Lecture II Supplement

Wave Adiabatics — Variational Approach
(c.f. Whitham).

i.e. from Wave train Lagrangian → Wave kinetics.

Wave Adiabatic Theory / Wave Kinetics

— frequently encounter continuum problems with slowly varying parameters ⇒ adiabatic theory needed

⇒

— $\boxed{\text{wave kinetic equation}}$ (consequence of Liouville Thm.)

$$\partial_t N + (\underline{v}_{gr} + \underline{V})\cdot\underline{\nabla} N - \partial_{\underline{x}}(\omega + \underline{k}\cdot\underline{V})\cdot\partial_{\underline{k}} N$$

$$= \cancel{C(N)}$$

; obvious analogy to Boltzmann Eqn.

$\underset{\frown}{N} \equiv \mathcal{E}/\omega_{\underline{k}} \equiv$ wave action density / wave quanta density

$\mathcal{E}$ ↓ wave energy density

$$\mathcal{E} \equiv \frac{\partial}{\partial\omega}(\omega\epsilon_{\underline{k}})\bigg|_{\omega_{\underline{k}}} \frac{|E_{\underline{k}}|^2}{8\pi}$$, for e.s. waves

$N \Leftrightarrow \mathcal{E}$.

Characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial\omega}{\partial\underline{k}}\hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial\underline{x}}(\omega + \underline{k}\cdot\underline{V})$$

refraction by shear ↓ ($\underline{k}\cdot\underline{V}$)

↑ ($\frac{\partial}{\partial\underline{x}}$) refraction by parametric variation

— need:

$$\omega < \frac{1}{\lambda}\frac{d\lambda}{dt} \qquad \lambda \equiv \text{parameter}$$

⇝ $\boxed{\text{space and time scale separation}}$

$$\frac{1}{N}(\underline{v}_{gr}\cdot\nabla N) \ll \omega \quad \Rightarrow \quad \underline{\nabla}\cdot\underline{v}_{gr} < \omega$$

1a.

Transport Eqn - QM

$$\frac{\partial n}{\partial t} + v_{gr}\cdot\nabla n - \frac{\partial \omega}{\partial \underline{x}}\cdot\nabla_{\underline{k}}\, n = C(n)$$

$$\frac{\partial n}{\partial t} + \frac{\partial \hbar\omega}{\partial \hbar \underline{k}}\cdot\nabla n - \frac{\partial \hbar\omega}{\partial x}\cdot\nabla_{\hbar k}\, n = C(n)$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial \epsilon}{\partial \underline{p}}\cdot\nabla n - \frac{\partial \epsilon}{\partial \underline{x}}\cdot\nabla_{p}\, n = C(n)}$$

Used for:

$$\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial x} < 1/\lambda_{DB}$$

$$\lambda_{DB} = \hbar/p.$$

(CN) → interaction with comparable scale.
ignore here.

Examples :

- linear theory of Langmuir turbulence i.e. when will phonon grow?
- QL theory of Langmuir turbulence i.e. determine evolution of plasmon energy → net impact?!
- drift waves and sheared flow.
- transport equation, super-fluids $N = \frac{\varepsilon}{\omega}$ → dynamics?!

Fundamentals of wave kinetics

→ where does conservation of action emerge from?

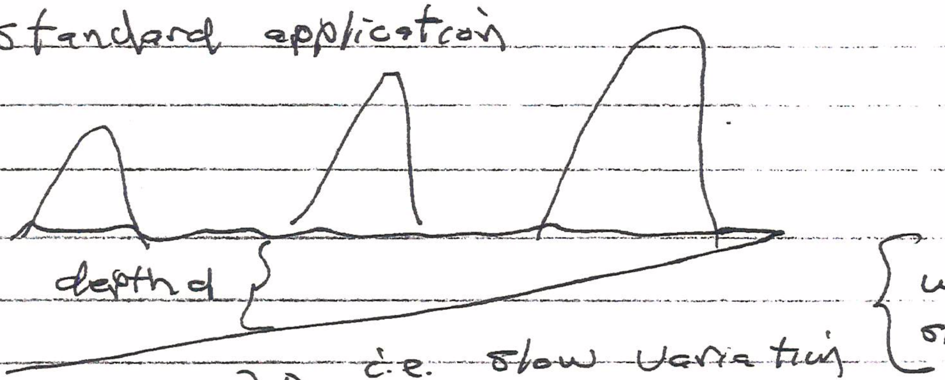
→ answer: phase symmetry of wave train underlies wave kinetics

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

⇒ standard application

⇒



beach ⟷

waves in shallow water

i.e. slow variation

⇒ $\frac{1}{d}\frac{d}{dx}d(x) \ll k$

- influx of wave energy

- depth $H(x,y)$ decreases

⇒ wave amplification, breaking.

- Derivation

Consider a system, [like ideal fluid MHD,] acoustics which can be described in terms of displacement $\underline{\xi}$;

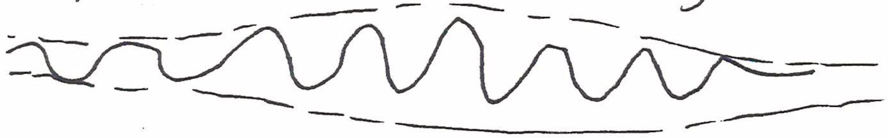
$$\text{i.e.} \quad \underline{\xi} = \mathrm{re}\left\{ A e^{i\overset{\text{phase}}{\phi}} + A^{*} e^{-i\phi} \right\}$$

displacement can be viewed as excitation level

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x} \; \mathcal{L}(\xi)$$

- Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, ala WKB :



$$S = \int dt \int d\underline{x} \; \mathcal{L}(\omega, \underline{k}, \underset{\text{amplitude}}{a}) \qquad \begin{cases} \underline{k} = \underline{\nabla}\phi \\ \omega = -\partial_t \phi \end{cases}$$

$$= \int dt \int d\underline{x} \; \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

- neglect all corrections to eikonal theory. (no higher order WKB).

→ where L corresponds to period-averaged Lagrangian

– ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\left.\begin{aligned} \delta S/\delta a &= 0 \\ \delta S/\delta \phi &= 0 \end{aligned}\right\} \rightarrow 2\text{ eqns}$$

[illegible]

$\downarrow$

[$G(k,\omega) \Leftrightarrow$ [illegible] G]

Now, in linear theory:

– $\mathcal{L} = \underset{\text{continuum,}}{G(\omega, k)}\, a^2$

$\left\{\begin{array}{l} G(\omega, \underline{k}) = 0 \\ \omega^2 = k^2 c_s^2, \end{array}\right.$ $\underline{\underline{\text{dispn}}}$

i.e. for MHD, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho\dot{\underline{\xi}}^2 - \frac{1}{2}\rho\left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

↳ eikonal form of stiffness matrix (→ potential energy)

$$\Rightarrow \underline{\xi}\cdot\underline{\underline{D}}\cdot\underline{\xi}$$

$\downarrow$

$D(k, \omega, \theta)$, as for linear wave

if: $\underline{\xi} = \underline{A}e^{i\phi} + A^* e^{-i\phi}$

$\underline{\underline{5_i}}$

$\underline{\underline{i}}$ $$\sigma(\omega, \underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2\right]$$

Now, 1) $\delta S/\delta a = 0$

$$\Rightarrow \quad G(\omega, k) = 0 \qquad \rightarrow \text{dispn relation}$$

but

$$G(\omega, k) = \rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2$$
$$= \rho\omega^2 - \underset{\hookrightarrow \text{ stiffness fctn.}}{D^2}$$

$\Rightarrow$ dispn. relation

2) $\delta S/\delta\phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)}\delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})}\delta(\phi_{\underline{x}})\right\}$$

end pts fixed, i b p

$$= \int dx \int d^3x \left\{ \partial_t\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial \underline{x}}\cdot\left(\frac{\partial \mathcal{L}}{\partial(\underline{\nabla}\phi)}\right)\right\}\delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\pm\partial(-\phi_t)}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0 \qquad \text{conservation eqn. !}$$

Now, have: $G(\underline{k}, \omega) = 0$ (dispn. reln.)

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} \cdot d\underline{k} = 0$$

$$\therefore \quad v_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$ (akin ω)

$\therefore$

$$\partial_t\left((\partial G/\partial \omega)\, a^2\right) + \underline{\nabla}\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\, \frac{\partial G}{\partial \omega}\, a^2\right] = 0$$

and so $N \equiv \dfrac{\partial G}{\partial \omega}\, a^2$

$$\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left(\underline{v}_{gr}\, N\right) = 0$$

though: (N not <u>yet</u> action)

$$G = \rho\omega^2 - \Lambda^2$$

$$\partial G/\partial \omega = 2\rho\omega = \frac{2\rho\omega^2}{\omega^2}$$

$$\frac{\partial G}{\partial \omega}\, a^2 \rightarrow \mathcal{E}/\omega \;!$$

Also note energy is conserved ⟺ G invariant to time translations.

so, Noethers thm ⇒ there exists an [redacted] energy conservation equation

have $\mathcal{L} = G(\underline{k},\omega)\, a^2$

$$\partial \mathcal{L}/\partial a = 0 \Rightarrow G(\omega,k) = 0$$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

and of course:

$$\underline{\nabla}\times\underline{k} = 0 \quad , \text{ as } \underline{k} = \underline{\nabla}\phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad , \text{ as } \partial_t \underline{\nabla}\phi = -\underline{\nabla}\left(-\frac{\partial}{\partial t}\phi\right) \checkmark$$

Now, $\mathcal{L} = 0$, as $G(\underline{k},\omega) = 0$

as expect $\frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N$, $\omega\frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$ $\xrightarrow{} 0$, [redacted]ectively → dispn relation satisfied

$$\rightarrow \frac{\partial}{\partial t}\left(\omega\frac{\partial \mathcal{L}}{\partial \omega} - \text{[redacted]}\underset{\downarrow 0}{\mathcal{L}}\right) + \nabla\cdot\underset{\downarrow \text{[redacted] } \underline{v}_{g}\mathcal{E}}{\left[-\omega\frac{\partial \mathcal{L}}{\partial \underline{k}}\right]} = 0$$

$$-\frac{\partial G/\partial k}{\partial G/\partial \omega}\,\frac{\partial G}{\partial \omega}a^2$$

$$\partial_t\left(\omega\mathcal{L}_\omega - \mathcal{L}\right) + \underline{\nabla}\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial\underline{u}}\right) = 0$$

check:

$$(\partial_t\omega)\mathcal{L}_\omega + \omega\,\partial_t(\mathcal{L}_\omega) - \partial\mathcal{L}/\partial t$$
$$+\underline{\nabla}\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial\underline{u}}\right) = 0$$

$\Rightarrow$

but $\partial_t\mathcal{L}_\omega = \underline{\nabla}\cdot(\mathcal{L}_{\underline{u}})$

$$\therefore\ (\mathcal{L}_\omega)(\partial_t\omega) + \omega\,\cancel{\nabla\cdot(\mathcal{L}_u)} - \omega\left(\cancel{\underline{\nabla}\cdot\mathcal{L}_{\underline{u}}}\right)$$
$$-\left(\frac{\partial\mathcal{L}}{\partial\underline{u}}\right)\cdot\underline{\nabla}\omega - \frac{\partial\mathcal{L}}{\partial t} = 0$$

but $\partial_t\underline{u} = -\underline{\nabla}\omega$ (order derivs)

$$(\partial_t\omega)(\mathcal{L}_\omega) + (\partial_t\underline{u})\cdot\frac{\partial\mathcal{L}}{\partial\underline{u}} - \frac{\partial\mathcal{L}}{\partial t} = 0 \quad \checkmark$$

(identity)

$$\Rightarrow \boxed{\partial_t\left\{\omega\frac{\partial\mathcal{L}}{\partial\omega} - \mathcal{L}\right\} + \underline{\nabla}\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial\underline{u}}\right) = 0}$$

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t\left\{\omega\frac{\partial\mathcal{L}}{\partial\omega}\right\} + \nabla\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

Poynting Form

so $\boxed{\mathcal{E} \equiv \omega\dfrac{\partial\mathcal{L}}{\partial\omega}}$ → {wave energy density

so $\dfrac{\partial\mathcal{L}}{\partial\omega} = \mathcal{E}/\omega$ → {wave action density ;

$= N(\underline{k}, \underline{x}, t)$ → (∼) adiabatic invariant for wave packet.

so have:

$$\boxed{\partial_t(N) + \nabla\cdot(\underline{v}_{gr}N) = 0}$$

wave-kinetic

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla}N - \frac{\partial\omega}{\partial\underline{x}}\cdot\nabla_{\underline{k}}N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla\cdot(\underline{v}_{gr}N) + \nabla_{\underline{k}}\cdot\left(-\partial\omega/\partial\underline{x}\,N\right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$ (i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{v}_{gr} N \right] = 0$$

Observe:

$\rightarrow$ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

$\rightarrow$ continuity-type equation in $\underline{x}$-space for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot \left(\underline{v}_{gr} N \right) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

Now $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian) (partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian) (total) relation, following packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla}\underline{k}$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\underline{k}}{\partial\underline{x}}$$

$\therefore$

$$\frac{\partial\underline{k}}{\partial t} = -\frac{\partial D}{\partial\underline{x}}$$

agree!

→ Now, can convert from N to E !

i.e. $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{\text{rays}} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\Big|_{\text{rays}} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\Big|_{\text{rays}} = 0$$

Now $\frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{d\underline{k}}{dt}$

from eikonal eqns:

$$= \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{\partial\omega}{\partial\underline{k}} - \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\omega}{\partial\underline{x}}$$

so if $\partial_t\omega = 0$ → energy conserved.

$\therefore\ \frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0$.

so $\partial_t\mathcal{E} + \underline{v}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\mathcal{E} = 0$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{v}_{gr}\,\mathcal{E}\right] = 0}$$

conserved energy density

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot [\underline{v}_{gr} \mathcal{E}] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

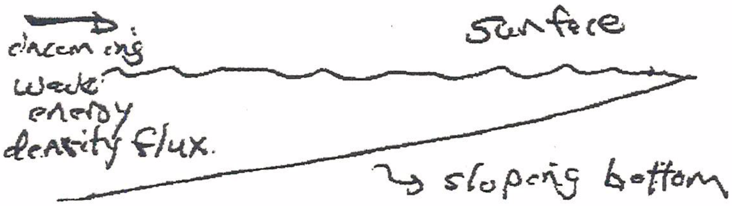
$$\Rightarrow \boxed{\underline{\nabla} \cdot [\underline{v}_{gr} \mathcal{E}] = 0}$$

incompressible wave energy flux !

$\Rightarrow$ v_{gr} drops $\Rightarrow$ $\mathcal{E} \uparrow$ $\Rightarrow$ blocking, breaking

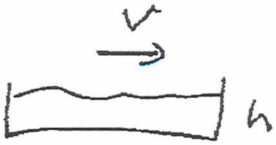
③ The beach....

Consider:



$H = H(x)$

Now, in shallow water $(\lambda > H)$



$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0$$

slope ↓

shallow water eqns.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

⟼ replaces pressure

$v = \cancel{v_0} + \tilde{v}$, $h = H + \tilde{h}$

$$\Rightarrow \quad -i\omega\tilde{h} + ikH\tilde{v} = 0$$
$$-i\omega\tilde{v} = -ikg\tilde{h}$$

$\therefore \Rightarrow \omega^2 = k^2 gH$ is dispersion relation

$\underline{gH \Leftrightarrow c_s^2}$

→ analogy with acoustics is obvious

$h \Leftrightarrow \rho \qquad c_s^2 = gH$

$v \Leftrightarrow v \qquad$ etc.

energy ⇒

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$$\Rightarrow \quad (1) \times \tilde{v} \;+\; (2) \times \left(g \cdot \frac{\tilde{h}}{H} \right)$$

$$\therefore \quad \frac{\partial}{\partial t} \tilde{v}^2 = -g\, \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial}{\partial t} \tilde{h}^2 = -g \frac{\cancel{H}}{\cancel{H}} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial t} \left(\frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \right) + \frac{\partial}{\partial x} \left(g \tilde{h} \tilde{v} \right) = 0$$

is energy theorom

$$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H}$$ is wave energy density.

$\omega / k = (gH)^{1/2}$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(v_{gr}\, \mathcal{E} \right) = 0$$

$$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty\,\mathcal{E}_\infty = \underset{\substack{\downarrow \\ \text{incoming} \\ \text{wave flux}}}{I}$$

$V_g = \sqrt{gH(x)}$

↳ shallow water waves

have zero dispersion

$$\therefore \quad \sqrt{g(H(x))}\;\mathcal{E}(x) = I$$

as $x \to$ shore $V_g \downarrow$ so wave energy must increase.

$$\text{Now} \quad \mathcal{E}(x) = \frac{\overline{\tilde{v}^2}}{2} + \frac{g\overline{\tilde{h}^2}}{2H} \simeq \frac{g\tilde{h}^2}{\cancel{2}H}$$

$$\sqrt{gH(x)}\;\frac{g\tilde{h}^2}{\cancel{2}H(x)} = I$$

$$\frac{\tilde{h}^2}{H(x)^2} = \frac{\cancel{2}I}{(\sqrt{g})^3}\left(\sqrt{H(x)}\right)^{-3}$$

i.e. $\boxed{(\tilde{h}/H)^2 \sim (\text{const})\,I/(H(x))^{3/2}}$

i.e. $\tilde{h}/H \to 1 \Leftrightarrow$ breaking $\Leftrightarrow$ as $H(x)$ drops.

N.B.:

→ if know bottom profile, can deduce displacement profile and <u>approximate</u> breaking point.

→ 2 D bottom contours ⇒ wave refraction

$$\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}} = -kg\left(\frac{\partial}{\partial \underline{x}} H(x,y)\right)$$

<u>i.e.</u> wavefronts tend to align with bottom contours approaching shore.