Relaxation, Instabilities and Quasilinear (Mean Field) Theory

i.) General ideas on relaxation

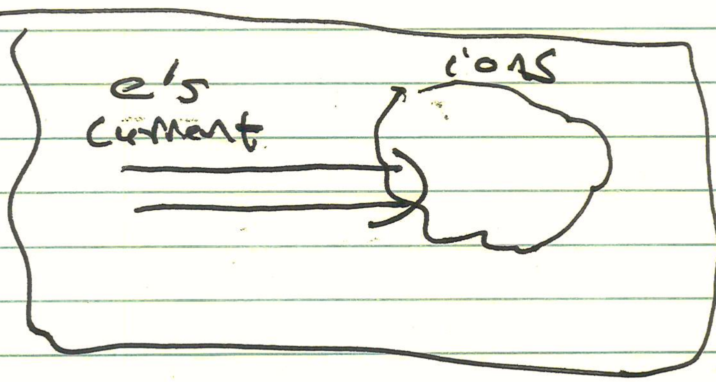
- focus on relaxation in much of plasma physics

- relaxation – evolution of distribution function
  $\Downarrow$ reduction in free energy
  – usually toward homogeneous Maxwellian
  – disposition of free energy

- types of relaxation:

$$\left\{\begin{array}{l}\text{collisional – via collisions}\\ \text{collective – via wave modes}\end{array}\right\} \text{disparate time scales!}$$

– i.e.



Free energy: $\underline{J} = -n_0 |e| \underline{V} \Rightarrow \underline{V}_e$

$$\Rightarrow \mathcal{E} \sim 1/2\, m_e n\, V_e^2$$

Relaxation $\Rightarrow$ electrons slow down.

How:

- collisions $\rightarrow$ $\underline{J} = \sigma \underline{E}$

σ: collisional resistance

i.e. $\rightarrow$ electron momentum transferred to ions

$\Leftrightarrow$ constraint: total momentum conserved.

n.b. calculate via Boltzmann / Chapman-Enskog

$$\underbrace{\partial_t f}_{\omega} + \underbrace{v\,\partial_x f}_{v/L} + \underbrace{\frac{q}{m} E}_{1/\tau_{acc}} \frac{\partial f}{\partial v} = \underbrace{C(f)}_{\nu_{ei}}$$

$C(f)$: collision operator

ν_{ei} (momentum transfer to ions.)

for DC:

$$0 = C(f) \qquad \text{l.o.}$$

$f = f_0 \rightarrow$ Maxwellian $\left\{\begin{array}{l}\text{shifted} \\ \text{in homogeneous}\end{array}\right.$

here, for weak E,

$$f_0 = \frac{1}{\sqrt{2\pi}\, v_{th}} \exp\left[-v^2/2v_{th}^2\right]$$

then:

Krook (Crook) model

$$\frac{q}{m} E \frac{\partial f_0}{\partial v} = C(f_1) = -\nu\, \delta f$$

1st O.

$\delta f = f - f_0$

so

$$\delta f = -\frac{1}{\nu}\frac{e}{m}E\frac{\partial f_0}{\partial v}$$

then:

symmetry

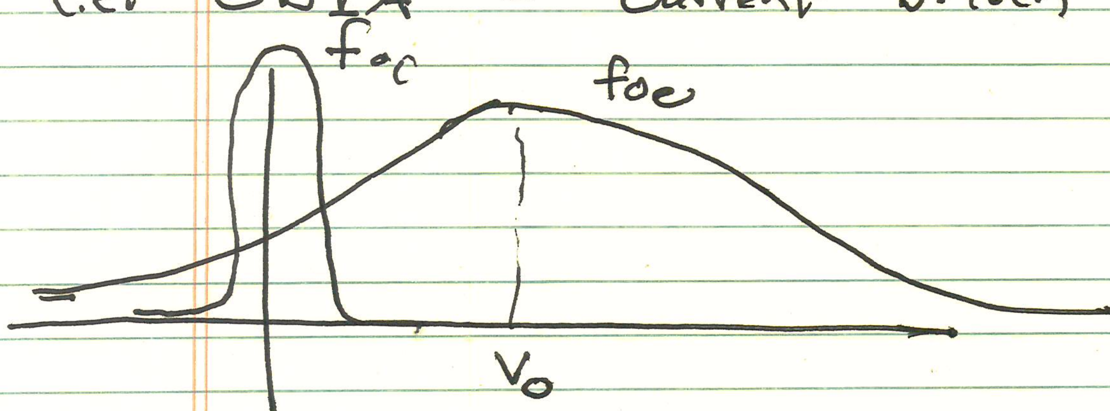
$$J = -n_0 e_{el} \int dv\, v\left[\cancel{f_0} + \delta f\right]$$

$$= +n_0 e_{el} \int dv\, \frac{v}{\nu}\frac{e}{m}E\frac{\partial f_0}{\partial v}$$

$$\equiv \sigma E.$$

- Collective Processes - Instabilities

i.e. CDIA - Current Driven Ion-Acoustic



- $\partial f_{0e}/\partial v > 0$ → growth
  $\partial f_{0i}/\partial v < 0$ → damping

- $V_0 > V_{crit} \sim C_s$ → instability
  inverse Landau damping

- momentum transferred to ions via wave

i.e. – inverse Landau damping → extracts electron momentum

anomalous resistivity – momentum deposited on ions via Landau damping

– rate transfer → {Quasilinear Theory}

[mean field evolution]

upcoming

→ Instability saturation is major issue
- use up free energy
- couple to dissp.

i.e.

→ Instabilities

– collective routes to relaxation

– usually faster than collisions

– Components:

→ free energy – relaxes
– converted to fluctn. energy
– i.e. V_0

→ trigger mechanism
– details of growth,

– can be quite complex and variable

– multiple mechanisms operate in different regimes

→ Best to approach instabilities 'ala' zoology

i.e.

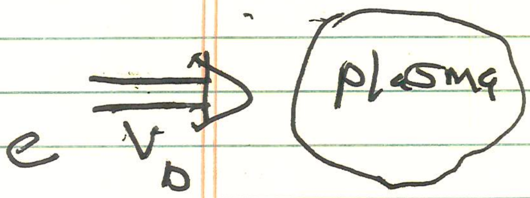
Family → Species
Felines ← all Cats

- Lion
- Tiger
- Domestic (Felis Sylvestris)

~ Families usually grouped by free energy source.

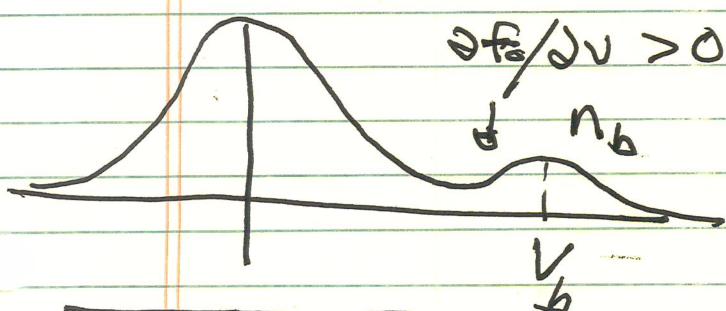
→ i.e. Beam + Plasma → beam K.E.

B-P ⇒
(a) B-O-T – inverse Landau Damping
(b) Two Stream – bunching



– Bump-on-Tail

(write as normalized sum of two Maxwellians)



$\boxed{\text{mechanism: inverse Landau Damping}}$

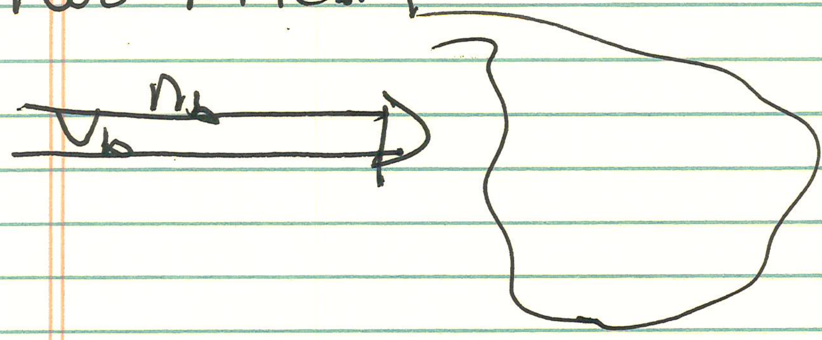
– bump $V_b > V_{Th}$

– wave → bulk
bump → resonant particles

– evol. ⇒ slow down bump

- V_b sub-critical to 2 stream
- techn: bumps as contrib to ϵ_{ION}, only.

(b) Two Stream



$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kV_b)^2}$$

Can be unstable by $\ominus$ energy wave in beam coupled to $\oplus$ energy wave in bulk.

need $- V_b > V_{b\,crit}$.

- if $\gamma > \gamma_{B\text{-}O\text{-}T}$ $\rightarrow$ ignore details of resonance.

bunching $\leftrightarrow$ mechanism

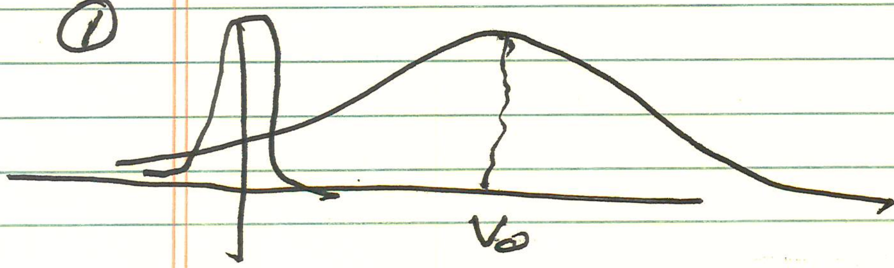
Critical: (1) B-O-T and Two-stream are same fundamental instability, in different regimes, with different mechanism.

(2) B-O-T persists when 2-stream stable.

(3) Instability saturates by beam slow-down.

Similarly: Current-Driven

(1)



CDIA → already covered

- inverse ELD vs ILD
- $V_0 > C_s$ for free energy
- $f_e = -\frac{|e|\phi}{T}\langle f\rangle + g$

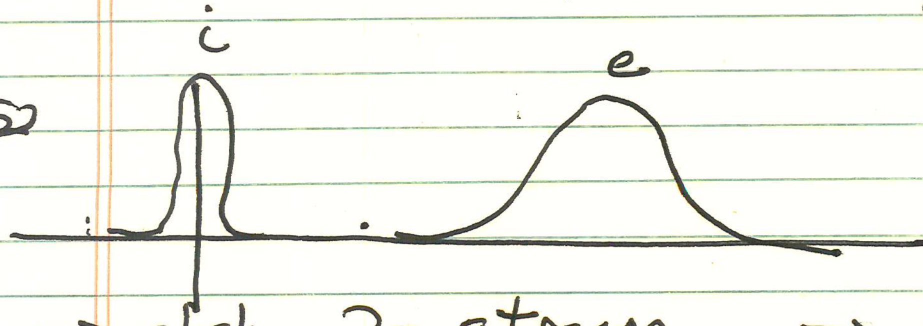
$\frac{|e|\phi}{T}\langle f\rangle$ → Boltzmann

g → correction (non-adiabatic)

$$\frac{\partial g}{\partial t} \sim \underbrace{(\omega - kV_0)}_{\text{sgn.}}\,(\quad)$$

$\omega = kC_s$ ⇒ V_0 vs C_s competition

(2)



→ ala 2-stream → macroscopic
→ "Buneman"!

iii) Weakly NL Theory → Framework developed for Hydro

Landau - Stuart Theory

- How do linear theories evolve?

- $\boxed{\text{Seek characterize weakly nonlinear evolution i.e. flow shear / KH !}}$

$\underline{V}_0(\underline{r})$ → base state

∃ Re_{crit} : critical Reynolds number for instability.

then:

$$Re = Re_{crit} + \delta Re$$

$$\delta Re / Re_{crit} \lll 1.$$

- Can one represent dynamics in some general form, especially near marginal stability?

- Analogy: Ginzburg - Landau Theory

- Leverage: Symmetry (time variation)

Now, if consider Navier-Stokes eqn. retaining nonlinear terms:

$$\partial_t \underline{\tilde{V}} + \underset{\substack{\downarrow \\ \text{driver}}}{\underline{\tilde{V}} \cdot \underline{\nabla} V_0} + \underline{V_0} \cdot \underline{\nabla} \tilde{V}$$

driver
∇V_0 is free energy source

$$+ \underline{\tilde{V}} \cdot \underline{\nabla} \tilde{V} = - \underline{\nabla} \tilde{P} + \nu \nabla^2 \tilde{V}$$

$\rightarrow$ fast osc.

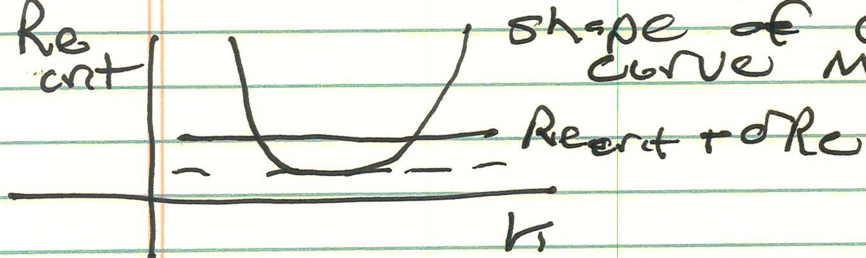
$$\underline{\tilde{V}} = f_i(\underline{r}) e^{-i\omega t} e^{\gamma t} \rightarrow \text{slow growth}$$

$$= \underset{\substack{\downarrow \\ \text{envelope}}}{f_1^{\prime}(r)} \underset{\substack{\downarrow \\ \text{carrier}}}{e^{i\underline{k}\cdot\underline{r}}} e^{-i\omega t} e^{\gamma t}$$

$$\boxed{\gamma < \omega} \qquad \gamma \sim \delta Re$$

$$\tilde{V} = \underset{\substack{\downarrow \\ \text{amplitude}}}{A(t)} f_i(\underline{r})$$

– for $\delta Re \ll Re_{crit}$, expect few modes relevant just above marginality



shape of criticality curve matters

then,

$$\frac{d}{dt}|A|^2 = 2\gamma |A|^2 + O(A^3) + O(A^4)$$

- Exploit time scale separation

$$\overline{A^2} = \int_0^{2\pi/\omega = T} \frac{dt}{T} \; A^2 \quad \rightarrow \text{period avg.}$$

∞ : $\overline{\underline{V}\cdot\underline{V}\cdot\underline{\nabla}\,\underline{V}} \rightarrow 0$ for single mode (no way eliminate fast oscillation)

so $O(A^3) \rightarrow 0$

For $O(A^4)$:

$$\overline{\tilde{V}\cdot\tilde{V}^{(2)}\cdot\nabla\,\tilde{V}} \neq 0$$

$$\tilde{V}^{(2)} \sim \tilde{V}\tilde{V}$$

$$\sim -\alpha|A|^4$$

α must be computed.

$$\Rightarrow \boxed{\partial_t |A|^2 = 2\gamma|A|^2 - \alpha|A|^4}$$

Landau Eqn.

→ Landau Eqn. has obvious structural similarity to Ginzburg – Landau Theory

→ Physics is mode feedback on profile ⇒ modifies so as turn off growth.

i.e.

$$\partial_t |A|^2 = 2\gamma |A|^2 - \alpha |A|^4$$
$$= (2\gamma - \alpha |A|^2) |A|^2$$
$$= 2\gamma_{eff}(|A|^2) |A|^2$$

$$2\gamma_{eff} = 2\gamma - \alpha |A|^2$$

finite amplitude modification

→ i.e. $\overline{\underline{\tilde{v}} \cdot \underline{\tilde{v}} \cdot \nabla (\underline{V}_0 + \tilde{v})}$ $\sim \tilde{v}\tilde{v}$

$$= \overline{\tilde{v} \cdot \underline{\tilde{v}} \cdot \nabla V_0 + \tilde{v} \cdot \tilde{v} \cdot \underline{\nabla} \nabla \tilde{v} \tilde{v}}$$

$$= \overline{\tilde{v} \cdot \underline{\tilde{v}} \cdot \underline{\nabla} (\overline{V_0} + \nabla \tilde{v} \tilde{v})}$$

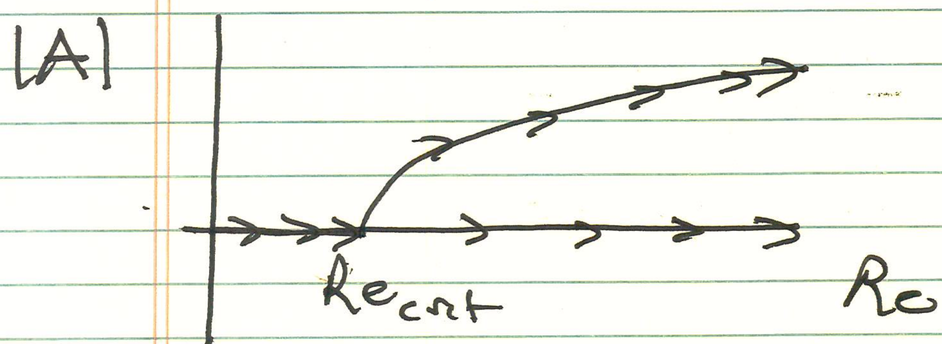
$$= 2\gamma |A|^2 - \alpha |A|^4$$

nonlinearity acts deplete free energy.

→ Predicts saturation at:

$$|A|^2 \simeq 2\gamma/\alpha \cong (Re - Re_{crit})$$

$$|A| \sim (Re - Re_{crit})^{1/2} \rightarrow \text{stationary state}$$



super-critical bifurcation

- super-critical bifurcation is usual for linear instability
- α need be calculated by perturbation theory.
- α need not be positive ⇒ $O(A^4)$ destabilizing

⇒ sub-critical processes.
(NL instability)

Then:

$$\partial_t |A|^2 = 2\gamma |A|^2 - \alpha |A|^4 - \underbrace{\beta |A|^6}_{\text{to saturate}}$$

$\Rightarrow$

$$= \underbrace{-2|\gamma|\,|A|^2}_{\substack{\text{damping}\\ \text{linearly}}} + \underbrace{|\alpha|\,|A|^4}_{\text{NL growth}} - \underbrace{\beta |A|^6}_{\text{satn.}}$$

- subcritical instability if:

$$|A|^2 > \frac{2|\gamma|}{|\alpha|}$$

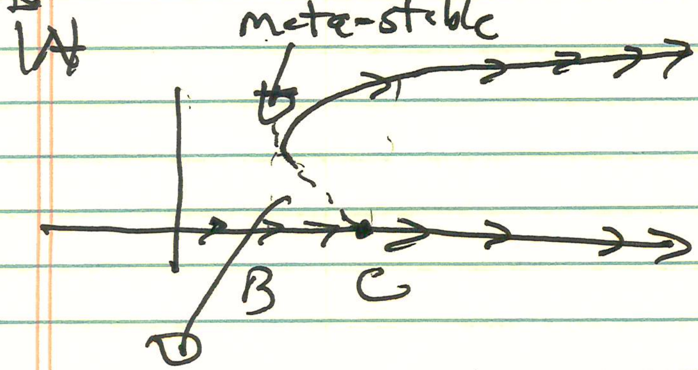
- stationary state:

$$|A|^2 = 0$$

$$|A|^2 = \frac{|\alpha|}{2\beta} \pm \left(\frac{|\alpha|^2}{4\beta^2} - 4\frac{(2|\gamma|)}{\beta}\right)^{1/2}$$

3 roots

$\Rightarrow$



unstable (|A| insuff to saturate)

Now:

- Super-critical:

$$\partial_t |A|^2 = (2\gamma - \alpha |A|^2)|A|^2$$

- QL:

$$\partial_t |E|^2 = 2\gamma(\langle f \rangle)|E|^2$$

$$\partial_t \langle f \rangle = \partial_v D(|E|^2)\,\partial_v \langle f \rangle$$

i.e. $\langle f \rangle = f_0 + \Delta\langle f \rangle$

$$\gamma(\langle f \rangle) \rightarrow \gamma_0 + \frac{\delta\gamma}{\delta\Delta\langle f \rangle}\underbrace{\Delta\langle f \rangle}_{O(E)^2}$$

QL ↔ Landau Eqn. address same point

⇒ instability feedback on mean to switch off growth.