Lecture I

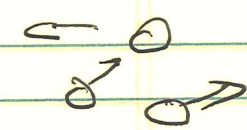
Plasma on a Back-of-an-Envelope

⇒ Basic Ideas

① What is a Plasma?
- dilute gas ($T \gg U_{int}$)
- net neutral ensemble of charged particles
- inter-particle interaction Coulombic
  - ⇒ long range
- screening

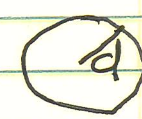
② Contrast: (Collisional) dilute gas

Scale ordering:



$$\boxed{d < \bar{r} < l_{mfp} < L}$$

- d → range of force (radius hard sphere)
- $\bar{r}$ → mean interparticle spacing
- l_{mfp} → mean free path
- L → system size



long mean free path:

$$d < \bar{r} < L \lesssim l_{mfp}$$

Key Orderings:

a) – $n d^3 \ll 1 \Leftrightarrow (d/\bar{r})^3 \ll 1$

→ diluteness – particles free, mostly non-interacting

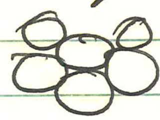
(b) $l_{mfp} = 1/n\sigma$

$$\cong \bar{r}\,(\bar{r}/d)^2$$

so

$$l_{mfp}/\bar{r} \sim (\bar{r}/d)^2 \gg 1$$

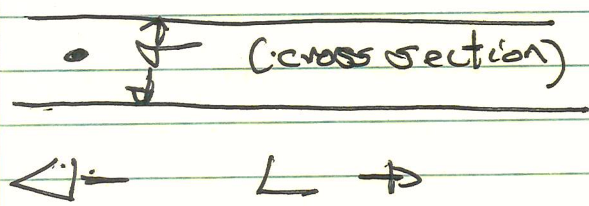
- collisions rare $\Leftrightarrow$ diluteness
- contrast liquid: $l_{mfp} \sim \bar{r}$



$l_{mfp}/d \sim (\bar{r}/d)^3 \rightarrow$ l_{mfp} exceeds force range

Aside: Why $l_{mfp} \sim 1/n\sigma$?

$\rightarrow$ define interaction cylinder for particle:



particle sweeps out cylinder of area σ, length L

$V_{int} = \sigma L$

if α = # collisions in cylinder of length L

then $\alpha = n \bar{V}_{int} = n\sigma L$

then mean length between collisions is

$l_{mfp} = L/\alpha = 1/n\sigma$.

$$\boxed{l_{mfp} = 1/n\sigma}$$

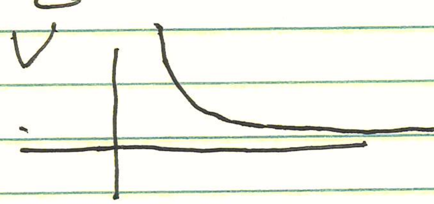
(c) $\nu_c \sim v_{th} / l_{mfp} \sim v_{th} n\sigma$.

collision frequency

(d) basic diffusivity (characteristic)

$D \sim v_{th} l_{mfp} \sim v_{th}^2 / \nu_c$

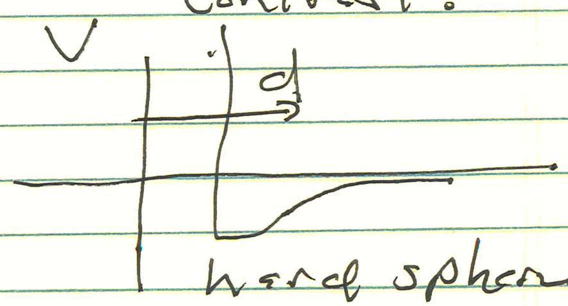
(3) For plasmas:



- force is Coulomb

$V = e\phi \sim 1/r$

Contrast:



scale free!
long range!

no "d" to characterize interaction.

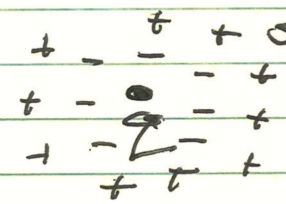
- but screening occurs:

Screening? - Enter the Debye length!

λ_D

• q_T

test charge



effect/field screened — plasma adjusts to screen charge

— needs energy T

so

$$1/r \to e^{-r/\lambda_D}/r$$

How? $\boxed{\text{Solve Poisson's Equation with Screening Response!}}$

$$\nabla^2\phi = -4\pi\rho$$

$$= -4\pi\left[\rho_{\text{screening response}} + 4\pi q_T\,\delta(\underline{x}-\underline{x}_T)\right]$$

$$= \underbrace{-4\pi n_0|e|\left[\frac{\delta n_i}{n_0} - \frac{\delta n_e}{n_0}\right]}_{\text{media/plasma response}} + 4\pi q_T\,\delta(\underline{x}-\underline{x}_T)$$

For density perturbation (response to test charge):

$\delta n_i / n_0 = \exp[-|e|\phi/T_i]$ $\quad k_B \to 1$

$\delta n_e / n_0 = \exp[+|e|\phi/T_e]$

so

charge neutrality

$$\nabla^2\phi = -4\pi n_0 |e| \left[\cancel{1} - \frac{|e|\phi}{T_i} - \cancel{1} - \frac{|e|\phi}{T_e} \right] + 4\pi q_T \delta(\underline{x} - \underline{x}_T)$$

so

$$\boxed{\nabla^2\phi = 4\pi n_0 |e|^2 \left(\frac{1}{T_i} + \frac{1}{T_e} \right)\phi + 4\pi q_T \delta(\underline{x} - \underline{x}_T)}$$

$\underline{x}_T \to 0$

note: T needed to screen

$$\boxed{\phi = \exp[-r/\lambda_D]/r}$$

→ screened Coulomb field

$$\boxed{1/\lambda_D^2 = 4\pi n_0 |e|^2 \left(1/T_e + 1/T_i \right)}$$

screening, Debye length.

$$\lambda_D^2 = \left[4\pi n_0 e^2 \left(1/T_i + 1/T_e \right) \right]^{-1}$$

~~Note~~ Debye length scale is key for plasma!

Which brings us the length scale ordering for plasmas:

(4)
$$\boxed{\bar{r} < \lambda_D < l_{mfp} < L}$$ (collisional)

(a)
Key Point: A plasma has many particles in Debye sphere!

i.e. $\boxed{n\lambda_D^3 \gg 1}$ $\Rightarrow$ $\lambda_D > \bar{r}$

$1/n\lambda_D^3 \ll 1$
$\Downarrow$
plasma parameter

Why? $\Rightarrow$ $\boxed{\text{Diluteness!}}$
$\rightarrow T > e^2/r$

i.e. - thermal energy must exceed electrostatic energy

- dilute plasma $\not\Leftrightarrow$ Crystal
(conceptual opposite)

– Checking:

$$T > e^2/\bar{r}$$

$$\frac{n T \bar{r}}{n e^2} > 1$$

$$\lambda_D^2 \bar{r} n > 1$$

$$\boxed{\lambda_D^2 > \bar{r}^2}$$ ✓

(b) Plasmas Classical !? (but QM tricks handy!)

Diluteness ⇒ Thermal fluctuations exceed QM fluctuations in energy

Thermal → T (energy)

QM → $\epsilon \sim p^2/2M$

$$\sim \hbar^2 k^2/2M$$

$$\sim \hbar^2/2m\bar{r}^2 \quad \sim \frac{\hbar^2}{m} n^{2/3}$$

so need:

$$T \gg \frac{\hbar^2}{m} n^{2/3}$$

have already for diluteness,

$$T \gg e^2 n^{1/3}$$

so, sufficient to have:

$$e^2 n^{1/3} > \hbar^2 n^{2/3}/m$$

⇒

$$e^2 n^{1/3} / \hbar^2 n^{2/3}/m \cong \frac{me^2}{\hbar^2 n^{1/3}} > 1$$

$$\Rightarrow \boxed{\frac{\bar{r}}{a_B} \gg 1}$$

$a_B = \hbar^2/me^2$ ($\to 4\pi\hbar^2/me^2$)

Bohr radius

- For classical plasma, mean interparticle spacing must exceed Bohr radius
- otherwise, quantum or degenerate plasma.

so Conditions for classical dilute plasma:

$$\boxed{\begin{array}{c} n\lambda_D^3 \gg 1 \\ \lambda_D^3/a_B \gg 1 \end{array}}$$

$$\boxed{\bar{r} < \lambda_D < l_{mfp} < L}$$

③ Frequencies / Resonances

- much of plasma physics deals with collective dynamics of plasmas
  $\Rightarrow$ waves / instabilities and their evolution ....

- collective resonances: (electrostatic) (EM generalization clear)

$$\nabla \cdot \underline{D} = 4\pi \rho_{ext}$$

dielectric function

$$\underline{D} = \underline{E} + 4\pi \underline{P} = \epsilon(\omega)\,\underline{E}$$

$\underline{P}$: polarization

$\epsilon(\omega) \Rightarrow \epsilon(\omega,k)$ more general

$\epsilon(\omega) \to 0 \Rightarrow$ collective resonance

Now,

- high frequency, so electrons respond (low inertia), ions stationary (high inertia)

$$\underline{P} = n e\, \delta\underline{x}$$

$$m_e \frac{d^2 \delta \underline{x}}{dt^2} = e\,\underline{E}$$

$$\left(\underline{E} = \underline{E}_0 \, e^{-i\omega t} \right)$$

$$-\omega^2 m_e \, \delta \underline{x} = e \underline{E}$$

$$\delta \underline{x} = -e \underline{E} / m_e \omega^2$$

$$4\pi \underline{P} = -\omega_{pe}^2 / \omega^2$$

$$\boxed{\underline{D} = \underline{E} + 4\pi \underline{P} = \left(1 - \frac{\omega_p^2}{\omega^2} \right) \underline{E} = \epsilon(\omega) \, \underline{E}}$$

$\boxed{\omega_{pe}^2 = 4\pi n_0 e^2 / m_e \Rightarrow \text{plasma frequency}}$ ← fundamental characteristic frequency.

→ space charge oscillation (no spatial dispersion)

$$\boxed{\frac{v_{th}^2}{\omega_{pe}^2} = \lambda_D^2}$$

→ $\delta n \Rightarrow \delta \underline{E}$ → restoring force.

Now, $\quad \underline{\nabla} \cdot \underline{D} = \left(1 - \omega_p^2 / \omega^2 \right) \underline{\nabla} \cdot \underline{E} = 4\pi \rho_{ext}$

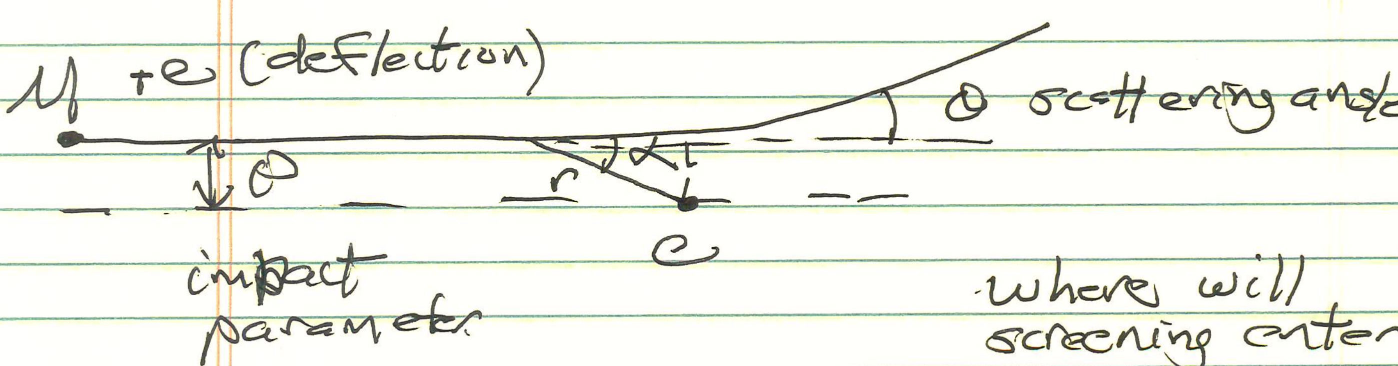
$$- \rho_{ext} = \rho_{ext}(\omega \sim \omega_{pe}) \nrightarrow 0$$

$\Rightarrow E$ response large $\Leftrightarrow \epsilon \to 0$

collective resonance!

⑥ Transport ⟷ Coulomb Collisions

- back to Rutherford scattering;



where will screening enter?

- Cross-section?
  - interaction is weak deflection
    "momentum transfer cross section)
  - <u>many</u> glancing collisions occur
  - central force, so
    p conserved, but direction changed.

for deflection:

$$M \Delta V_\perp = \Delta p_\perp = \int_{-\infty}^{+\infty} dt \, F_\perp$$

$$= \int_{-\infty}^{+\infty} dt \, \frac{e^2}{r^2} \sin\alpha$$

$$= \int_{-\infty}^{+\infty} dt \, \frac{e^2}{r^2} \frac{\rho}{r}$$

$$\Delta p_\perp = e^2 \int_{-\infty}^{+\infty} \frac{\rho \, dt}{(\rho^2 + v^2 t^2)^{3/2}}$$

$$\sim e^2 \rho \int_{-\infty}^{+\infty} \frac{1}{\rho^3} \frac{dt}{(1 + v^2/\rho^2 t^2)^{3/2}}$$

$$\sim e^2/\rho v$$

but $\Delta p_\perp \sim \mu v \sin\theta$ (defn.)

$\sim \mu v \theta$ weak defl.

so $\boxed{\theta \sim e^2/\mu v^2 \rho}$ scattering angle (weak deflection)

Now, $\underbrace{d\sigma = \rho \, d\rho}_{\text{interaction cylinder}} = d\left(e^2/\mu v^2 \theta\right)^2$.

N. B.: - Cross section heavily weights weak deflections

- long range character of force (weak, but many interactions)

$\rightarrow$ cylinder

so $$d\sigma = \left(\frac{e^2}{\mu v^2}\right)^2 \frac{d\Theta}{\Theta^3}$$

weak deflection divergence.

n.b. small $\Theta \Leftrightarrow$ large ρ (more cylinder volume).

- long ranged character of Coulomb force!

- screening will enter as cut-off.

Now, seek:

momentum transfer cross section

i.e. take out forward scattering, as no transfer.

so $$d\sigma_{+} = \underbrace{(1-\cos\Theta)}_{\text{removes forward scatt}} d\sigma$$

$$\simeq \Theta^2 \left(\frac{e^2}{\mu v^2}\right)^2 \Big/ \Theta^3$$

$$\boxed{d\sigma \simeq \left(\frac{e^2}{\mu v^2}\right)^2 \frac{1}{\Theta}}$$

if $\theta_0 \Rightarrow$ low angle cut-off
$\Rightarrow$ large b
$\rightarrow$ screening

$$\theta_0 \sim e^2/mv^2 b \sim e^2/mv^2 \lambda_D$$

Max range force, Debye Length.

and:

$$\boxed{\sigma_t \cong \left(\frac{e^2}{mv^2}\right)^2 \ln\left(1/\theta_0\right)}$$

(resolved systematically by dynamic b in orbits)

so

$$\boxed{\ln\Lambda} = \ln(1/\theta_0) = \ln\left(T\lambda_D/e^2\right)$$

$$\boxed{\sigma_t \sim \left(\frac{e^2}{T}\right)^2 \ln\Lambda}$$

screened Coulomb cross-section

Coulomb Logarithm

Note:

$$\boxed{\sigma_t \sim \bar{r}^2\left(e^2/\bar{r}T\right)^2 \ln\Lambda}$$

And now:

$$\boxed{\sigma_t \sim \bar{r}^2 \left(e^2/\bar{r}T\right)^2 \ln \Lambda}$$

$$\left(e^2/\bar{r}T\right)^2 \rightarrow \left(\left[1/n\lambda_D^3\right]^{2/3}\right)^2$$

$$\sim \bar{r}^4/\lambda_D^4$$

$$\sim 1/\left(n\lambda_D^3\right)^{4/3}$$

$$\boxed{\sigma_t \sim \bar{r}^2 \left(1/n\lambda_D^3\right)^{4/3} \ln \Lambda}$$

Check:

- $l_{mfp} \sim 1/n\sigma_t$

<u>so</u>

$$l_{mfp} \cong \bar{r}\left(\lambda_D/\bar{r}\right)^4/\ln \Lambda$$

$$l_{mfp}/\lambda_D \sim \left(\lambda_D/\bar{r}\right)^3/\ln \Lambda$$

$$\sim \left(n\lambda_D^3\right)/\ln \Lambda$$

$$n \lambda_D^3 \gg \ln \Lambda$$

$$\rightarrow \boxed{l_{mfp} \gg \lambda_D}$$

consistent with screening

$$\Rightarrow \boxed{\bar{r} < \lambda_D < l_{mfp} < L}$$

collisional plasma ordering

Further points about transport:

- no mass M scaling in τ_T, l_{mfp} (apart $\ln \Lambda$)
- so $\nu_{col} \sim M^{-1/2}$ $\rightarrow v_{th}$

$$\tau_{col} \sim M^{1/2}$$

$$\tau_{ee}/\tau_{ii} \sim (M_e/M_i)^{1/2} \ll 1.$$

electron collisions (ei) more rapid.

thing as before:

→ thermal conductivity

$$\lambda \sim n \, v_{th} \, l_{mfp}$$

n ↓ C_V; v_{th} ↓ bigger electrons; l_{mfp} ↓ indep n

⇒ electrons control thermal conductivity

→ viscosity

$$\eta_i \sim M_i n \, v_{th_i} \, l_{mfp}$$

$$l_{mfp} \sim 1/n\sigma$$

$$(\eta_e \sim m_e n \, v_{th_e} \, l_{mfp})$$

$$\eta \sim M \, v_{th_i} / \sigma_t$$

$$\eta \sim (M_i T)^{1/2} / \sigma_t$$

ions control flow.