\\ \omega}}\right|^2 \left(\phi_{\substack{-\boldsymbol{k}'\\-\omega'}} f_{\substack{\boldsymbol{k}'\\ \omega'}}\right)\right\} = 0, \tag{4.45}$$

by anti-symmetry under $\boldsymbol{k},\omega \leftrightarrow \boldsymbol{k}',\omega'$. Thus, we see that the renormalization is consistent with $\left\langle \boldsymbol{E}_\perp^* \cdot \boldsymbol{J}_\perp \right\rangle = 0$. Second, it is straightforward to show that $\lim_{f\to f_{\mathrm{B}}} N_{\boldsymbol{k},\omega} \to 0$, so the renormalized nonlinearity vanishes in the limit of the Boltzmann response. Hence, we see that the perturbative renormalization procedure respects both properties of $N_{\boldsymbol{k},\omega}$, as it should. The presence of the background renormalization $\bar{f}_{\boldsymbol{k},\omega}$ is essential to this outcome! For the sensitive and subtle case of drift-kinetic turbulence, the perturbative renormalization approach, derived from the idea of extracting the piece of $N_{\boldsymbol{k},\omega}$ phase coherent with $\alpha_{\boldsymbol{k},\omega}$, clearly is more successful than is resonance broadening theory.

4.4.5 Physics content and predictions

Having addressed some of the questions concerning the formal structure of the renormalized theory of drift wave turbulence, we now turn to more interesting issues of physics content and predictions. Of course, the principal goals of *any* renormalized theory of plasma turbulence, in general, or of drift wave turbulence, in particular, are:

1. to identify and understand nonlinear space-time scales;
2. to identify the relevant nonlinear saturation mechanisms and calculate the corresponding nonlinear damping rates;
3. to identify and predict possible bifurcations in the saturated state.

In the context of the specific example of drift-kinetic turbulence, goals (1) and (2) may be refined further to focus on the specific questions:

1. What is the *physical* meaning of the decorrelation rate and propagator renormalization $d_{\boldsymbol{k},\omega}$, and how is it related to mixing, transport and heating?
2. What is the rate of nonlinear ion heating? Note that ion heating is required for saturation of drift wave turbulence in order to balance energy input from electron relaxation.

Answering and illuminating these two questions is the task to which we now turn.

4.4.5.1 Propagator renormalization and mixing

The propagator renormalization $d_{\boldsymbol{k},\omega}$ is a measure of the rate at which the response $f_{\boldsymbol{k},\omega}$ to a test wave fluctuation $E_{\boldsymbol{k},\omega}$ is mixed or scrambled by the ensemble of turbulent fluctuations. The term $d_{\boldsymbol{k},\omega}$ is defined recursively, i.e.

$$d_{\boldsymbol{k},\omega} = \sum_{\boldsymbol{k}',\omega'} \frac{c^2}{B_0^2} \left|\phi_{\boldsymbol{k}',\omega'}\right|^2 \left(\boldsymbol{k}\cdot\boldsymbol{k}'\times z\right)^2 \frac{i}{(\omega+\omega') - \left(k_z + k_z'\right) v_z + i d_{\boldsymbol{k}+\boldsymbol{k}',\omega+\omega'}}, \tag{4.46}$$

since $d_{\boldsymbol{k},\omega}$ results from the beat interactions of the test wave with background modes which themselves undergo turbulent decorrelation. (A method based on the recursion is explained in Chapter 6, where the closure model is discussed.) For simplicity, we further specialize to the isotropic turbulence, long wavelength, low frequency limit where,

$$d_{\boldsymbol{k},\omega} \to k^2 D = \sum_{\boldsymbol{k},\omega} \frac{c^2}{B_0^2} \left|\phi_{\boldsymbol{k}',\omega'}\right|^2 \left(\boldsymbol{k}\cdot\boldsymbol{k}'\times z\right)^2 \frac{i}{\omega' - k_z' v_z + i d_{\boldsymbol{k}',\omega'}}. \tag{4.47}$$

This decorrelation rate corresponds to 'resonance broadened' quasi-linear theory. Function $d_{\boldsymbol{k},\omega}$ behaves rather differently in the weak and strong turbulence regimes, i.e. if $|(\mathrm{d}\omega/\mathrm{d}k_\perp)\,\Delta k_\perp| + \left|\left(v_{\mathrm{gr}\parallel} - \omega/k_\parallel\right)\Delta k_\parallel\right| > d_{\boldsymbol{k}',\omega'}$ or $< d_{\boldsymbol{k}',\omega'}$, respectively. In the first case, which corresponds to weak turbulence theory and resembles the simple quasi-linear prediction,

$$\begin{aligned}\operatorname{Re} d_{\boldsymbol{k},\omega} &= \sum_{\boldsymbol{k}',\omega'} \frac{c^2}{B_0^2} \left|\phi_{\boldsymbol{k}',\omega'}\right|^2 \left(\boldsymbol{k}\cdot\boldsymbol{k}'\times z\right)^2 \pi\delta\left(\omega' - k_z' v_z\right) \\ &= k^2 \left\langle \tilde{v}^2 \right\rangle \tau_{\mathrm{ac}}.\end{aligned} \tag{4.48}$$

Here, the irreversibility inherent to quasi-linear diffusion results from wave–particle resonance and the auto-correlation time τ_{ac} is just $(|(\mathrm{d}\omega/\mathrm{d}k_\perp)\,\Delta k_\perp| + \left|\left(v_{\mathrm{gr}\parallel} - \omega/k_\parallel\right)\Delta k_\parallel\right|)^{-1}$, which is determined by the spectrum and the wave dispersion properties. As is usual for such cases, $D \sim \langle\tilde{v}^2\rangle$, and the physical meaning of $d_{\boldsymbol{k},\omega}$ is simply decorrelation due to resonant diffusion in space. However, on drift wave turbulence k_z is often quite small, so as to avoid the strongly stabilizing effects of ion Landau damping. With this in mind, it is interesting to examine the opposite limit, where $k_z' v_z$ is negligible but propagator broadening is retained. Assuming spectral isotropy, assuming a spectrum of eigenmodes where $\omega = \omega\left(\boldsymbol{k}\right)$ and ignoring the $\boldsymbol{k}'$, ω' dependence of $d_{\boldsymbol{k}',\omega'}$ then gives,

$$k^2 D \cong \sum_{\boldsymbol{k}'} \frac{c^2}{B_0^2} \left|\tilde{\phi}_{\boldsymbol{k}'}\right|^2 k^2 k'^2 \frac{k'^2 D}{\omega'^2 + \left(k'^2 D\right)^2}, \tag{4.49a}$$

which reduces to,

$$1 \cong \sum_{\boldsymbol{k}'} \frac{c^2}{B_0^2} \left|\tilde{\phi}_{\boldsymbol{k}'}\right|^2 \frac{k'^4}{\omega'^2 + \left(k'^2 D\right)^2}, \tag{4.49b}$$

with understanding a simple scaling as the goal in mind, we throw caution to the winds and boldly pull the right-hand side denominator of Eq.(4.49b) outside the mode summation to obtain,

$$\left(k^2 D\right)^2 + \omega^2 \cong k^2 \left\langle \tilde{v}_{E\times B}^2 \right\rangle, \tag{4.49c}$$

or equivalently,

$$\left(k^2 D\right)^2 \cong k^2 \left\langle \tilde{v}_{E\times B}^2 \right\rangle - \omega^2. \tag{4.49d}$$

Equation (4.49d) finally reveals the physical meaning of D in the limit where $d_{\boldsymbol{k},\omega} > |k_z v_z|$, since it relates the mean squared decorrelation rate $\left(k^2 D\right)^2$ to the *difference* of the mean squared $E \times B$ Doppler shift $\left(k^2 \langle\tilde{v}_{E\times B}^2\rangle\right)$ and the mean squared wave frequency $\langle \omega^2 \rangle$. Of course, since the electric field is turbulent, the $E \times B$ Doppler shift is stochastic, and we tacitly presume the second moment $\langle \tilde{v}_{E\times B}^2 \rangle$ is well defined. Thus, Eq.(4.49d) states that there is a critical level of fluctuating $\tilde{v}_{E\times B}$ needed to scatter a test particle *into* resonance, and so render $D \neq 0$. That level is $(\tilde{v}_{E\times B})_{\rm rms} \sim \omega/k_\perp$, i.e. a stochastic perpendicular velocity which is comparable to the perpendicular phase velocity of the drift wave. The particle in question is called a 'test particle', since nowhere is the field forced to be self-consistent by the imposition of quasi-neutrality.

Given that $(\tilde{v}_{E\times B})_{\rm rms} \sim \omega/k_\perp$ defines the threshold for stochastization or mixing of a test particle, it is natural to discuss the relationship of this criterion to the familiar "mixing length estimate" for turbulence saturation levels. Note that for drift waves, the stochastic Doppler resonance criterion becomes $\tilde{v}_{E\times B} \sim \omega/k_\perp \sim V_{de}$. Since $\tilde{v}_{E\times B} \sim k_\perp \rho_s c_s \left(|e|\,\phi/T\right)$ and $V_{de} = \rho_s c_s/L_n$, $\tilde{v}_{E\times B} \sim V_{de}$ occurs for fluctuation levels of $e\tilde{\phi}/T \sim 1/k_\perp L_n$, which, noting that $\tilde{n}/n \sim e\tilde{\phi}/T$ for electron drift waves, is *precisely* the traditional mixing length estimate of the saturation level. This occurrence is not entirely coincidental, as we now discuss.

Basically, all of the standard drift wave type plasma instabilities are *gradient driven*, and (in the absence of external drive) tend to radially mix or transport the driving gradient, and so *relax* or *flatten* the gradient thus turning off the gradient drive. Thus, electron drift waves tend to mix density n or electron temperature $T_{\rm e}$ and so to relax ∇n and $\nabla T_{\rm e}$, ion temperature gradient driven modes tend to mix T_i and relax ∇T_i, etc. The essence of the mixing length estimate is that the growth of an instability driven by a local gradient will cease when the 'mixing term' or nonlinearity grows to a size which is comparable to the gradient drive. Thus, if one considers advection of density in the context of a drift wave, the density fluctuation would satisfy an equation with the generic structure,

$$\frac{\partial n}{\partial t} + \tilde{v}_{E\times B} \cdot \nabla n + \cdots = -v_{E\times B,r} \frac{\partial \langle n \rangle}{\partial r}. \tag{4.50a}$$

A concrete example of such a balance would be that between the linear term $(\tilde{v}_r\, \partial \langle n \rangle / \partial r)$ and the nonlinear term $(\tilde{v}_r\, \partial \tilde{n}/\partial r)$ of Eq.(4.50a), which yields,

$$\frac{\tilde{n}}{\langle n \rangle} \sim \frac{l_\perp}{L_n}, \tag{4.50b}$$

which is the conventional mixing length 'estimate' of the saturation level. Note that Eq.(4.50b) relates the density fluctuation level $\tilde{n}/n$ to the ratio of two length scales, namely $l_\perp$, the "mixing length" and L_n, the density gradient scale length. This then begs the question of just what precisely *is* the mixing length $l_\perp$. Intuitively speaking, it is the length over which a fluid or plasma element is scattered by instability-induced fluctuations. The mixing length, $l_\perp$ is often thought to correspond to the width of a typical convection cell, and motivated by concerns of calculation, is frequently estimated by the radial wavelength of the underlying linear instability. We emphasize that this is *purely* an approximation of convenience, and that there is absolutely *no* reason why $l_\perp$ for a state of fully developed turbulence should be tied to the scale of the original linear instability. In general, $l_\perp$ is unknown, and the accuracy to which it can be calculated varies closely with the depth of one's understanding of the fundamental nonlinear dynamics. For example, in the core of Prandtl mixing length theory discussed in Chapter 2, the choice of the distance to the wall as the mixing length most likely was motivated by an appreciation of the importance of self-similarity of the mean velocity gradient and the need to fit empirically determined flow profiles. Thus, mixing length theory should be considered *only* as a guideline for estimation, and practitioners of mixing length theory should keep in mind the old adage that "mixing length theory is always correct, *if* one knows the mixing length". Finally, we should add that local mixing length estimation, of the form described above, is also based upon the tacit presumption that $l_\perp \ll L_n$, so there are many cells within a gradient scale length. In this sense, the system is taken to be more like a sandpile than Rayleigh–Benard convection in a box, which is dominated by a single big convection cell, so $l_\perp \sim L_n$. Non-local mixing length models have been developed, and resemble in structure those of flux-limited transport, where $l_{\rm mfp} \sim L_n$. However, applications of such models has been limited in scope. (The multiple-scale problem is discussed in Chapter 7 and in Volume 2. The interested reader should also see the discussion in Frisch (1995).)

4.4.5.2 Nonlinear heating and saturation mechanism

We now turn to the second physics issue, namely that of nonlinear heating and the saturation mechanism. In this regard, it is useful to recall the key points of the previous discussion, which were:

(a) while turbulent test particle scattering and decorrelation occur at a rate given by $k_\perp^2 D$, where for $k_z \to 0$, $D \neq 0$ requires that a threshold in intensity be exceeded, i.e. $k^2 \left\langle \tilde{v}_{\rm E\times B}^2 \right\rangle \gtrsim \left\langle \omega^2 \right\rangle$;

(b) actual *heating* occurs only via *parallel* $\boldsymbol{E} \cdot \boldsymbol{J}$ work, i.e. $\left\langle E_\parallel J_\parallel \right\rangle$, since the energy density satisfies the conservation equation (compare Eq.(3.62a)),

$$\frac{\partial E}{\partial t} + \frac{\partial Q}{\partial x} + \langle E_z J_z \rangle = 0,$$

this is a consequence of the fact that $\langle E_\perp \cdot J_\perp \rangle = 0$ for drift kinetics.

Thus, *any nonlinear heating which leads to saturation **must** be in proportion to a power of* k_z, since $\langle E_z J_z \rangle \rightarrow 0$ for $k_z \rightarrow 0$. Equivalently, any 'action' from the nonlinearity to saturate the turbulence occurs via k_z.

4.4.5.3 Description by moments of the drift-kinetic equation

Given this important observation, and the fact that $k_z v_{\mathrm{Ti}}/\omega < 1$ for drift wave turbulence, it is useful to work from moments of the drift-kinetic equation, i.e. Eq.(4.26c). Assuming a Maxwellian $\langle f \rangle$, the drift-kinetic equation may be written as,

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B_0} \nabla \phi \times z \cdot \nabla f = -i \left(k_z v_z - \omega_{*i} \right) \frac{|e|\, \phi}{T_{\mathrm{i}}} \langle f \rangle . \tag{4.51}$$

Then the relevant moments are:

$$n = \int \mathrm{d}^3 v \, f \qquad \text{: density} \tag{4.52a}$$

$$J = |e| \int \mathrm{d}^3 v \, v_z f \qquad \text{: parallel current} \tag{4.52b}$$

$$p = m \int \mathrm{d}^3 v \, v^2 f \qquad \text{: energy} \tag{4.52c}$$

and satisfy the fluid equations,

$$\frac{\partial n}{\partial t} + \frac{1}{|e|} \frac{\partial J}{\partial z} - \frac{c}{B_0} \nabla \phi \times z \cdot \nabla n = \frac{c}{B_0} \nabla \phi \times z \cdot \nabla n_0 \tag{4.53a}$$

$$\frac{\partial J}{\partial t} + \frac{\partial\, |e|}{\partial z} \frac{p}{m} + \frac{e^2}{m} n \frac{\partial \phi}{\partial z} - \frac{c}{B_0} \nabla \phi \times z \cdot \nabla J = 0 \tag{4.53b}$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial z} nm \overline{v_z v^2} - E_z J_z - \frac{c}{B_0} \nabla \phi \times z \cdot \nabla p = 0. \tag{4.53c}$$

Equations (4.53a)–(4.53c) may be further simplified by noting that,

$$\frac{\tilde{n}}{n} \simeq \frac{|e|\, \phi}{T_{\mathrm{e}}}, \tag{4.54a}$$

$$p = m \frac{v_{\mathrm{Ti}}^2}{2} n, \tag{4.54b}$$

so

$$-\frac{c}{B_0} \nabla \phi \times z \cdot \nabla n \rightarrow 0 \tag{4.54c}$$

$$-\frac{c}{B_0} \nabla \phi \times z \cdot \nabla p \rightarrow 0. \tag{4.54d}$$

In this limit, the density equation is *strictly linear*, so the system of fluid equations reduces to

$$(\omega - \omega_{*e}) \frac{|e|\,\phi_{\boldsymbol{k}}}{T} = \frac{k_z}{n_0\,|e|} J_{z\,\boldsymbol{k},\omega} \tag{4.55a}$$

and

$$J_{z\,\substack{\boldsymbol{k}\\ \omega}} = \frac{k_z v_{\mathrm{Ti}}^2}{\omega} \frac{n_0\,|e|^2}{T} \phi_{\boldsymbol{k},\omega} + \frac{-i}{\omega} \sum_{\boldsymbol{k}',\omega'} \frac{c}{B_0} \left(k \cdot k' \times z\right) \phi_{-\boldsymbol{k}'-\omega'} J_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}. \tag{4.55b}$$

Linear theory tells us that the waves here are drift-accoustic modes, with

$$\omega = \omega_{*e} + \frac{k_z^2 v_{\mathrm{Ti}}^2}{\omega}. \tag{4.56}$$

This structure, and the fact that ion heating requires finite k_z, together, strongly suggest that *shear viscosity of the parallel flow will provide the requisite damping*. Since turbulent shear viscosity results from $\boldsymbol{E} \times \boldsymbol{B}$ advection of J, we now focus on the renormalization of the current equation.

In $\boldsymbol{k}$, ω space, the equation for parallel flow or current is,

$$-i\omega J_{\substack{\boldsymbol{k}\\ \omega}} + \frac{c}{B_0} \sum_{\boldsymbol{k}',\omega'} \left(\boldsymbol{k} \cdot \boldsymbol{k}' \times z\right) \left\{ \phi_{\substack{-\boldsymbol{k}'\\ -\omega'}} J^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} - \phi^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} J_{\substack{-\boldsymbol{k}'\\ -\omega'}} \right\} = -i \frac{k_z}{\omega} v_{\mathrm{Ti}}^2 n_0 \frac{|e|^2}{T} \phi_{\substack{\boldsymbol{k}\\ \omega}}. \tag{4.57}$$

As before, since we are concerned with heating, we focus on nonlinear wave–particle interaction, and so neglect the $\phi^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}$ contribution. For $J^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}$ we can then immediately write,

$$\left(-i\left(\omega+\omega'\right) + d_{\substack{\boldsymbol{k}''\\ \omega''}}\right) J^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = \frac{c}{B_0} \left(\boldsymbol{k} \cdot \boldsymbol{k}' \times \boldsymbol{z}\right) \left(\phi_{\substack{\boldsymbol{k}'\\ \omega'}} J_{\substack{\boldsymbol{k}\\ \omega}} - J_{\substack{\boldsymbol{k}'\\ \omega'}} \phi_{\substack{\boldsymbol{k}\\ \omega}}\right), \tag{4.58a}$$

so

$$J^{(2)}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = L_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} \frac{c}{B_0} \left(\boldsymbol{k} \cdot \boldsymbol{k}' \times \boldsymbol{z}\right) \left(\phi_{\substack{\boldsymbol{k}'\\ \omega'}} J_{\substack{\boldsymbol{k}\\ \omega}} - J_{\substack{\boldsymbol{k}'\\ \omega'}} \phi_{\substack{\boldsymbol{k}\\ \omega}}\right), \tag{4.58b}$$

where

$$L^{-1}_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} = -i\left(\omega+\omega'\right) + d_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} \tag{4.58c}$$

is the propagator. Substitution of Eq.(4.58b) into Eq.(4.57) then gives the renormalized parallel flow or current equations as,

$$\left(-i\omega + d_{\boldsymbol{k},\omega}\right) J_{\substack{\boldsymbol{k}\\ \omega}} = i\frac{k_z}{\omega} v_{\mathrm{Ti}}^2 n_0 \frac{|e|^2}{T}\phi_{\boldsymbol{k},\omega} + \beta_{\boldsymbol{k},\omega}\phi_{\boldsymbol{k},\omega} \tag{4.59a}$$

$$d_{\boldsymbol{k},\omega} = \sum_{\boldsymbol{k}',\omega'} \left(k \cdot k' \times z\right)^2 \frac{c^2}{B_0^2} \left|\phi_{\substack{\boldsymbol{k}'\\ \omega'}}\right|^2 L_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}} \tag{4.59b}$$

$$\beta_{\boldsymbol{k},\omega} = \sum_{\boldsymbol{k}',\omega'} \left(\boldsymbol{k} \cdot \boldsymbol{k}' \times z\right)^2 \frac{c^2}{B_0^2} \phi_{\substack{-\boldsymbol{k}'\\ -\omega'}} J_{\substack{\boldsymbol{k}'\\ \omega'}} L_{\substack{\boldsymbol{k}+\boldsymbol{k}'\\ \omega+\omega'}}. \tag{4.59c}$$

Furthermore, since both linear response theory and more general considerations of energetics suggest that $J_{z\,\boldsymbol{k}} \sim k_z \phi_{\boldsymbol{k}}$, we have,

$$\beta_{\boldsymbol{k},\omega} \approx \sum_{\boldsymbol{k}',\omega'} \left(\boldsymbol{k}' \cdot \boldsymbol{k}' \times z\right)^2 \frac{c^2}{B_0^2} \left|\phi_{\boldsymbol{k}',\omega'}\right|^2 k_z L_{\substack{\boldsymbol{k}+\boldsymbol{k}''\\ \omega+\omega''}} \\ \rightarrow 0, \tag{4.60}$$

since the integrand is odd in k_z. Thus, we see that Eq.(4.59a) simplifies to,

$$\left(-i\omega + d_{\boldsymbol{k},\omega}\right) J_{\boldsymbol{k},\omega} = -ik_z v_{\mathrm{Ti}}^2 n_0 \frac{|e|^2}{T}\phi_{\boldsymbol{k},\omega}. \tag{4.61}$$

Note this simply states that in a system of drift wave turbulence, the response of the parallel flow is renormalized by a shear viscosity, and that this is the leading-order nonlinear effect. Combining Eq.(4.61) and Eq.(4.55a) then gives,

$$\omega - \omega_{*e} = \frac{k_z^2 v_{\mathrm{Ti}}^2}{\omega + d_{\boldsymbol{k},\omega}}, \tag{4.62a}$$

so for low or moderate fluctuation levels we find,

$$\omega \cong \omega_{*e} + \frac{k_z^2}{\omega} v_{\mathrm{Ti}}^2 d_{\boldsymbol{k},\omega}, \tag{4.62b}$$

which says that turbulent dissipation is set by the product of the shear viscous mixing rate and the hydrodynamic factor $k_z^2 v_{\mathrm{Ti}}^2/\omega^2$. Note that nonlinear damping enters in proportion to the non-resonance factor $k_z^2 v_{\mathrm{Ti}}^2/\omega^2$, since the only heating which can occur is *parallel heating*, $\sim \langle E_z J_z \rangle$. Also note that since $k_z^2 v_{\mathrm{Ti}}^2/\omega^2 < 1$, the size nonlinear damping rate $\gamma_{\boldsymbol{k}}^{\mathrm{NL}}$ is $\left|\gamma_{\boldsymbol{k}}^{\mathrm{NL}}\right| < d_{\boldsymbol{k},\omega}$. Thus, the nonlinear damping rate is reduced relative to naive expectations by the factor $k_z^2 v_{\mathrm{Ti}}^2/\omega^2$. Finally for $k_z \rightarrow 0$, $\gamma_{\boldsymbol{k}}^{\mathrm{NL}} \rightarrow 0$, as it must.

This simple case of weakly resonant drift-kinetic turbulence is a good example of the subtleties of renormalized turbulence theory, energetics, symmetry, etc. Further application of related techniques to other propagator renormalization problems may be found in (Kim and Dubrulle, 2001; Diamond and Malkov, 2007). The moral of this story is clearly one that physical insight and careful consideration of dynamical constraints are essential elements of the application of any renormalized theory, no matter how formally appealing it may be.