1

see next

Cmnt Landau

$\partial_t |A|^2 = 2\gamma_0 |A|^2 - \sigma |A|^4$

$\gamma_0 + \Delta\gamma' \to -\sigma |A|^2$

$\Delta\gamma' \to \Delta \partial \langle f \rangle / \partial v$

Quasilinear Theory - Vlasov Plasma

i.) Motivation and Overview

$\rightarrow$ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$

$\Rightarrow$ growth/damping rate $\gamma_k = \gamma_k[\langle f \rangle]$

but $\langle f \rangle$ evolves... If $\langle f \rangle$ evolves slowly:

i.e. "slowly" $\Rightarrow \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t) \rangle] \rightarrow \left\{ \begin{array}{l} \text{evolution driven} \\ \text{by instabilities} \end{array} \right.$

Physics: mean distribution evolution....

$\Rightarrow$ driven by relaxation.

$\Rightarrow$ $\boxed{\text{quasilinear theory is concerned with describing and understanding the slow evolution of } \langle f \rangle \ldots}$

To connect to Landau Theory:

in Landau theory, for weakly nonlinear evolution:

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

$\gamma_0 \sim V_0'$ (i.e. shear inst.)

but

$$\partial_t \langle V \rangle = -\partial_x \langle \tilde{V}_x \tilde{V}_y \rangle$$

so: $V_0' \rightarrow V_0' + \Delta V$

$$\Delta V' \sim A^2 \text{, necessarily}$$

write $\Delta V' \sim -\sigma' A^2$

↳ QL-like feedback on driving profile.

$$\partial_t A^2 = 2\gamma A^2$$
$$= 2(\gamma_0 + \Delta\gamma) A^2$$

but

$$\Delta\gamma \sim \Delta V' \sim -\sigma' A^2$$

so

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

③ quasilinear theory is "mindless mean field theory", i.e.

$\langle f \rangle \equiv \langle f(v,t) \rangle$ where → $\langle\ \rangle$ eliminates spatial dependence
→ t understood "slow"

so if:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f} \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$) for mean of conserved order parameter ($\langle f \rangle$)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} J_v = 0 \Rightarrow$$ phase space continuity equation

$$J_v = \Gamma_v = \left\langle \frac{q}{m} E f \right\rangle$$
$$= \frac{q}{m}\langle \tilde{E}\tilde{f} \rangle$$

for: $E = \tilde{E}$

$$f = \langle f \rangle + \tilde{f}$$

elementary closure problem
i.e. relate $\langle f \rangle$ to $\langle \tilde{E}\tilde{f} \rangle$ → hierarchy!

How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(i) $\tilde{f} \to \tilde{f}_{\text{Linear}}$ (i.e. linear response of f)
plug in linear response

i.e. $\dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} + \dfrac{q}{m}E\dfrac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

so $$J_v = -\frac{q^2}{m^2}\sum_{k,\omega}|\tilde{E}_{k,\omega}|^2\frac{i}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only) i.e. contrast approach to criticality in usual phase transitions (2nd order)
→ here growth of order parameter in broken symmet. phase ... not noise driv "

Q.L. equation is:

$$\left.\begin{aligned} &\frac{\partial\langle f\rangle}{\partial t} = \frac{\partial}{\partial v}D\frac{\partial\langle f\rangle}{\partial v} \\ &D = \frac{q^2}{m^2}\sum_k|\tilde{E}_k|^2\frac{i}{\omega - kv + i0} \end{aligned}\right\} \text{“Q.L. equation.”}$$

i.e. mindless mean field theory ...

$$\left.\begin{aligned} &\text{with } \epsilon(k,\omega) = 0 \\ &\partial_t|E_k|^2 = 2\gamma_k|E_k|^2 \end{aligned}\right\} \to \text{advance fields.}$$

But,

Surprisingly: Q.L.T. works quite well!

Key issue: why?

N.B.: In contrast to critical phenomena, external noise ignored → instability driven ... deterministic

④ Some questions to keep in mind:

→ (i) why is QL equation a diffusion equation? When is this valid?

↔ nature of "irreversibility" ....

→ (ii) can Q.L. equation be derived from Fokker-Planck theory?

↔ also "irreversibility" related ...

→ (iii) how does Q.L. equation balance the energy-momentum budgets?

→ (iv) when / how does Q.L. theory fail?

↔ related (i) ..... What is "Ginzburg Criterion" for Q.L.T.? Can such a criterion be formulated?

→ v.) what is dynamics of quasilinear relaxation?

i.e. physics?

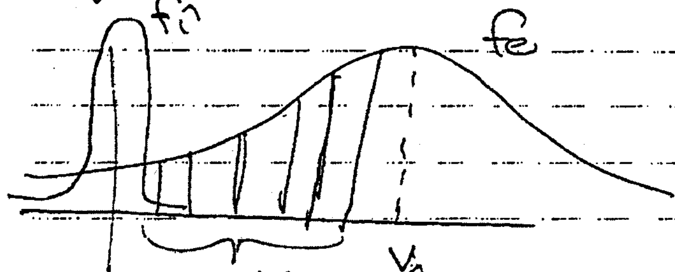
ii) Basic Scales / Regime Definition

① → Generally, Q.L.T. concerned with

i) 'broad' spectrum of:
→ how broad?

ii) unstable waves

i.e. for current-driven ion-acoustic (C.D.I-A.) turbulence:



unstable spectrum → why?

②
→ In finite system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$\omega_m/k_m = \omega(k_m)/k_m = V_{ph,m}$

- wave-particle resonance occurs when

$V = V_{ph,m}$

7.

then Sin Isaac ⇒

$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t)$$

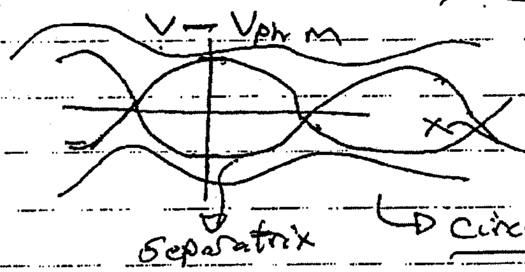
n.b. { deterministic, no RPA

and 1 resonance dominant ⇒

$$m\ddot{x} \cong q E_{m_0} \cos\left(k_m x_0 + (k_m v - \omega_m) t\right)$$

⇒ each resonant velocity defines a phase space island

i.e.

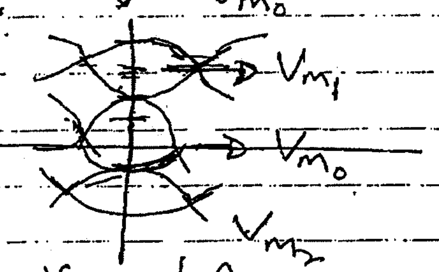


$$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$$

∴ QLT is concerned with the case of:

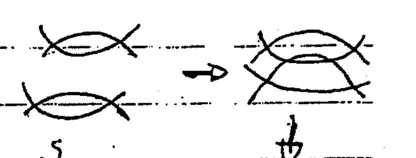
→ multiple, $\boxed{\text{overlapping resonances}}$ → { separatrix proximity ⇒ destruction

→ i.e.



trapped → wandering

i.e.



if overlap, hop

∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v / $D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$

Δv ~ resonance width
τ_{ac} → pattern time
↳ what is it?

⇓

ergodicity → mixing

See Phys. 200B Notes 2014, 2015 — Google directly.

overlap $\Leftrightarrow$ $\exists$ 1 $h_k > 0$. (Positive Lyapunov exp.)

Overlap condition (B. V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \gtrsim V_{ph_{m+1}} - V_{ph_m}.}$$

$\Delta v_T \sim \sqrt{\frac{eE_0}{m}}$

underlies diffusion eq'n. – QL

∴ → particle motion stochastic ⇔ irreversibility

→ fundamental irreversibility ⇒ orbit stochasticity (not dissipation, Landau damping ... ⇔ contrast critical phenomena).

→ underpinning of diffusion equation.

③ → But, a swindle !?! ⇒ [use of un-perturbed orbit in estimate !]

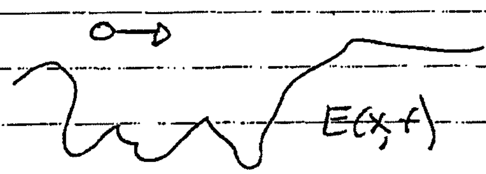
i.e. is $x \to x_0 + vt$ valid ?

Consider: linear, un-perturbed orbit ?! ?

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

∴ particle "sees" instantaneous pattern of electric field, from modal superposition.

i.e.



∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern

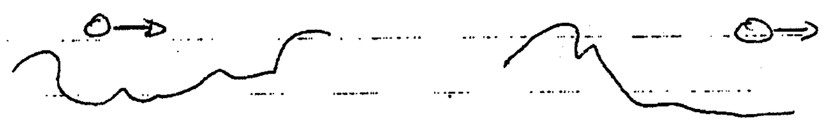
obviously, ① $\tau_L \ll \tau_b \rightarrow$ unperturbed orbit is satisfactory approximation

(pattern changes prior ⟷ bouncing)

② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes,

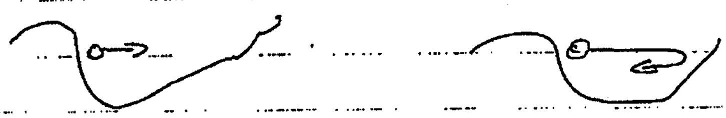
so must consider orbit perturbation. ...

i.e.



$\tau = 0$ $\qquad \tau_b$

vs.



∴ quasilinear theory relevant to evolution when:

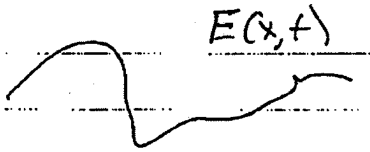
① → orbits stochastic (Chirikov condition satisfied)

② → $\tau_{Life} < \tau_{bounce} \rightarrow$ unperturbed orbits ⊙ valid.

3)

But, how relate $\tau_{lifetime}$, τ_{bounce} to physical quantities?

Key point: Superposition patterns disperse!



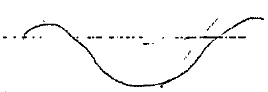
$$\Leftrightarrow \sum_k E_k\, e^{i(kx-\omega_k t)}$$

$$= \sum_k E_k \exp\left[i\left(k\left[x-\underbrace{\omega_k/k}_{v_{ph}(k)}\right]t\right)\right]$$

$\therefore$ $\Delta(\omega_k/k) \equiv$ spread in phase velocities

$\Downarrow$ sets dispersal ~~rate~~ speed.

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength $\Rightarrow$ FOM



$$1/\tau_{life} = k\,\Delta(\omega_k/k)$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)$$

$$(v_g - v_{ph})$$

$$\boxed{1/\tau_{life} = \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k} = \left(v_g(k) - v_{ph}(k)\right)\Delta k$$

n.b. $\tau_{life} \to \infty$ for non-dispersive waves!

Generally, QLT / weak turbulence encounters trouble for {non dispersive, weakly dispersive} waves.

How systematize? — E-field correlation Fctn

Consider: $\langle E(x_1,t_1)\, E(x_2,t_2)\rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, \tau)$, for $\left\{\begin{matrix}\text{homogeneous}\\ \text{stationary}\end{matrix}\right\}$ fluctuations

relative coords, space/time

$x_1 = x_+ + x_-$ $\qquad t_1 = t_+ + t_-$
$x_2 = x_+ - x_-$ $\qquad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \underline{\langle\ \rangle_{x_+,t_+}}$ $\qquad$ what the bracket means

so

$$C(x_-,\tau) = \Big\langle \sum_{k,k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k+\omega_{k'})t_+} \, e^{i(k-k')x_-} e^{-i(\omega_k-\omega_{k'})t_-} \Big\rangle_{x_+,t_+}$$

x_+, t_+ average $\Rightarrow k = -k'$ $\quad \omega_k = -\omega_{k'}$

so

$$C(x_-,\tau) = \sum_k |E_k|^2 e^{ikx_-} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum — i.e. post-overlap

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \underbrace{\left[\left(\frac{k-k_0}{\Delta k}\right)^2+1\right]}_{\text{width}}$$

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

∴

$$\langle E^2\rangle = \int dk\, \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2}+1\right]}\, e^{ikx_{0-}}\, e^{i(kv-\omega_k)\tau}$$

integrating:

phase info - irrelevant

$$\sim E_0^2\, e^{ik_0x_-}\, e^{-|\Delta k\, x_{0-}|} \;*$$

$$e^{i(k_0v-\omega_{k_0})\tau}\, e^{-|\Delta(kv-\omega_k)|\tau}$$

oscillation (→ 0 on resonance)

correlation decay due dispersal and its interplay with resonance.

n.b.: note that spread in doppler-shifted ω is critical

13.

now $\Delta(kV-\omega_k) = V\Delta k - V_{gr}\Delta k$

$= |(V-V_{gr})\Delta k|$

$V_{gr} = \frac{d\omega}{dk}$

so
$\langle E^2\rangle = C(x_-, \tau)$

$= E_0^2 e^{ik_0x_-} e^{i(k_0V-\omega_{k_0})\tau} e^{-|\Delta k x_0|}$

$* \exp[|(V-V_{gr})\Delta k|\tau]$

sets lifetime

$1/\tau_L = |(V-V_{gr}(k))\Delta k| \equiv (\text{Autocorrelation Time})^{-1}$

Note:

$\equiv 1/\tau_{ac}$.

- for resonant particles, $V = \omega_k/k$

$1/\tau_L = |(V_{ph}-V_{gr})\Delta k|$ → recovers earlier!

- can think: $|V\Delta k| \rightarrow 1/\tau_{ac}^{\text{wave-particle}}$

$|V_{gr}\Delta k| \rightarrow 1/\tau_{ac}^{\text{wave}}$. packet dispersal generally, shorter time dominates, except for non-dispersive waves.

So, can enumerate key time scales

→ $\tau_{ac} = |\Delta k (v_{ph} - v_{gr})|^{-1}$

$\equiv$ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles

→ γ^{-1} = growth/damping time

→ $\tau_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1}$ $\equiv$ trapping time

→ $\tau_{relax} = \left(\dfrac{1}{\langle f \rangle}\dfrac{\partial \langle f \rangle}{\partial t}\right)^{-1}$ $\equiv$ avg. distribution relaxation time

So

$\tau_{ac} < \tau_{Tr}$ → u.p.o. valid

$\left.\begin{matrix}\tau_{ac} \\ \gamma^{-1}\end{matrix}\right\} < \tau_{relax}$ → $\langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tau_{relax}$ → Q.L.T. valid.

iii) Energy – Momentum Budgets

→ key Point: There are two ways of implementing the book-keeping and accounting

i.e. $\left\{\begin{array}{lll} \text{resonant particles} & \text{vs.} & \text{'waves'} \\ \underline{\text{or}} & & \\ \text{particles} & \underline{\text{vs.}} & \text{fields} \end{array}\right.$

keep in mind: Wave = Field + Non-resonant Particles

i.e. for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle.

(show!)

16.

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f\rangle}{\partial t} = -\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$\frac{\partial}{\partial t}\int dv\, \frac{mv^2}{2}\langle f\rangle = -\int dv\, \frac{mv^2}{2}\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$= \int dv\, mv\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

plugging in $\tilde{f}_k^{\text{Linear}}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t}\mathcal{E}_{kin} = -i\int dv\, \frac{vq^2}{m}\sum_k |E_k|^2\left(\frac{P}{\omega - kv} - i\pi\delta(\omega - kv)\right)\frac{\partial\langle f\rangle}{\partial v}$$

resonant

$$\frac{\partial}{\partial t}\mathcal{E}_{kin}^{res} = -\int dv\, \frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\delta(\omega/k - v)\frac{\partial\langle f\rangle}{\partial v}|E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

[As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

For wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$

$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$

$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega}$; $\cancel{i}\gamma_k = -\cancel{i}\frac{\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$

$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$

Now, $W \equiv$ Wave Energy Density

action density

$$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi} = \sum_k \omega_k N_k$$

$$= \sum_k \omega_k \frac{\partial\epsilon^r}{\partial\omega}\bigg|_{\omega_k}\frac{|E_k|^2}{8\pi}$$

$|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k \omega_k \frac{\partial\epsilon^r}{\partial\omega}\bigg|_{\omega_k}\frac{|E_k|^2}{8\pi}$$

$$= \sum_k 2\left(-\frac{\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k\frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\epsilon_{IM} = \frac{\omega_p^2}{k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore \frac{\partial W}{\partial t} = \sum_k \frac{\pi q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

$$\Rightarrow \boxed{\partial_t E^{\text{resonant}}_{\text{kinetic}} + \partial_t W = 0}$$

$$\cancel{\partial_t \int dv \tfrac{1}{2} m v^2 \langle f\rangle = q\langle E \tilde{f}\rangle}$$

Note:

— this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\frac{\partial}{\partial t}W + \nabla\cdot\underline{S} + Q = 0$$

$\frac{\partial}{\partial t}W$: wave energy

$\nabla\cdot\underline{S}$: divergence of wave energy density flux

$Q \to \langle \tilde{\underline{E}}\cdot\tilde{\underline{J}}\rangle$ coupling: resonant particle heating

19.

For homogeneous system: $\underline{\nabla} \cdot \underline{S} = 0$

so $\frac{\partial W}{\partial t} + Q = 0$

$\langle E \cdot J \rangle$ mediated by resonant particles (DC field)

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(RPKED) = 0}$$

resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$$W = NRPKED + FED$$

non-resonant particle kinetic energy density

field energy density

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}(RPKED + NRPKED) = 0$$

$\underline{PKED}$ → total

so $\boxed{\frac{\partial}{\partial t}\,FED + \frac{\partial}{\partial t}\,(\,PKED\,) = 0}$ Energy Thm. II

i.e. fields and particles conserve energy.

What is the physics of all this ? !

$D = \mathrm{Re}\sum_k \frac{q^2}{m^2}|E_k|^2 \left(i/\omega - kv \right)$

QL diffusion

for general, weakly non-stationary state ---

$= \sum_k \frac{q^2}{m^2}|E_k|^2 \left(|\gamma_k| \big/ \left((\omega - kv)^2 + |\gamma_k|^2 \right) \right)$

n.b. causality $\Rightarrow$ no negative diffusion for damped waves

$\cong \sum_k \frac{q^2}{m^2}|E_k|^2 \Big\{ \underbrace{\pi\delta(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}} \Big\}$

Resonant Diffusion $\rightarrow$ irreversible — resonance overlap is underpinning

$\rightarrow$ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy ..... ?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{|\gamma_k|}{\omega_k^2}$$

take $\gamma_k > 0$.

ponderomotive energy

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \quad \text{where} \quad |V_k|^2 = \frac{q^2}{m^2} \frac{|E_k|^2}{\omega_k^2}$$

~ ponderomotive energy

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka "fake diffusion"

→ vanishes in stationary state ($\partial_t = 0$)

Point is that can count non-resonant diffusion as:

$$D_{\text{non-resonant}} \left\langle \begin{array}{c} \dfrac{\text{particle sloshing energy}}{\text{part of wave energy density}} \\ \text{or} \\ \text{part of total particle kinetic energy density} \end{array} \right.$$

so two forms of energy conservation.

Note: Physically, the picture of plasma as gas $\left\{ \begin{array}{l} \text{- resonant particles} \\ \text{- waves} \end{array} \right.$ or equivalently

resonant particles + quasi-particles

$$\Downarrow \quad \text{waves} \left\langle \begin{array}{l} N(k, x, t) \\ WKE, \text{ etc.} \end{array} \right.$$

is appealing and will pervade this course.

N.B.: For Direct Proof of $\partial_t (PKED + FED) = 0$

see below.

23.

From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv \, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} (\underbrace{kv - \omega}_{} + \omega) * \frac{1}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$\hookrightarrow$ $\left\{ \begin{array}{l} \text{cancels denom} \\ \text{residual odd in} \\ k \end{array} \right.$

$$= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

$\omega_k = \omega_k^r + i\gamma_k$

$$= -\sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= -\partial_t (FED)$$ ✓

24. Further:

⇒ "Numbers" and "Ratios" ......

Numbers – as in Re. ?!

– what is assumed in QLT?

→ linear response adequate – no NL distortion

→ resonant diffusion – Markov process, a la' FPE

→ stochasticity / irreversibility

→ RPA (?)

Exercise:

a.) Derive QL (resonant diffusion) equation from Fokker-Planck theory

b.) use Hamiltonian structure of dynamics to eliminate dynamical friction term (c.f. Lichtenberg and Lieberman)

⇒ 2 dimensionless numbers:

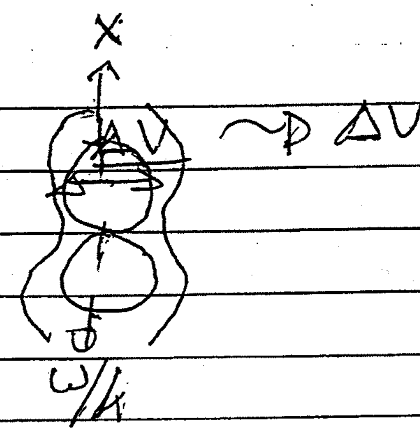
1)

$$\boxed{\underset{\text{Chirikov \#}}{S_c} = \Delta V / |v_{ph_i} - v_{ph\,i+1}|}$$

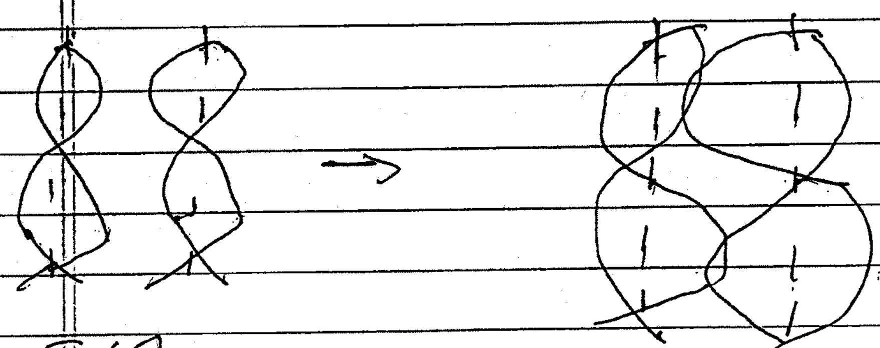
→ [measures stochasticity of particle orbits

i.e. resonances overlap $S_c > 1$

25.



so if multiple waves:



$\delta < 1$

⇒ (n) independent resonant behaviors

$\delta > 1$

⇒ resonance overlap

⇒ diffusive "kicking" from 1 resonance to another.

(2) $\boxed{\text{Kubo \#}}$ / Strouhal # → $\boxed{\text{measures memory of flow}}$

→ field correlation time

i.e.

$$K = \frac{(qE/m)\,\tau_c}{(\Delta V)_0}$$

∂_t vs $\frac{ZE}{m}\partial v$

↳ velocity correlation length

Kubo number of prime importance

for spatial scattering:

$$S = \frac{\tilde{V}\, \tau_c}{l_c}$$

τ_c → scatterer correlation time

l_c → correlation length

Note: For $\Delta V_c \sim \Delta V_T \sim \sqrt{T/m}$

$$K \sim k(\Delta V_T)\,\tau_c$$

$$\sim \frac{\omega_b \tau_c}{J}$$

ω_b → bounce freq.

$K < 1 \Rightarrow$
$\tau_c\, \omega_b < 1$
→ [pattern changes prior bounce

⇒

$K_0 S > 1$ → { ordered flow with memory; persistent scatterer pattern

$K_0 S < 1$ → { random short lived scatterers; random flow — suggests RPA

can treat $\tilde{E}$ as random variable

usually $K_0 S > 1$ ⇒ linear unperturbed trajectory poor approximation

⇒ need incorporate scattering field.

so, usual wisdom is that QLT valid if:

→ $\sigma_c > 1$ → need stochastic orbits

→ $K < 1$ → need avoid trapping, strong distortion, etc.

but: ?

① → is field / scatterer correlation time the relevant time scale ?

② → what of $\sigma_c > 1$, $K \sim 1$? → often realistic

③ → what of non-resonant piece ?

Regarding ①,

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty e^{i(\omega - kv)\tau} d\tau$$

$$|E_k|^2 = E_0^2 \, \Delta k \Big/ \left[(k - k_0)^2 + (\Delta k)^2\right]$$

⇒

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$= \int dk\, \frac{q^2}{m^2} \frac{E_0^2\, \Delta k}{(k-k_0)^2 + (\Delta k)^2} \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

flip order

$$\approx \frac{q^2}{m^2} E_0^2 (2\pi) \int_0^\infty d\tau\, e^{i(\omega_{k_0} - k_0 v)\tau}$$

$$\exp\left[- \left| \frac{d\omega}{dk} - v \right| \Delta k\, \tau \right]$$

→ sets correlation decay

$\Rightarrow$ if $D \sim \frac{q^2}{m^2} \langle E^2 \rangle \tau_c$

wave-particle correlation time

then:

$$1/\tau_{ac} \sim \left| \frac{d\omega}{dk} - v \right| \Delta k \rightarrow \left\{ \begin{array}{l} \text{wave-particle} \\ \text{decorrelation rate} \end{array} \right.$$

if $v \sim \omega/k$ – resonance

$$1/\tau_{ac_{pkt}} \sim \left| \frac{d\omega}{dk} - \frac{\omega}{k} \right| \Delta k \rightarrow \text{packet dispersal rate}$$

Generally: $\tau_{ac_{w-p}} \neq \tau_{ac_{pkt}} \rightarrow$ {differences more pronounced in 3D}

<u>So</u>, really need:

→ specify velocity for wave-particle decorrelation being considered.

→ need: $1/\tilde{\tau}_{ac} > \omega_b, 1/\tau_{scatt}$
$$\delta \quad (k^2 D)^{1/3}$$

→ $\tilde{\tau}_{ac} \sim \tilde{\tau}_c$ only for resonant particles in 1D.

→ broad spectrum alone is <u>not</u> sufficient to justify QLT → <u>effective</u> dispersion significant!

$$1/\tilde{\tau}_{ac}\Big|_{res} \sim \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right| \Delta k \rightarrow \underline{0}; \ \underline{\text{for}}$$

<u>non-dispersive waves</u>

→ $K<1$ criterion most accurate if one takes $\tilde{\tau}_c \sim \tilde{\tau}_{ac,pkt}$

$$K = \frac{2}{m} E \, \tilde{\tau}_{ac,pkt} / \Delta V_i = (k \Delta V_i) \tilde{\tau}_{ac} < 1$$

agrees with intuition.

→ "phase randomization" irrelevant $\Rightarrow$

- can have $\sigma > 1$, $k < 1$ with coherent phases
- QL known to well describe stochastic trajectory divergence in standard map/magnetic field lines, even for static fields/fixed phases, cf: Rechester, Rosenbluth, White PRL '80.
- phases fixed in Tsunoda/Malmberg experiments

→ often QLT seems to work reasonably well in limit of $\sigma > 1$, $k \sim 1$

- unclear why.....
- corrections due granulations needed (?!)

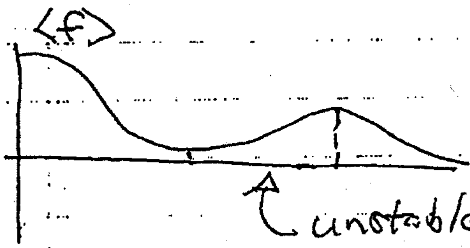
  i.e. $k \sim 1 \Rightarrow$ $\Rightarrow$

  phase space eddys formed....

→ strong non-stationarity can "boost" applicability of QLT.

(iv.) Applications of Quasilinear Theory (I)

→ Bump on Tail



↑ unstable phase velocities (bump on tail)

$\omega_k = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right)^{1/2}$

Quasi-linear Equations:

$$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k),\ \gamma(k) \quad \text{from } \langle f \rangle$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi \delta(\omega_k - kv) + \frac{|\gamma_k|}{\omega_k^2} \right\}$$

$$\frac{\partial}{\partial t}\left(|E_k|^2/8\pi\right) = 2\gamma_k |E_k|^2/8\pi$$

Observe: – resonant diffusion describes dynamics of tail particles

– non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: – tail flattening
with $\downarrow$ $\uparrow$
– adjustment of core/bulk profile (i.e. effective "temperature")

, ow first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v} \qquad * \langle f \rangle \text{ and } \int \text{ibp} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{\text{res}} \frac{\langle f \rangle^2}{2} = - \int_{\text{res}} dv \, D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2$$

$\left\{ \begin{array}{l} \text{generalization} \Rightarrow \\ \text{Zeldovich Thm.} \end{array} \right.$

stationarity $\Rightarrow$

$$D^R (\partial \langle f \rangle / \partial v)^2 = 0$$

Now "res" → some finite interval of phase velocities

So

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay and damp

or

$\partial \langle f \rangle / \partial v = 0$; plateau forms, removing growth

N.B.: – In 1D → plateau
– can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2} \int dk \, \mathcal{E}_F(k) \, \delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}_F(\omega_{pe}/v)$$

so

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v} (2\gamma_{\omega_{pe}/v}) \mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} / \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}$

$$\gamma_k = \gamma_{\omega_p/V} = \pi V^2 \omega_p \frac{\partial \langle f \rangle}{\partial V}$$

So $$\frac{\partial D^R}{\partial t} = \frac{16\pi^2 q^2}{m^2 V}\left(2\pi V^2 \omega_p \frac{\partial \langle f\rangle}{\partial V}\right)\mathcal{E}(\omega_p/V)$$

$$= \left(\pi \omega_p V^2 \frac{\partial \langle f\rangle}{\partial V}\right) D^R \quad , \text{ using } D^R \text{ defn.}$$

So

$$D^R(V,t) = D^R(V,0)\exp\left[\pi \omega_{pe} V^2 \int_0^t dt' \frac{\partial \langle f\rangle}{\partial V}\right]$$

and:

$$\frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial V} D^R \frac{\partial \langle f\rangle}{\partial V}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial V}\left[\frac{D^R}{\pi \omega_{pe} V^2}\right] \qquad \left\{ \begin{array}{l} \text{using } \gamma_k, D \\ \text{definitions} \end{array} \right.$$

35,

so

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]$$

∴ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_{pe} v^2\int_0^t dt' \frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right] \end{aligned}}$$

Now, recall seek to know if:

i) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\Big|_{t\to\infty} < 0$ (fluctuations damp)

ii) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p v^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

but $\frac{16\pi^2 q^2}{m^2 V} \frac{\mathcal{E}(0)}{\pi \omega_p V^2} = 2E_F(0) / (n m v_0^2/2)$

$\ll 1$, as $n \gg n_b$

$\therefore \langle f(v,t)\rangle \cong \langle f(v,0)\rangle$, to good approx.

but, for resonant velocities,

$\rightarrow$ linear instability $\Rightarrow \partial\langle f\rangle/\partial v > 0$

$\rightarrow D^R \xrightarrow[t\to\infty]{} 0 \Rightarrow \partial\langle f\rangle/\partial v < 0$

but have (for $D^R \rightarrow 0$) $\langle f(t)\rangle = \langle f(0)\rangle$!

$\therefore$ $\boxed{\text{contradiction follows from assumption of } D^R(v,t) \rightarrow 0}$

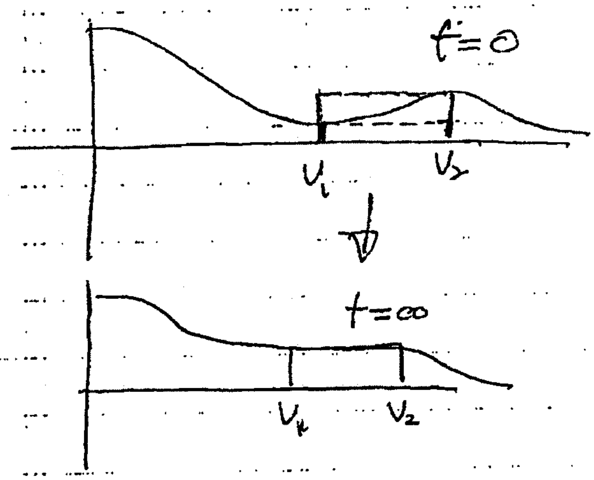
$\therefore$ have established that

$\partial\langle f\rangle/\partial v\Big\}_{res} \rightarrow 0 \Rightarrow$ plateau forms!

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(\text{RPKED}) + \frac{\partial}{\partial t}(\text{WED}) = 0$$

i.e.



t→∞

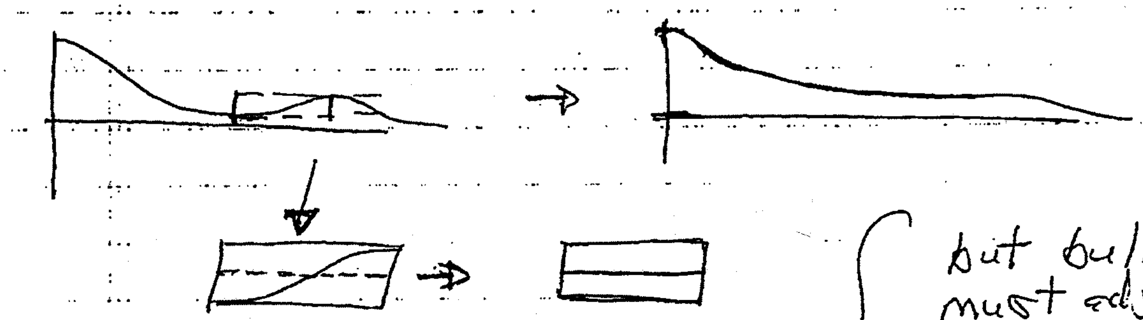
$k = \omega_p/v$

$$\therefore \quad \Delta\left(\int_{v_1}^{v_2} \frac{mv^2}{2}\langle f\rangle\right) = -\Delta \int_{k_1}^{k_2} W_k \, dk$$

but $W_k = 2\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv \, \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta \int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

→ can estimate A (RPKED) analytically, via construction



i.e. beam slows down

but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial \langle f \rangle}{\partial v}$$

$\gamma > 0$

$$\cong \frac{8\pi q^2}{m^2} \int dk \, \mathcal{E}(k) \frac{\gamma_k}{\omega_{p0}^2} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial\langle f\rangle}{\partial t} = \left(\frac{1}{nm}\frac{\partial}{\partial t}\int dk\, \mathcal{E}(k)\right)\frac{\partial^2\langle f\rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e}\int dk\, \mathcal{E}(k,t)$

so $\Rightarrow$

$$\frac{\partial\langle f\rangle}{\partial T} = \frac{1}{2m}\frac{\partial^2\langle f\rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle f\rangle = \left[m/2\pi\left[T+T(t)-T(0)\right]\right]^{1/2}\exp\left[\frac{-mv^2/2}{\left[T+T(t)-T(0)\right]}\right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \rightarrow T/2 + \frac{1}{n}\int dk\left[\mathcal{E}(k,\infty)-\mathcal{E}(k,0)\right]$$

i.e. electrons 'heated' by net increase in field energy ...

– can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta(RPKED) = -2\Delta(FED)$

but $\Delta(PKED) = -\Delta(FED)$

so $\Delta(RPKED) = +2(\Delta(PKED))$

$\Rightarrow \quad 0 = \Delta(RPKED) + 2\Delta(NRPKED)$

and,

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (-2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above

→ heating is one-sided, to conserve momentum.

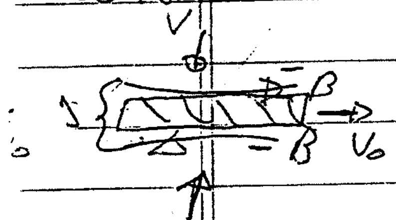
Applications II

Anomalous Resistivity → $\left\{ \begin{array}{l} \text{Application of} \\ \text{QLI} \end{array} \right.$ $\left\{ \begin{array}{l} \text{Follows / expands} \\ \text{on Sadeev/ Sagdeev} \\ \text{Rev. Pl Phys 7.} \end{array} \right.$

→ an instructive and important example of quasilinear theory is anomalous resistivity

→ here, try approach classic current-driven ion acoustic instability (CDIA) model of anomalous resistivity via coupled micro-macro dynamics

— consider Sweet-Parker model, i.e.



$$VL = V_{out}\Delta$$

$$V_{out} = V_A$$

cf. 218B notes

$\langle E \rangle = \langle \frac{v B}{c} \rangle < 0$

(into page)

$$2 V \frac{B^2}{8\pi} L = \eta \bar{J}^2 L \Delta$$

$$\frac{\Delta}{L} = \frac{V}{V_A} \Rightarrow \boxed{\Delta^2 = \frac{(L)(\eta)}{V_A}} \Rightarrow \text{layer width}$$

$$\frac{\Delta}{L} \sim 1/\sqrt{R_m}$$

What happens as η decreased ?

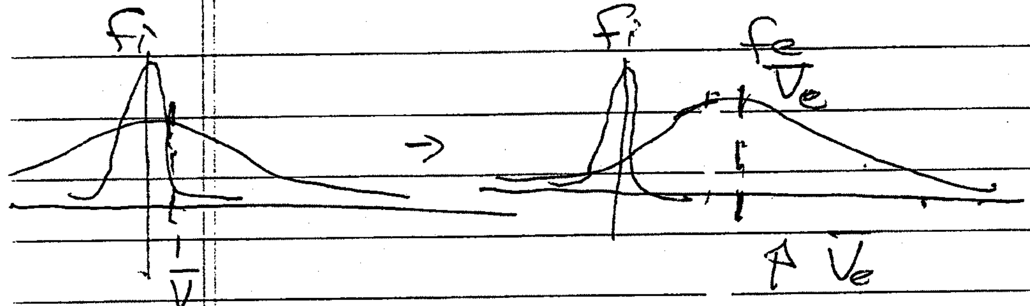
$$-\frac{c}{4\pi\Delta} B = J = n q \bar{V}_e$$

electron drift speed

so $\bar{V}_e = cB/4\pi n q \Delta = \frac{d_{skin}^2}{\Delta} \Omega_e$

Now, $\bar{V}_e \sim B/\Delta n \Rightarrow \bar{V}_e \uparrow$ as $\Delta \downarrow \Rightarrow$ narrow layer; $n \downarrow \Rightarrow$ few charge carriers; $B \uparrow \Rightarrow$ stronger field (drive)

$\Delta \uparrow \Rightarrow$

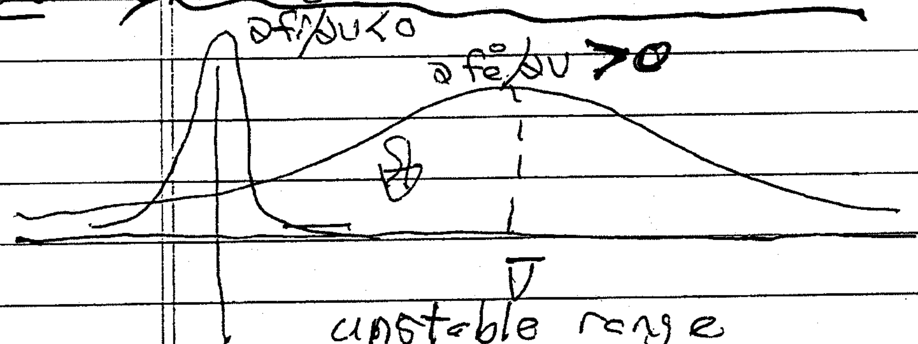


$\Rightarrow$ decreasing Δ raises $\bar{V}$

$\rightarrow$ up-shifts f_e centroid relative to f_i

$\Rightarrow$ destabilizes CDIA !

i.e. classic scenario of CDIA



$\therefore$ expect CDIA will:

$\rightarrow$ exchange momentum between electrons and waves

so $\rightarrow$ slow down electrons, reduce $\bar{V}_e$

$\rightarrow$ act as "anomalous" "turbulent" resistivity i.e. $\begin{cases} \Delta^2 = \frac{L}{V_A}\left(\eta + \eta_A(\bar{V})\right) \\ \quad\quad \hookrightarrow \text{anomalous resistivity} \\ \bar{V} = \frac{c}{4\pi} B / nq\Delta \end{cases}$

How calculate:

① Brute Force

- confining oneself to 1D model, ignoring layer structure, have:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = C(f)$$

here $x \to$ vertical
$v \to$ vertical velocity

vertical $\to$ $\perp$ to layer

$m_e v$ * $\Rightarrow$

$$\frac{\partial \langle p_e \rangle}{\partial t} - q\left\langle E \int dv\, f \right\rangle = -\nu_{e,i} n_0 m_e \bar{V}_e$$

collisional loss to ions

$$\frac{\partial \langle p_e \rangle}{\partial t} - q n_0 \langle E \rangle - q \langle \tilde{E}\tilde{n} \rangle = -\nu_{e,i} n_0 m_e \bar{V}_e$$

so

$$\langle E \rangle + \frac{\langle \tilde{E}\tilde{n} \rangle}{n_0} - \frac{1}{n_0 q}\frac{\partial \langle p_e \rangle}{\partial t} = + \frac{\nu_{ei} m_e n_0 q \bar{V}_e}{n_0 q^2}$$

at ⓒ stationary state,

$$\langle E \rangle + \left\langle \tilde{E}\frac{\tilde{n}}{n_0} \right\rangle = \eta \langle J \rangle$$

driving field $\sim \langle VB \rangle$

electron acceleration by turbulence ⇒ "anomalous resistivity"

collisional resistivity

to calculate:

$$\left\langle \tilde{E}\frac{\tilde{n}}{n_0} \right\rangle = \sum_k +ik\,\hat{\phi}_{-k}\frac{\tilde{n}^e_k}{n_0}$$

$$= \int dv \sum_k ik\,\hat{\phi}_{-k}\,\tilde{f}^e_k$$

electron density perturbation

$f^e_k \rightarrow f^{e\,L}_k$

- quasilinear calculation
- stationarity ⇒ resonant transport.

ⓑ Conservation Argument

- as in ⓐ anticipate stationarity ⇒ resonant quasilinear evolution

– recall,

$$\frac{\partial}{\partial t}\left(\mathcal{E}^{RP} + \mathcal{E}^{wave}\right) = 0$$

$$\frac{\partial}{\partial t}\left(P^{RP} + P^{wave}\right) = 0$$

$$\mathcal{E}^{w}_{k} = \omega_k \left.\frac{\partial \epsilon}{\partial \omega}\right|_{k} \frac{|E_k|^2}{8\pi} \equiv \omega_k \underbrace{N_k}_{\#\ \text{quanta}}$$

$$P^{\omega}_{k} = \frac{k}{\omega} \mathcal{E}^{w}_{k} = k N_k$$

as waves (CDIA) electrostatic, can ignore field momentum.

so, for resonant electrons!

$$\frac{\partial}{\partial t} P^{RP}_{e} = -\frac{\partial}{\partial t} P^{w} = -\sum_{k} \left(2\gamma^{e}_{k}\right) \frac{k}{\omega_k} \mathcal{E}^{w}_{k}$$

$\gamma^{e}_{k} \equiv$ electron (resonant) growth rate

but

$\frac{\partial}{\partial t} P^{RP}_{electron}$ → slowing down

→ macro-representation as effective collision frequency

so $\frac{\partial P^{RP}_{electron}}{\partial t} = -n m_e \underbrace{\nu_{eff}}_{\text{effective collision frequency}} \bar{V}$

slowing down by resonant scattering (resonant particle interaction)

$$\boxed{n m_e \nu_{eff} \bar{V} = \sum_k (2\gamma_k^e) \frac{k}{\omega_k} \mathcal{E}_k^{\omega}}$$

- defines $\bar{V}$
- for macro-micro link

$$\bar{V} = \frac{cB}{4\pi n q \Delta}$$

* - n.b. if 2D, 3D theory, i.e. $\perp$ dynamics ⇒ non-resonant scattering ⇔ wave driven momentum flux i.e. $\Pi_{\perp,\parallel}$ → $\perp$ radiation $\parallel$ momentum. Relation to whistler interpretation of Bellan ?! There need include wave radiation in energy balance.

so now have

$$\boxed{\begin{aligned} n\, m\, \nu_{eff}(B_0, \Delta)\bar{V} &= \sum_k (2\gamma_k^e)\frac{k}{\omega_k}\, \mathcal{E}_k^w \\ \Delta^2 &= \frac{L}{V_A}\left(1 + \frac{c^3}{\omega_{pe}^2}\nu_{eff}\right) \end{aligned}}$$ (1)

$\Rightarrow$ need γ_k^e, $\mathcal{E}_k^w$ and $\langle f^e \rangle$ evolution

at simplest level, proceed via linear/quasilinear theory in 1D
- at more advanced level:
  - consider 1D phase space structures
    - $\rightarrow$ electron/ion clumps, momentum exchange
    - $\rightarrow$ electron scattering off ion hole
  - consider 3D $J_\parallel$ driven instability with electron viscosity

Now, proceed in usual fashion:

$\gamma_k^e \rightarrow$ linear theory

$\mathcal{E}_k^w \rightarrow$ nonlinear saturation

$\langle f_e \rangle \rightarrow$ QL equation — flattening

For linear theory of CDIA;

$$\nabla^2\hat{\phi} = -4\pi n_0|e|\left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0}\right)$$

$$\hat{n}_i/n_0 = \frac{k^2 c_s^2}{\omega^2}\frac{|e|\phi}{T}$$

$$\frac{\hat{n}_e}{n_0} = \frac{|e|\hat{\phi}}{T}\left[1 - i\Gamma(k)\right]$$

$\Gamma(k)$;

$$\frac{\partial\tilde{f}}{\partial t} + v\frac{\partial\tilde{f}}{\partial x} = -\frac{|e|}{m_e}\tilde{E}\frac{\partial\langle f\rangle}{\partial v}$$

$$\tilde{f} = \frac{|e|\hat{\phi}}{T}\langle f\rangle + g$$

$$\frac{\partial\tilde{g}}{\partial t} + v\frac{\partial\tilde{g}}{\partial x} = -v\frac{\partial}{\partial x}\left(\frac{|e|\hat{\phi}\langle f\rangle}{T}\right) + \frac{|e|}{m_e}\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial\langle f\rangle}{\partial v}$$
$$-\frac{\partial}{\partial t}\left(\frac{|e|\hat{\phi}\langle f\rangle}{T_e}\right)$$

$$= v\frac{\partial\tilde{\phi}}{\partial x}\frac{|e|}{T}\langle f\rangle + \frac{|e|}{\cancel{m_e}}\frac{\partial\tilde{\phi}}{\partial x} - \frac{(v-\bar{v})}{T/\cancel{m_e}}\langle f\rangle - \frac{\partial}{\partial t}\frac{|e|\hat{\phi}\langle f\rangle}{T_e}$$

$$= -\frac{\partial}{\partial t}\frac{|e|\hat{\phi}}{T}\langle f\rangle + \bar{v}\frac{\partial}{\partial x}\frac{|e|\hat{\phi}\langle f\rangle}{T}$$

50.

$$\Rightarrow g_k = \frac{i(\omega - k\bar{V})}{-i(\omega - kV)} \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$= -\left(\frac{\omega - k\bar{V}}{\omega - kV}\right) \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$\omega_k^{2\,(0)} = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2}$$

$$-i\,r(k) = \int dV + \frac{(\omega - k\bar{V})}{(\omega - kV)} \langle f \rangle$$

$$\bar{f} = \frac{1}{\sqrt{\pi}} \exp\left[\frac{-(\omega/k - \bar{V})^2}{V_{Th}^2}\right]$$

$$= -(\omega - k\bar{V}) \frac{(i\pi)}{|k| V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$$1 + k^2 \lambda_D^2 = \frac{k^2 c_s^2}{\omega^2} + \frac{(\omega - k\bar{V})}{|k| V_{Th}} (i\pi) \bar{f}\Big|_{\omega/kV_{Th}}$$

$$\omega \to \omega + \delta\omega$$

$$0 = -\frac{2\delta\omega}{\omega} + \frac{(\omega - k\bar{V})(-i\pi)}{|k| V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$$\frac{\delta\omega}{\omega} = -\frac{i\pi}{2} \frac{(\omega - k\bar{V})}{|k| V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$\delta\omega \Rightarrow i\gamma_k$ growth rate

(2) $$\boxed{\gamma_k \sim \frac{-\pi}{2} \omega_k \frac{(\omega - k\bar{V})}{|k| V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}} \Rightarrow$$

$\gamma > 0$ for
$\bar{V} > c_s$
$\Rightarrow$ critical velocity

for $\langle f \rangle$ evolution,

$$\frac{\partial \langle f \rangle}{\partial t} = +\frac{\partial}{\partial v} \sum_k \frac{+|e|}{m_e} \tilde{E}_{-k}\, \tilde{g}_k$$

$$= +\frac{\partial}{\partial v} \sum_k \frac{|e|}{m_e} \tilde{E}_{-k} \left(\frac{-(\omega - k\bar{V})}{(\omega - kV)} \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle \right)$$

$$= \frac{\partial}{\partial v} \sum_k \frac{T_e |e|}{m_e} \frac{+ik\tilde{\phi}_k}{T_e} \left(-(\omega - kV)\, i\pi\delta(\omega - kV) \right) \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle$$

$$= \frac{\partial}{\partial v} \sum_k (-v_{th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{V})\, \pi\delta(\omega - kV) \langle f \rangle$$

$$\boxed{\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} \sum_k (-v_{th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{V})\, \pi\delta(\omega - kV) \langle f \rangle} \quad (3)$$

— mean evolution

Note:

— really only assumed $\langle f \rangle = \langle f((V - \bar{V})^2 / 2v_{th}^2) \rangle$

so $\frac{\partial \langle f \rangle}{\partial V} = \left(\frac{V - \bar{V}}{v_{th}^2} \right) \langle f \rangle$

and

$\langle f \rangle' = -\langle f \rangle$

→ minimal assumption on structure

– can write as $\bar{v}$ evolution

$$\bar{v} = \int dv\, v \langle f\rangle \Big/ \int dv \langle f\rangle$$

$$\Rightarrow \boxed{\frac{\partial \bar{v}}{\partial t} = + \int dv \sum_{k} v_{Th}^2 \left|\frac{e\phi_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \bar{v}\right) \pi\delta(\omega - kv) \langle f\rangle}$$

$\omega/k < \bar{v} \Rightarrow \partial\bar{v}/\partial t < 0$

$\quad > \bar{v} \Rightarrow \partial\bar{v}/\partial t > 0$

Remains to determine fluctuation intensity level

Generically, can write:

$$\frac{\partial}{\partial t}\mathcal{E}_k^\omega = \gamma_k \mathcal{E}_k^\omega - \left(\sum_{k'} \omega_k C(k,k') \frac{\mathcal{E}_{k'}^\omega}{nT}\right)\mathcal{E}_k^\omega$$
$$- \left(\sum_{k',k''} \omega_k C_2(k,k',k'') \frac{\mathcal{E}_{k'}^\omega}{nT} \frac{\mathcal{E}_{k''}^\omega}{nT}\right)\mathcal{E}_k^\omega$$

spectral equation constituents:

a – linear growth

b – quadratic nonlinearity → 3 wave coupling / NL ion-wave interaction

c – cubic "NL" → wave coupling

Now, for ion-acoustic wave:

- 3 wave coupling effects negligible
  ⇒ can't satisfy resonance
- NL wave-particle effects weak ⇒ intrinsically

⇒ consider 4 wave process

$$\boxed{\frac{\partial \mathcal{E}^{\omega}_{k}}{\partial t} = \left[\gamma_k - \omega_k \, B(\omega, k) \left(\frac{\mathcal{E}^{\omega}_{k}}{nT} \right)^2 \right] \mathcal{E}^{\omega}_{k}} \qquad (5)$$

- 'cartoon' NL saturation equation

Now, (1)-(5) ⇒ $\left\{ \begin{array}{l} \text{coupled, (2)-stationary} \\ \text{micro-macro system} \end{array} \right.$

⇒ describe anomalous resistivity dynamics and its effect on reconnection

⇒ coupled solution corresponds to solution of the problem

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$$①\begin{cases} n m_e \nu_{eff}(B,\Delta)\,\bar{V} = \sum_k 2\gamma_k^e \frac{k}{\omega_k} \mathcal{E}_k^w \\ \Delta^2 = \frac{L}{V_A}\left(M + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right), \quad \bar{V} = cB/4\pi n q \Delta \end{cases}$$

$$②④\begin{cases} \gamma_k^e = -\frac{\pi}{2}\,\omega_k \frac{(\omega - k\bar{V})}{|k|v_{Th}}\,\bar{f}\Big|_{\omega/kv_{Th}} \\ \frac{\partial \bar{V}}{\partial t} = \int dv \left(\sum_k v_{Th}^2 \left|\frac{e\tilde{\phi}_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \bar{V}\right)\pi c\,\delta(\omega - kv)\langle f\rangle\right) \end{cases}$$

$$⑤\left\{ \frac{\partial \mathcal{E}_k^w}{\partial t} = \left[\gamma_k - \omega_k B(\omega,k)\left(\frac{\mathcal{E}_k^w}{nT}\right)^2\right]\mathcal{E}_k^w \right.$$

Now, stationarity $\Rightarrow$

$$\mathcal{E}_k^w = nT\left(\gamma_k/\omega_k B\right)^{1/2}$$

$$\gamma_k = +\frac{\pi}{2}\frac{(\bar{V} - c_s)}{|kv_{Th}|}\,k\omega_k\,\bar{f}\Big|_{\omega/kv_{Th}}.$$

So, for scalings:

55.

$$\gamma_{eff} = \frac{1}{nm\bar{V}} \sum_k 2\gamma_k^e \frac{k}{\omega_k} \varepsilon_k^\omega$$

$$\sim \frac{1}{nm\bar{V}} \frac{(\bar{V}-C_s)}{|k|v_{th}} k\cancel{\omega_k} \bar{f}\Big|_{\frac{\omega_k}{kv_{th}}} \frac{k}{\cancel{\omega_k}}(nT)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim \frac{(1-C_s/\bar{V})}{\cancel{n}m} \frac{k^2 \cancel{n}T}{|k|v_{th}} \left(\frac{\gamma_k}{\omega_k B}\right) \bar{f}\Big|_{\frac{\omega_k}{kv_{th}}}$$

$$\sim (1-C_s/\bar{V})\,\bar{f}\Big|_{\omega/|k|v_{th}} \left(\frac{k^2 v_{th}}{|k|}\right)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim (1-C_s/\bar{V})\left(\frac{k\cancel{\omega_k}}{|kv_{th}|}\frac{(\bar{V}-C_s)}{\cancel{\omega_k}B}\bar{f}\right)^{1/2} \bar{f}\,\frac{k^2 v_{th}}{|k|}$$

$$\sim \left[(\bar{V}-C_s)k\right]^{3/2} \bar{f}^{3/2} \frac{k(v_{th}/\bar{V})}{|k|\,|kv_{th}|^{1/2}}$$

$$\boxed{\gamma_{eff} \sim \frac{\left[(\bar{V}-C_s)k\right]^{3/2}}{|kv_{th}|^{1/2}} \frac{v_{th}}{\bar{V}} \bar{f}^{3/2}\left(\frac{k}{|k|}\right)}$$

[Turbulent collision frequency

so have

$$\boxed{\begin{aligned} \Delta^2 &= \frac{L}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right) \\ \overline{V} &= cB_0/4\pi n_0 q \Delta \\ \nu_{eff} &= \frac{\left[(\overline{V}-C_s)k\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\overline{V}} \overline{F}^{3/2}\left(\frac{k}{|k|}\right) \end{aligned}}$$

with:

$$\boxed{\begin{aligned} \frac{\gamma_k}{\omega_k} &= \frac{\pi}{2}\left(\overline{V}-C_s\right)\frac{k}{|k| v_{Th}} \overline{F} \\ \overline{F} &= \frac{1}{\sqrt{\pi}} \exp\left[-\left(\omega/k - \overline{V}\right)^2/2 v_{Th}^2\right] \end{aligned}}$$

→ characterize micro-macro coupling with anomalous resistivity

→ now can envision structure

- finite current, $\Delta \sim (L\eta/V_A)^{1/2}$

  $\nu_{eff} = 0$

so if:

- decrease η ⇒ Δ decreases

- Δ decreases ⇒ $\overline{V}$ increases

- $\bar{V}$ increases $\Rightarrow \gamma_{11} > 0$
- $\gamma_{11} > 0 \Rightarrow \begin{cases} \varepsilon^{\omega}_{11} > 0 \\ \nu_{eff} > 0 \end{cases}$
- $\nu_{eff} > 0 \Rightarrow \begin{cases} \Delta \text{ increases} \\ \bar{V} \text{ decreases} \end{cases}$

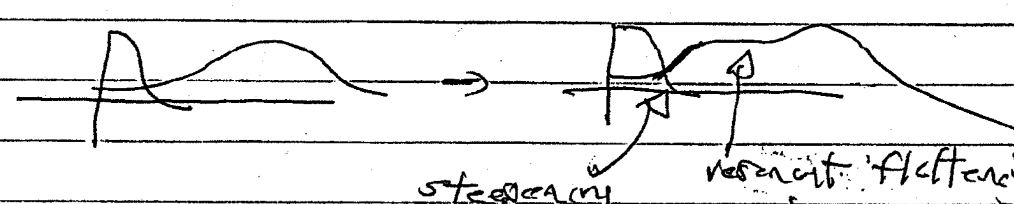
$\Rightarrow$ decreasing ν so Δ decreases triggers feedback so Δ increases $\rightarrow$ self-regulation / $\oplus$ feedback

$\therefore$ in this model, can expect:

- at low $\nu_{collisional}$, so $\bar{V}\big|_c \sim cB_0/4\pi n_0 e \Delta_c \sim C_s$

$\Rightarrow$ CDIA "hovers" near marginal stability

- for stronger drive (i.e. $B_0 \uparrow$)
  - $\Rightarrow$ ion interaction important
  - $\Rightarrow$ strong ion distortion possibly significant
  - $\Rightarrow$ granulation formation important
  - $\Rightarrow$ distortion of electron distribution function need by considered

i.e. 

58.

⇒ useful extensions:

- 1D avalanche model → avalanching ⇒ jitter effects on Δ, $\bar{V}$

- nonlinear noise effects via fluctuations

- 2D, 3D ⇒ wave radiation, esp. wave momentum flux ⊥ layer.

* - granulation effects ⇒ strong distortion hole (cf. later in course)

Comment:

This simple problem is surprisingly poorly understood. Excellent example of:

⇒ micro - macro feedback

⇒ self - regulation

⇒ marginal stability ...