

Lecture III - Aspects of Nonlinear Waves, I

Recall:

- dispersionology of basic plasma waves
- most interesting → ion acoustic wave

$$\omega^2 = k^2 c_s^2 / (1 + k^2 \lambda_D^2)$$

⇒ - nonlinear waves ? → evolution past crit. to finite amplitude

- shocks
- solitons

⇒ ~~Gasdynamic~~ Gasdynamic Waves / shocks

observe, for quasi-neutral ion acoustic wave: $(k^2 \lambda_D^2 \ll 1)$

$$n_e = n_0 \exp\left[\frac{e|\phi|}{T}\right]$$



$$\frac{\partial n_i}{\partial t} + \partial_x (v n_i) = 0$$

or

$$\frac{\partial n_i}{\partial t} + v \frac{\partial n_i}{\partial x} = -n_i \frac{\partial v}{\partial x}$$

and

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -|e| \frac{\partial \phi}{\partial x}$$

$$= -\frac{\partial}{\partial x} \left(\frac{-|e| T}{|e|} \ln \left(\frac{n_e}{n_0} \right) \right)$$

$$= -\frac{T_e}{n_e} \frac{\partial n_e}{\partial x}$$

$$= \frac{T_e}{n_i} \frac{\partial n_i}{\partial x} \quad (\text{PN})$$

$$\infty \left\{ \frac{\partial n_i}{\partial t} + \partial_x (n_i u) = 0 \right.$$

$$\left. \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{T_e}{n_i} \frac{\partial n_i}{\partial x} \right\}$$

→ isomorphic to 1D gas-dynamics with isothermal EOS

→ ∞ , study gas dynamics!

1D Gas Dynamics - 'Simple Waves'

of { Landau/Lifshitz Fluids
Whitham } → steepening shocks etc.

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t v + v \partial_x v + \frac{1}{\rho} \partial_x p = 0$$

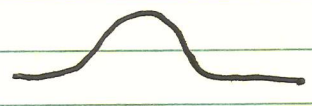
ideal

$$p = p(\rho)$$

Entropy: - S initially homogeneous

- $S = \text{const}$, till shock forms

1D



$$A = p(\rho)$$

{ speed vs ρ left
flow

so

$$\frac{d\rho}{dt} = 0$$

along

$$\partial_t \rho + \frac{d(\rho v)}{d\rho} \partial_x \rho = 0$$

$$\left(\frac{dx}{dt} \right)_\rho = \frac{d(\rho v)}{d\rho}$$

$$\partial_t v + \left(v + \frac{1}{\rho} \frac{dp}{dv} \right) \partial_x v = 0$$

$$p = p(\rho) \\ v = v(\rho)$$

so

$$\left(\frac{dx}{dt} \right)_\rho = \frac{d}{d\rho} \frac{d(\rho v)}{d\rho} = v + \rho \frac{dv}{d\rho}$$

n.b set on $\frac{d\rho}{dt} = 0$

$$\frac{dv}{dt} = 0$$

conservation for

{ characteristic eqn. → all info

likewise:

$$\left(\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho} \frac{d\rho}{dv}$$



but

$$v = v(\rho)$$

$$\rho = \rho(v)$$

via



$$\boxed{\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_v}$$

$$\rho \frac{dv}{d\rho} + v = v + \frac{1}{\rho} \frac{d\rho}{dv}$$

$$d\rho = c_s^2 d\rho$$

$$\rho \frac{dv}{d\rho} = \frac{c_s^2}{\rho} \frac{d\rho}{dv}$$

$$\left(\frac{dv}{d\rho}\right)^2 = c_s^2 / \rho^2$$

$$\boxed{v = \pm \int \frac{c_s}{\rho} d\rho = \pm \int \frac{d\rho}{\rho c_s}}$$

solved for flow speed
in terms density

Note:

$$- v = \pm \int \frac{c_s}{\rho} d\rho$$

$$- \rho \rho^{-\gamma} = \text{const.} \quad \gamma = 5/3$$

$$d\rho = c_s^2 d\rho = \alpha \gamma \rho^{\gamma-1} d\rho$$

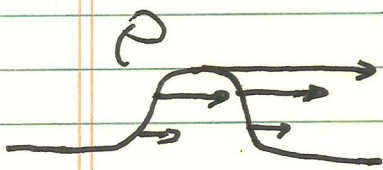
$$\boxed{c_s^2 = \alpha \gamma \rho^{\gamma-1}}$$

$$\alpha = \text{const}$$

$$v = \pm \int \frac{c_s^2}{\rho c_s} d\rho = \pm \alpha \gamma \int \rho^{1/3} d\rho$$

$$\boxed{v = \pm 3\sqrt{\alpha \gamma} \rho^{1/3}}$$

- Flow speed increases with density



- high density elements go faster.

- they:

$$\frac{dx}{dt} = v + \frac{1}{\rho} \frac{d\rho}{dt}$$

$$v = \pm \int \frac{d\rho}{\rho c_s}$$

$$dv = \frac{d\rho}{\rho c_s}$$

$$\underline{\rho} \quad \frac{dP}{dv} = \rho c_s$$

$$\rho = \rho(v)$$

$$\frac{\partial x}{\partial t + v} = v \pm c_s(v)$$

$$x = t [v \pm c_s(v)] + f(v)$$

"simple wave" solution

general
nonlinear.

Check: linearized limit

$$x = t [v \pm c_s(v)] + f(v) + x_0$$

$$x = x_0 \pm c_s(v)t$$

→ Why "simple" } — No characteristic scale

⇒ alternate approach

so if no characteristic scale
all quantities depend only on

$$\xi = x/t$$

$$\rho = f(\xi) \quad \xi = x/t \quad \underline{7.}$$

↳ Velocity formed by x, t in absence/scalar

so $\partial_x = 1/t \partial/\partial \xi$

$$\partial_t = -\frac{\xi}{t} \frac{\partial}{\partial \xi}$$

$$\Rightarrow \partial_x \rho + \rho \partial_x v + v \partial_x \rho = 0$$

$$\partial_t v + v \partial_x v = -1/\rho \partial_x p$$

become:

$$-\frac{\xi}{t} \rho' + \frac{\rho}{t} \dot{v} + \frac{v}{t} \rho' = 0$$

$$1 = \partial/\partial \xi$$

$$-\frac{\xi}{t} \dot{v} + \frac{v}{t} \dot{v} = -\frac{c_s^2}{t} \frac{\rho'}{\rho}$$

so

$$(v - \xi) \rho' + \rho \dot{v} = 0$$

$$(v - \xi) \dot{v} = -c_s^2 \frac{\rho'}{\rho}$$

like linear dispersion theory, treat ξ as eigenvalue.

$$(v - \xi) \rho' + \rho \dot{v} = 0$$

$$-c_s^2 \frac{\rho'}{\rho} + (v - \xi) \dot{v} = 0$$

$$(v - \varepsilon)^2 = c_s^2$$

$$\# \quad \Sigma = v \pm c_s$$

$$\frac{x}{t} = v \pm c_s \quad \checkmark$$

$$x = (v \pm c_s) t.$$

From "eigenvector", relate v , ρ etc.

$$v - \varepsilon = -c_s$$

$$(v - \varepsilon) \rho' + \rho v' = 0$$

$$-c_s \rho' + \rho v'$$

$$\rho dv = c_s d\rho$$

$$dv/d\rho = c_s/\rho$$

$$v = \int c_s d\rho/\rho = \int d\rho/c_s \rho$$

- equivalent to previous, with $f(v) = 0$

- corresponds to Kundt's version of simple wave

- can also write:

$$v = \int + (-dp/dV)^{1/2}$$

$$d(1/\rho) = dV = -\frac{1}{\rho^2} d\rho, \quad dp = c_s^2 d\rho$$

$$\left(v = \int \left(\frac{1}{\rho^2} d\rho c_s^2 d\rho \right)^{1/2} \right. \\ \left. = \int \frac{d\rho}{\rho} c_s \right)$$



→ Physics of Simple Wave

- have established simple wave in gas dynamics, with key element of scale invariance
- now elucidate simple wave physics ⇒ shocks.

Similarity flow → simple wave with $f(v) = 0$

$$x = t [v \pm c_s(v)]$$

wave
front
ref'n

Can write general solution,
for adiabatic process:

$$\rho \delta^{-\gamma} = \text{const}$$

$$T \delta^{-(\gamma-1)} = \text{const}$$

$$T = c_s^2 = \rho^{\gamma-1}$$

$$\frac{d\rho}{\rho} = c_s^2 = T$$

$$\rho = \rho T$$

$$\rho T^{1/\gamma-1} = \text{const}$$

as $c_s^2 \sim T$

$$\rho = \rho_0 \left(\frac{c}{c_0} \right)^{2/\gamma-1}$$

(eliminates
→ density)

but $v = \pm \int c \frac{d\rho}{\rho}$

so

$$v = \pm \frac{2}{\gamma-1} (c - c_0)$$

so finally can write:

$$c = c_0 \pm \frac{\gamma}{2} (\gamma - 1) v$$

$$\rho = \rho_0 \left(1 \pm \frac{(\gamma - 1) v}{c_0} \right)^{2/\gamma - 1}$$

$$p = p_0 \left(1 \pm \frac{\gamma}{2} (\gamma - 1) v / c_0 \right)^{2\gamma / \gamma - 1}$$

so then:

$$x = t [v \pm c_s(v)] + f(v)$$

$$x = t \left[v \pm \left(c_0 \pm \frac{\gamma}{2} (\gamma - 1) v \right) \right] + f(v)$$

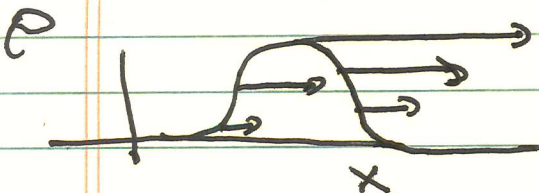
Now:

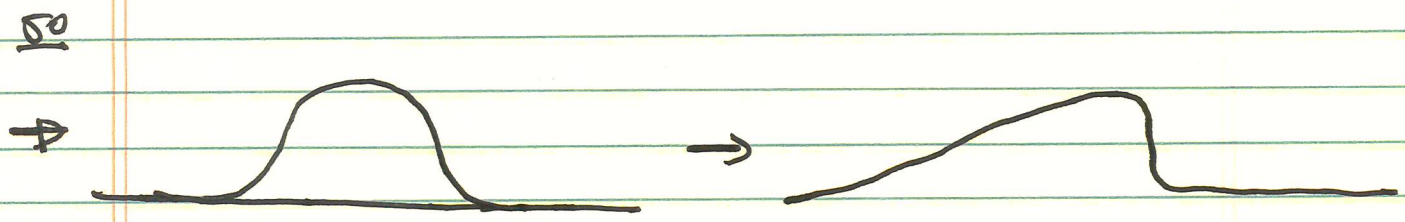
— point on wave profile moves at

$$u = \frac{\partial x}{\partial t} = v \pm c_s$$

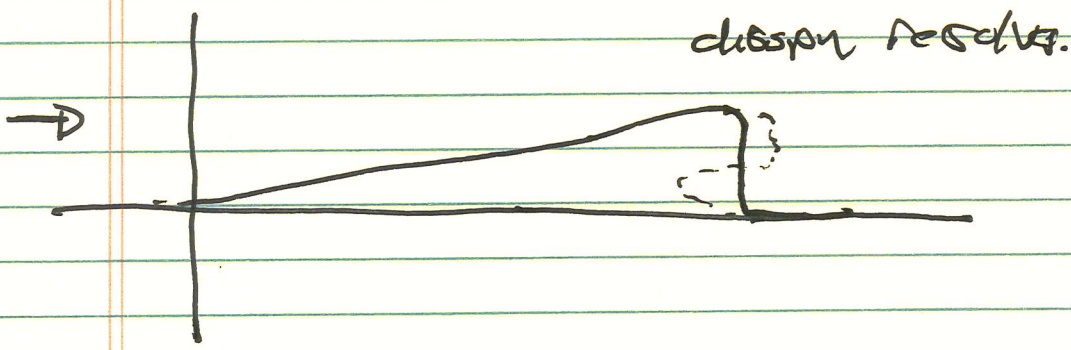
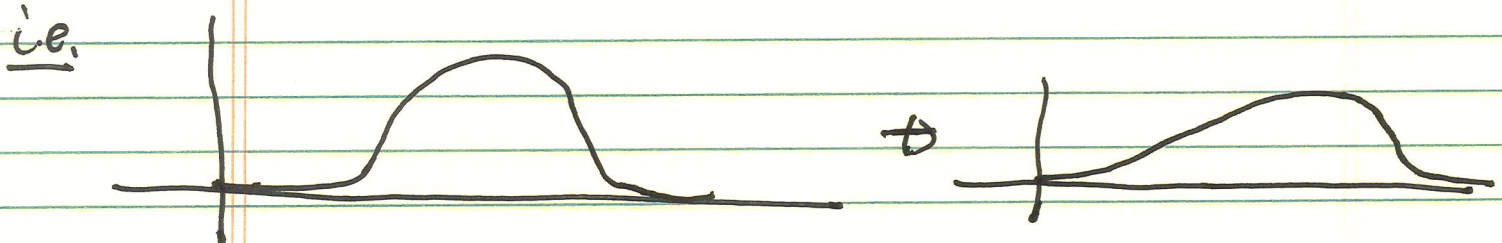
and $du/d\rho > 0$

speed increases
with density!



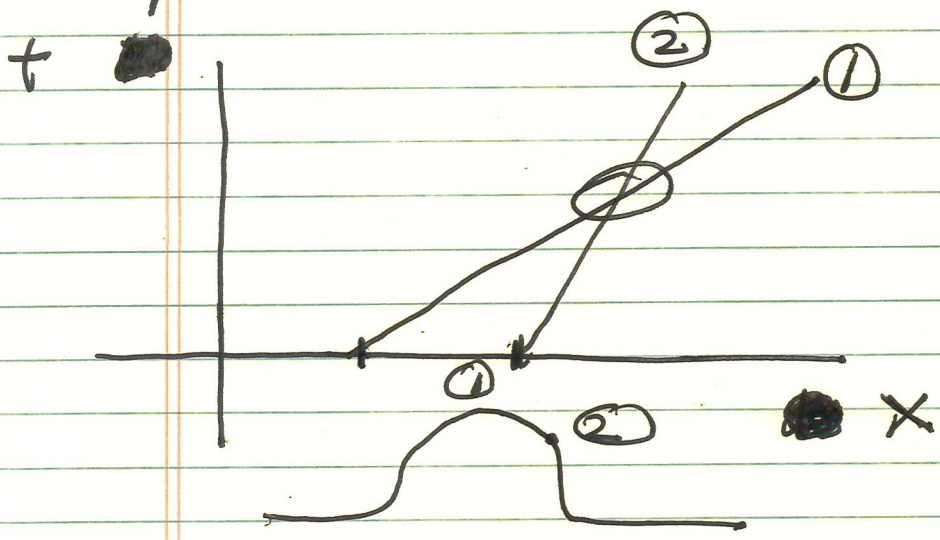


if $dp/dx \neq 0$ anywhere in i.v.d.
 \Rightarrow { overtaking and shock formation
 { discontinuity forms
 { dissipation no longer negligible



When does breaking / shock formation occur?

→ Breaking occurs when 2 critical points arrive at same position at same time.



Overtaking
↓
characteristics crossing

$$x = t \left[c_0 + \frac{1}{2} (\gamma + 1) v \right] + f(v)$$

$$x = t \left[c_0 + \frac{1}{2} (\gamma + 1) (v + \delta v) \right] + f(v + \delta v)$$

~~$$c_0 t + \frac{1}{2} (\gamma + 1) v t + f(v)$$~~

~~$$= c_0 t + \frac{1}{2} (\gamma + 1) t v + \delta v \frac{1}{2} (\gamma + 1) t + f(v) + \delta v f'(v)$$~~

$$= \delta v \left[\frac{1}{2} (\gamma + 1) t + f'(v) \right]$$

shock

Difference $\rightarrow 0$ 8

$$t_{shock} = \frac{-2f'(v)}{(\gamma+1)}$$

- need $f'(v) < 0$ for overtaking

- Generally:

need:

$$\left(\frac{\partial x}{\partial v}\right)_+ = 0$$

$$\left(\frac{\partial^2 x}{\partial v^2}\right)_+ = 0 \quad \text{inflection}$$

$$\left(\frac{\partial x}{\partial v}\right)_+ = \frac{1}{2}(\gamma+1)t + t f' = 0 \quad \checkmark$$

~ formation

→ Shocks — Flows with Discontinuity

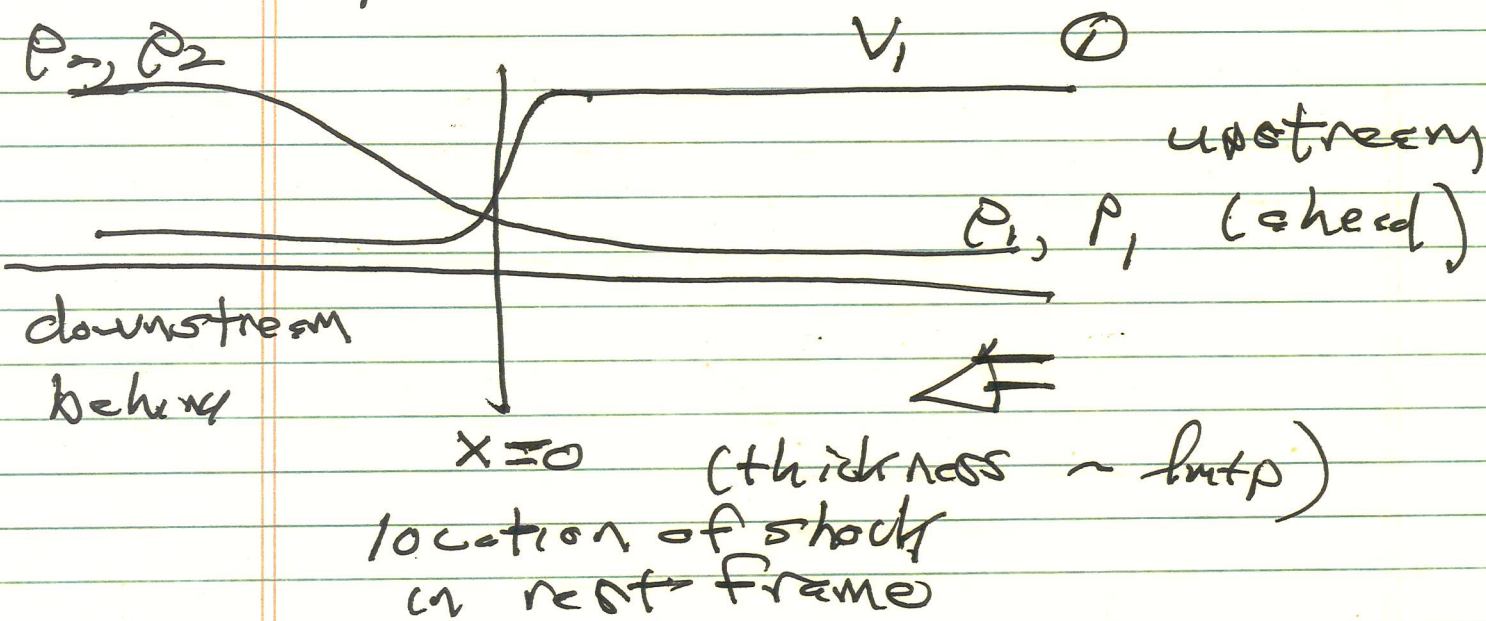
— once wave steeper and breaks

→ discontinuity appears

— "shock" — localized region of rapid change/discontinuity

— dissipation essential in shock
(thickness $\sim \lambda_{mp}$)

Classic picture:



→ Shock converts kinetic energy into thermal energy, compression

Conservation relations \Rightarrow jump conditions

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad \begin{array}{l} \text{continuity,} \\ \text{mass} \end{array}$$

$$\partial_t \left(\frac{1}{2} \rho v^2 + \rho e \right) + \underline{\nabla} \cdot \left(\rho \underline{v} \left(\frac{v^2}{2} + \frac{\gamma p}{(\gamma-1)\rho} \right) \right)$$

\Rightarrow

energy

$$\frac{\gamma p}{\gamma-1} = \frac{p}{\gamma-1} + p \quad \begin{array}{l} \hookrightarrow p v \text{ work on} \\ \text{surroundings} \end{array}$$

\downarrow
 energy density

$$\frac{\partial \rho v_i}{\partial t} = - \frac{\partial \pi_{ik}}{\partial x_k} \quad \hookrightarrow \text{stress tensor}$$

$$\pi_{ik} = p \delta_{ik} + \rho v_i v_k$$

\leadsto tangential components v continuous

= so integrating,

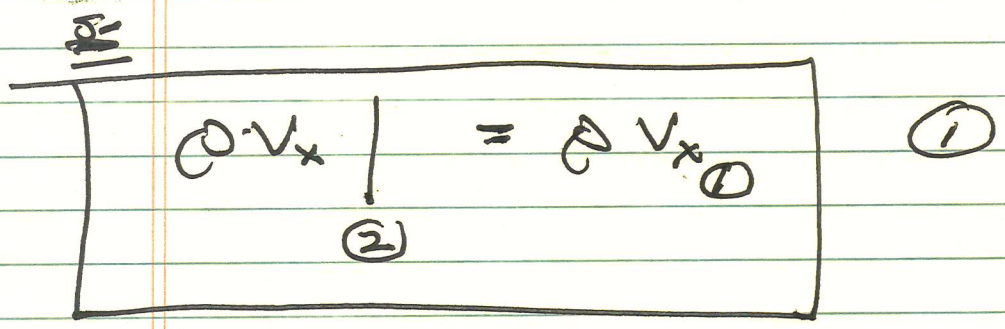
Work in shock frame - $U=0$

$$\partial_t \rho = -U \partial_x \rho$$

$$-\int U \partial_x \rho = (\rho_2 - \rho_1) U$$

but $U=0$,

mass



continuity
mass flux

energy

$v_n \equiv v_{nL}$ plane

$$\rho v_n \left(\frac{1}{2} v^2 + \frac{\gamma P / \rho}{\gamma - 1} \right) \Big|_2 = \rho v_n \left(\frac{1}{2} v^2 + \frac{\gamma P / \rho}{\gamma - 1} \right) \Big|_1$$

but

$$\rho v \Big|_2 = \rho v \Big|_1$$

and continuity parallel \rightarrow

$$V_y^2 \textcircled{2} = V_y^2 \textcircled{1}$$

02

$$\left(\frac{V_x^2}{2} + \frac{\gamma P / \rho}{\gamma - 1} \right) \textcircled{2} = \left(\frac{V_x^2}{2} + \frac{\gamma P / \rho}{\gamma - 1} \right) \textcircled{1}$$

2

and momentum conservation:

$$\left(\rho V_x^2 + P \right) \textcircled{2} = \left(\rho V_x^2 + P \right) \textcircled{1}$$

normal component, only, varies.

03; have 3 Rankine-Hugoniot jump/continuity conditions \rightarrow

$$[] = () \textcircled{2} - () \textcircled{1}$$

$$[\rho v_x] = 0$$

$$\left[\frac{v_x^2}{2} + \frac{\gamma p}{\gamma - 1} \right] = 0$$

$$[\rho v_x^2 + p] = 0$$

in shock frame

in Fixed coordinate frame:

$$v_x = v_n - U$$

\downarrow
normal v
in fixed coords

\rightarrow shock velocity

then on to:

- shock adiabatic.

$$- v_2/v_1, \rho_2/\rho_1,$$

$$P_2/P_1$$

\downarrow
compression

\downarrow
~~compression~~
heating