Thermal Equilibrium Fluctuations

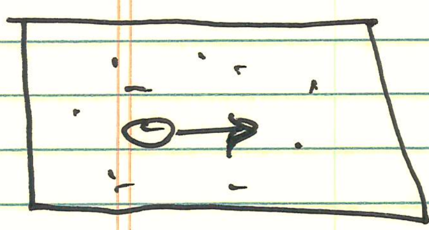
i.) Basic Ideas:

- dilute gas charged particles, quasi-neutral

  $T > e^2 n^{1/3} \quad \leftrightarrow \quad 1/n\lambda_D^3 \ll 1$

- ~ Thermal equilibrium fluctuations;

Balance: → fluctuation vs. dissipation, a la Brownian Motion
→ absorption vs. emission.



(i) emission

- discrete particle in Vlasov fluid emits waves

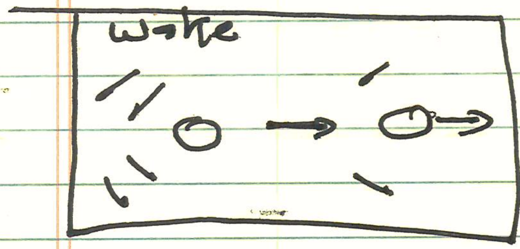
- Cerenkov emission, a la boat wake in water.

$$\underline{\nabla}\cdot\underline{D} = \underline{\nabla}\cdot\underline{\underline{\epsilon}}\,\underline{E} = 4\pi q\,\delta(x - \underline{x}(t))$$

- emission ⇒ fluctuation
- kinetic energy of particle ultimately coupled to collective response of the medium.
- particle slows down due to Cerenkov emission. → drag

(2) Cerenkov emitted waves damp via Landau damping

⇒ absorption



emitted waves scatter other particles.

So Thermal equilibrium fluctuations result from detailed balance of:

- Cerenkov emission waves from discrete particle motion.
- absorption of wave energy via Landau damping of Vlasov Fluid

Note: Particles are { schizophrenic? / double agent } :

- act as discrete particle emitter
- act as element of Vlasov fluid absorber.

Here:
- assume confined system → no outgoing waves, radiative damping
- $l_{damp} \sim \frac{\omega}{k} / |\gamma_{damp}| \ll L_{syst}$

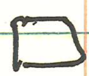
finite damping length

Double Agents:

- "emitter": a discrete particle moving along some specified (unperturbed) orbit

i.e.  an identifiable 'pea' in a 'pea soup' composed of other peas

- "absorber": an element of the Vlasov fluid responding to, and damping (by Landau resonance) other discrete particles.

i.e.  a 'crushed pea', element of the "pea soup" of the Vlasov Fluid.

$\therefore$

→ Thermal Equilibrium Plasma:

soup/gas of 'dressed' test particles + Vlasov fluid

⇒ Test Particle Model

Re: Pea soup model of Vlasov Fluid:

> "Every pea in the soup acts like (fluid) soup for all the other peas".

Useful Analogy:

Here: Brownian Motion $\leftrightarrow$

$$d\tilde{V}/dt + \beta \underline{V} = \underline{\tilde{f}}/M \qquad \beta = \gamma/M$$

	Brownian Motion	Plasma at Eqbm
Fluctuation	$\langle \tilde{V}^2 \rangle_\omega$	$\langle \tilde{E}^2 \rangle_{k,\omega}$
Excitation	ω — mode	$k, \omega(k)$ $\rightarrow$ plasma waves
Emission / Source	$\langle \tilde{f}^2 \rangle \rightarrow$ fluid thermal flctns	particle discreteness
Absorption	$\beta \rightarrow$ Stokes drag on particle	$\epsilon_{IM} \rightarrow$ Landau Damping of Collective Modes.

ii.) TPM for Fluctuation Spectrum

Basic ideas:

- dual identity as emitter and absorber
- fluctuations weak → unperturbed orbits valid (Contrast Turbulence).

A la Debye length calculation, with stationary ions:

$$\delta f = \underbrace{f^c}_{\substack{\text{coherent}\\ \text{Vlasov}\\ \text{response}}} + \underbrace{\tilde{f}}_{\substack{\text{discrete}\\ \text{source}}}$$

$$\delta f = \frac{\frac{|e|}{m} E_{k,\omega} \frac{\partial \langle f \rangle}{\partial v}}{-i(\omega - kv)} + \underbrace{|e|\, \delta(\underline{x} - \underline{x}(t))\, \delta(\underline{v} - \underline{v}(t))}_{\text{discreteness}}$$

i.e.

$$\underline{\nabla} \cdot \underline{D} = \underline{\nabla} \cdot \epsilon\, \underline{E} = 4\pi |e|\, \delta(\underline{x} - \underline{x}(t))$$

equiv to:

$$\nabla^2 \phi = 4\pi n_0 |e| \int dv\, \delta f = 4\pi n_0 |e| \int dv\, f^c + 4\pi n_0 |e| \int dv\, \tilde{f}$$

so

$$\hat{\phi}_{k,\omega} = \frac{4\pi n_0 |e|}{k^2} \int dv\, \tilde{f} \Big/ \epsilon(k,\omega)$$

(!?)

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

check:

$$\int dv\, \tilde{f}_{\underline{k}} = \int dv \int dx\, e^{-ikx} |e|\, \delta(x - x(t)))$$

$$\underline{x} = \cancel{\underline{x}_0} + \underline{v}t$$

$$\int dv\, \tilde{f}_{\underline{k}} = \int |e|\, e^{-ikvt}$$

so

$$\phi_k(t) = \underline{\underline{\epsilon}}^{-1}(k,t)\, \frac{4\pi n_0 |e|}{k^2} \int dv\, e^{-ikvt}$$

strictly:

$$\underline{\underline{\epsilon}}(\underline{k},t)\, \hat{\phi}_{\underline{k}}(t) = \frac{4\pi n_0 |e|}{k^2} \int dv\, e^{-ivt}$$

so

$$\phi_{\underline{k}}(t) = \underline{\underline{\epsilon}}^{-1}(\underline{k},t)\, \frac{4\pi n_0 |e|}{k^2}\, e^{-ikvt} + \phi_k^{\text{homog.}}\, e^{-i\omega_{\underline{k}} t}$$

$\omega_k = \omega_r(k) + i\,\omega_i(k)$ → eigenmode frequency.

Now:

- time asymptotically .....

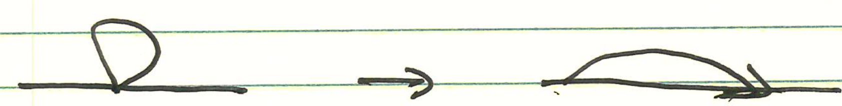
  $\omega_i < 0$ ⇒ collective modes (free response) damp.

- only discreteness driven flctns. survive

- moving toward, but not to marginal stability ⇒ $\tau_{relax} \to \infty$

- for sufficient source strength and/or weak enough damping, amplification to nonlinearity occurs

- if unstable modes require nonlinear damping to balance nonlinear noise (mode coupling)

i.e. 

so, for $\tilde{f}$ ≡ discreteness fluctn.

$$\boxed{\langle \hat{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2}\right)^2 \int dv_1 \int dv_2 \, \frac{\langle \tilde{f}(1) \tilde{f}(2) \rangle_{k,\omega}}{|\epsilon(k,\omega)|^2}}$$

→ thermal equilibrium fluctuation spectrum

→ all content → $\dfrac{\langle \tilde{f}^2 \rangle_{k,\omega}}{\epsilon(k,\omega)}$

What of Discreteness Noise?

$\langle \tilde{f}(1) \tilde{f}(2) \rangle$ ≡ discrete particle correlation function

will show:

$$\boxed{\langle \tilde{f}(1) \tilde{f}(2) \rangle = \text{[redacted]} \, \frac{\langle f \rangle}{n} \delta(\underline{x}_1 - \underline{x}_2) \, \delta(\underline{v}_1 - \underline{v}_2)}$$

iii) Discreteness Noise

For discreteness correlation function:

$$\tilde{f}(1) = \underbrace{\frac{1}{n}}_{\text{normalization}} \sum_{i=1}^{N} \delta(x_1 - x_i(t))\, \delta(v_1 - v_i(t))$$

$1 \equiv x_1, v_1 \rightarrow$ phase space point

$\tilde{f}(1) = \tilde{f}(x_1, v_1, t) \rightarrow$ discrete particle dist.

Can write $\tilde{f}(2)$, as well.

Now, for:

$$\langle A \rangle = \int dx_i \int dv_i \overset{\text{norm}}{\overset{\downarrow}{n}} \langle f \rangle A$$

i.e. avg. over distribution, uncorrelated particles. (uncorrelated ↔ dilute.)

$\Rightarrow$

$$\langle \tilde{f}(1) \tilde{f}(2) \rangle$$

$$= \int dx_i \int dv_i\, n \langle f \rangle \left(\frac{1}{n} \sum_{i=1}^{N} \delta(\underline{x}_1 - \underline{x}_i(t))\, \delta(\underline{v}_1 - \underline{v}_i(t)) \right)$$

$$* \left(\frac{1}{n} \sum_{j=1}^{N} \delta(\underline{x}_2 - \underline{x}_j(t))\, \delta(\underline{v}_2 - \underline{v}_j(t)) \right)$$

$$= \int dx_i \int dv_i \frac{\langle f \rangle}{n} \sum_{i,j=1}^{N} \Big[\delta(x_1 - x_i(t))\, \delta(x_2 - x_j(t)) * \delta(v_1 - \underline{v}_i)\, \delta(v_2 - \underline{v}_j) \Big]$$

$\langle \tilde{f}(1)\tilde{f}(2) \rangle \neq 0$ only if arguments interchangeable

so

$$\langle \tilde{f}(1)\,\tilde{f}(2)\rangle = \int dx_i \int dv_i \,\frac{\langle f\rangle}{n} \sum_{j\neq i}^{N} \Big[\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{x}_1'-\underline{x}_2')\,* \\ \delta(\underline{v}_1-\underline{v}_2)\,\delta(\underline{v}_1'-\underline{v}_2')\Big]$$

$$= \frac{\langle f\rangle}{n}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)$$

$$\boxed{\langle \tilde{f}(1)\,\tilde{f}(2)\rangle = \frac{\langle f\rangle}{n}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)}$$

- equal time discreteness correlation function.

- particles uncorrelated unless they are same particle. No width, no range.

Now, must obtain $\langle \tilde{f}(1)\,\tilde{f}(2)\rangle_{k,\omega}$

$\Rightarrow$ Transformed correlation function.

Now, need:

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_k = \int e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle$$

and for time transform:

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega} = \underbrace{\int_0^\infty d\tau\, e^{i\omega\tau}\, U(2,\tau)\int dx\, e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle}_{a}$$

$$+ \underbrace{\int_{-\infty}^0 d\tau\, e^{i\omega\tau}\, U(1,-\tau)\int e^{-ik(x_2-x_1)} \langle \tilde{f}(1)\tilde{f}(2)\rangle\, dx}_{b}$$

$$\langle \tilde{f}(1,t')\,\tilde{f}(2,t)\rangle = \langle \tilde{f}(1)\,\tilde{f}(2)\,[t-t']\rangle$$

so $\tau = t - t' \rightarrow$ can $\begin{cases} \text{push 2 forward } \tau \\ \text{push 1 back } \tau \end{cases}$

"push" $\equiv U \rightarrow$ operator evolving/pushing particle along unperturbed orbit

i.e.

$$U : \; x \rightarrow x + v\tau$$

so

(2) $\rightarrow$ (a)

$$(a) = \int_0^\infty d\tau \int e^{-ik(x_2-x_1)}\, e^{i(\omega - kv_2)} \langle \tilde{f}(1)\tilde{f}(2)\rangle\, dx$$

$$= \int_0^\infty d\tau\, e^{i(\omega-kv_2)\tau} \langle \tilde{f}(1)\,\tilde{f}(2)\rangle_k$$

$\otimes$

$$ⓐ = \frac{-1}{i(\omega - kv_2)} \langle \tilde{f}\tilde{f} \rangle_{k}$$

$$= \frac{i}{(\omega - kv_2)} \langle \tilde{f}\tilde{f} \rangle_{k}$$

real part

$$ⓐ = \pi \delta(\omega - kv_2) \langle \tilde{f}\tilde{f} \rangle_{k}$$

Similarly,

$$ⓑ = \pi \delta(\omega - kv_1) \langle \tilde{f}\tilde{f} \rangle_{k}$$

Now, ultimately seek:

$$\int dv_1 \int dv_2 \, \langle \tilde{f}(1)\tilde{f}(2) \rangle_{k,\omega}$$

$$\int dv_1 \int dv_2 = \int dv_+ \int dv_-$$

cm relative

$$\langle \tilde{f}(1)\tilde{f}(2) \rangle = \frac{\langle f \rangle}{n} \delta(x_-)\delta(v_-)$$

so

$$\boxed{\int dv_- \, \langle \tilde{f}(1)\, \tilde{f}(2) \rangle = 2\pi \frac{\langle f \rangle}{n} \delta(\omega - k v_+)}$$

and,

$$\left\langle \frac{\tilde{n}}{n_0} \frac{\tilde{n}}{n_0} \right\rangle_{k,\omega} = \int dv_+ \frac{\langle f \rangle}{n} 2\pi \delta(\omega - k v_+)$$

$$\equiv \underbrace{C(k,\omega)}_{\text{emission correlator}}$$

and so

v_{the} extracted.

$$C(k,\omega) = \frac{2\pi}{n|k|\, v_{the}} \left\langle \bar{f}(\omega / k v_{the}) \right\rangle$$

$$= \frac{2\pi}{n} \underbrace{\tilde{\tau}_{str}}_{\substack{\text{streaming time} \\ \sim (1/|k| v_{the})}} \left\langle \bar{f}(\omega / k v_{the}) \right\rangle$$

$\Rightarrow$ discreteness noise has Maxwellian Doppler spectrum (obvious).

so, Thermal Equilibrium Spectrum

$$\boxed{\langle\hat{\phi}^2\rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2}\right)^2 \frac{1}{n_0} \frac{2\pi}{|k| v_{the}} \frac{\langle \bar{f}(\omega/k v_{the})\rangle}{|\epsilon(k,\omega)|^2}}$$

Spectrum set by:

- particle emission distribution
  $\sim e^{-\omega^2/k^2 v_{the}^2}$

- collective screening, i.e.
  $\omega > k v_{the}$, $\quad \epsilon \cong 1 - \frac{\omega_p^2}{\omega^2} + i\epsilon_{IM}$

  $\omega < k v_{the}$, $\quad \epsilon = 1 + 1/k^2\lambda_D^2 + i\epsilon_{IM}$

- Coulomb factor $\left(\frac{4\pi n_0 |e|}{k^2}\right)$

- $\tau_{str} \rightarrow$ (time transform)

N.B:

- Collective response strongest at wave resonance ($\omega_p \sim \omega \sim kv$)

  $\Rightarrow$ expect peak.

- $\omega \gg \omega_p$, $\rightarrow$ high frequency limit

  Noise decouples from collective dynamics

  $\epsilon \to 1$

$$\langle \tilde{\phi}^2 \rangle_{k,\omega} \underset{\omega \gg \omega_p}{\cong} n_0 \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{2\pi}{|k| v_{the}} e^{-\omega^2/k^2 v_{the}^2}$$

- $\omega < \omega_p$ $\rightarrow$ low frequency limit

  $\Rightarrow$ static plasma screening, expect

$$\langle \phi^2 \rangle_{k\omega} \underset{\omega \ll \omega_p}{\cong} n_0 (4\pi |e|)^2 \frac{2\pi}{|k| v_{the}} \frac{e^{-\omega^2/k^2 v_{the}^2}}{\left(k^2 + 1/\lambda_D^2\right)^2}$$

Now, can write:

$$\frac{\langle E^2\rangle_{k,\omega}}{8\pi} = \frac{4\pi^2 n_0 |e|^2}{k^2 |k|} \frac{F(\omega/k v_{the})}{|\epsilon_r|^2 + |\epsilon_{IM}|^2}$$

$$F(\omega/k v_{the}) \equiv \langle \bar{f}\rangle / v_{Th}\sqrt{2\pi}$$

re-absorb factor.

$F = F(u)$
$F' = dF/du$
$u = \omega/k$

$$\epsilon_{IM} = \frac{\pi \omega_{pe}^2}{|k| k} F'$$

Now useful to make contact with usual $k_B T/2$ per d-o-f.

$$W_k = \int \frac{d\omega}{2\pi} \frac{\langle E^2\rangle_{k,\omega}}{8\pi}$$

How do frequency integral:

Pole Approximation!

$$\frac{1}{|\epsilon|^2} \cong 1 \Big/ \left[(\omega-\omega_k)^2 \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}^2 + |\epsilon_{IM}|^2\right]$$

real frequency sets location pole

→ width.

so

$$\frac{1}{|\epsilon|^2} = \frac{1}{|\epsilon_{Im}|}\left\{\frac{|\epsilon_{Im}|}{(\omega-\omega_k)^2\left|\frac{\partial\epsilon}{\partial\omega}\right|^2+|\epsilon_{Im}|^2}\right\}$$

$$= \frac{1}{|\epsilon_{Im}|}\left|\frac{\partial\epsilon}{\partial\omega}\right|^{-1}\pi\delta(\omega-\omega_k)$$

so integrating:

$$W_k = \frac{m_e\omega_{pe}}{2|k|}F/|F'|$$

$$= \frac{m_e\omega_{pe}}{2|k|}F\bigg/\frac{\omega_{pe}F}{|k|v_{The}^2} = T/2$$

so $\boxed{W_k = k_bT/2}$ ✓ for $k\lambda_D < 1$

→ but for $k\lambda_D \gg 1$

$$\epsilon \to 1 + 1/k^2\lambda_D^2$$

no collective resonance.

so

$$W_k \cong \frac{k_b T}{2} \Big/ k^2 \lambda_D^2 \qquad \text{for } k\lambda_D > 1$$

↳ strong cut-off beyond λ_D.

Now, for total energy density:
(3D)

$$\langle E^2/8\pi \rangle = \int d\underline{k}\; W_{\underline{k}}$$

$$\cong \left(\frac{k_b T}{2}\right) k_{max}^3$$

$k_{max} \sim 1/\lambda_D \rightarrow$ screening kills $k\lambda_D > 1$

$$\langle E^2/8\pi \rangle = n\left(\frac{k_b T}{2}\right) \Big/ n\lambda_D^3$$

$$= (KED) \Big/ \underbrace{n\lambda_D^3}$$

→ diluteness
→ # in Debye sphere.

KED: kinetic energy density.

so

$$\text{Field Energy Density} \sim \frac{\text{Kinetic Energy Density}}{n\lambda_D^3}$$

↳ diluteness factor.

so $\frac{FED}{KED} = 1/n\lambda_D^3$, as should.

→ Connect to Fluctuation - Dissipation Theorem. (Notation reverts).

Recall: $G_{IM} = -\frac{\omega_p^2 \pi}{|k| k} \frac{\partial \langle f \rangle}{\partial v}\Big|_{\omega/k}$

$$= \frac{2\pi\omega}{k^2 v_{tho}^2} \frac{\omega_p^2}{|k| v_{tho}} \langle \bar{f}(\omega/k) \rangle$$

Maxwellian

$\langle \bar{f}(\omega/k) \rangle = k^2 v_{tho}^2 |k| \, v_{tho} \, G_{IM} / 2\pi \omega \omega_p^2$

Now have:

$$\langle \phi^2 \rangle_{k,\omega} = \frac{2\pi n}{|k| v_{tho}} \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{\langle \bar{f}(\omega/k) \rangle}{|G(k,\omega)|^2}$$

so plugging in:

$$\boxed{\langle \phi^2 \rangle_{k,\omega} = \frac{8\pi T}{k^2 \omega} \frac{\operatorname{Im} G}{|G|^2}}$$

and,
$$\boxed{\frac{\langle \hat{E}^2 \rangle_{k,\omega}}{8\pi} = \frac{T}{\omega} \frac{\mathrm{Im}\,G}{|G|^2}}$$

- Fluctn - Dissipn Thm for plasma
- restates form of spectrum
- relates fluctuations to T and dissipation in collective modes. ($\mathrm{Im}\,G$)

- obviously consistent, by construction.

Comments:

- key to TPM is causality, and use of linear f^c or UPO.

- assumes weak fluctuations

  i.e. $\underline{x}(t) = \underline{x}(0) + vt + \cancel{\delta x}$

  How weak?

- $\tau_{ac} < \tau_T$ condition

-

For stable plasma, $F' \to 0$ from $F' < 0$, so $\langle \phi^2 \rangle \uparrow$, even for stable system.

- akin critical opalescence.

- TPM fails $\Rightarrow$ lack of damping.