Nonlinear Waves II

- initial study: gas-dynamic 'simple waves'
  - scale free
  - non-dispersive

- now: dispersive
  - scale
  - solitons ⇒ ion acoustic

Point: $\omega^2 = c_s^2 k^2 / 1 + k^2 \lambda_D^2$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 / 1 + k^2 \lambda_D^2$$

dispersion

$\lambda_D \to 0$: gas dynamic limit
non-dispersive

⇒ all harmonics have same phase velocity

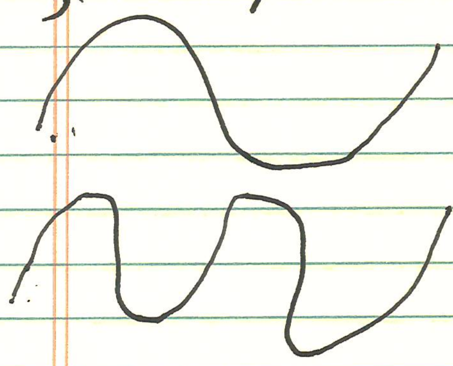
$$(\omega/k)^2 = c_s^2$$

self-beat:

$$(2\omega/2k)^2 = c_s^2$$

etc.

As wave steepens by nonlinear interaction, all harmonics move at c_s as phase velocity.



but with dispersion:

$$(\omega/k)^2 = c_s^2 / 1+k^2\lambda_D^2$$

$$\left(\frac{2\omega}{2k}\right)^2 = c_s^2 / 1+\underbrace{4k^2\lambda_D^2}$$

then fundamental and self-beats <u>disperse</u>.

$\therefore$ 2 routes to balance of steepened fronts:

n.l. steepening vs. dissipation

$\rightarrow$ shock,

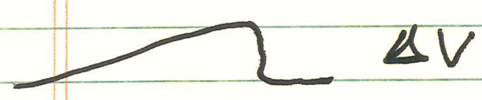
n.l. steepening vs. dispersion

$\rightarrow$ soliton (1D)

i.e. Schematic:

recall 1D compressible hydro - Burgers Eqn.

$$\partial_t V + V\partial_x V = \nu \partial_x^2 V$$



$$\Delta V \cancel{\frac{\Delta V}{\Delta x}} \sim \nu \frac{\cancel{\Delta V}}{(\Delta x)^{\cancel{2}}}$$

$$\boxed{\Delta x \sim \nu / \Delta V} \rightarrow \text{shock layer thickness}$$

dissipn sets.

Now, can generalize:

$$\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_D^2$$

$$\omega \cong k c_s \left(1 - \frac{k^2 \lambda_D^2}{2}\right)$$

so

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + c_s \partial_x \mathcal{E} + \mathcal{E}\partial_x \mathcal{E} = \nu \partial_x^2 \mathcal{E} + \frac{c_s \lambda_D^2}{2} \partial_x^3 \mathcal{E}}$$

$c_s, \lambda_D \to 0 \Rightarrow$ Burgers

$\nu \to 0 \Rightarrow$ KdV

KdV

$$\boxed{\partial_t \mathcal{E} + c_s \partial_x \mathcal{E} + \mathcal{E} \partial_x \mathcal{E} = \frac{c_s \lambda_D^2}{2} \partial_x^3 \mathcal{E}}$$

dispersion controls steepening

$$\frac{\mathcal{E}}{\Delta} \mathcal{E} \sim \frac{c_s \lambda_D^2}{2} \frac{\mathcal{E}}{\Delta^3}$$

$$\Delta^2 \sim \frac{c_s \lambda_D^2}{2 \mathcal{E}} \qquad \Delta \sim \left(\frac{c_s}{2\mathcal{E}} \right)^{1/2} \lambda_D$$

scale.

n.b. Where does KdV come from in hydro.

- surface wave: $\omega^2 = g k$

- surface wave with finite depth:

$$\omega^2 = g k \tanh(kd)$$

then expanding:

$$\omega^2 = k^2 g d + g k d \left(- \frac{k^2 d^2}{3} \right)$$

$$\omega^2 = k^2 g d \left(1 - \frac{k^2 d^2}{3}\right)$$

$$\Updownarrow$$

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2} \rightarrow k^2 c_s^2 \left(1 - \frac{k^2 \lambda_D^2}{2}\right)$$

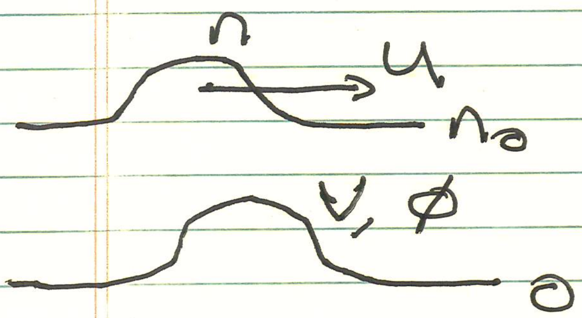
&

$$\boxed{\partial_t \mathcal{E} + v_0 \partial_x \mathcal{E} + \mathcal{E} \partial_x \mathcal{E} = - \frac{v_0 d^2}{3} \partial_x^3 \mathcal{E}}$$

$V_0 = (gd)^{1/2}$.

Now, for ion-acoustic:

$n_e = n_0 \exp\left[|e|\phi / T_e\right]$ Boltzmann electrons



$\partial_t n_i + \partial_x (n_i V) = 0$

$\partial_t V_i + V_i \partial_x V_i = -\frac{|e|}{M_i} \frac{\partial \phi}{\partial x}$

fluid ions

$$\left\{\begin{matrix} n_i \\ v_i \\ \phi \end{matrix}\right\} = \overset{\substack{\text{form} \\ \downarrow}}{F}\Big(x - \underset{\substack{\uparrow \\ \underline{\text{speed}}}}{u}t\Big) \rightsquigarrow \text{standard form of NL pulse.}$$

$$-u n_i' + (n_i v)' = 0$$

$$(v-u)v' = -\frac{q}{m_i}\phi'$$

integrate, $\left.\begin{matrix} \phi \to 0 \\ v \to 0 \\ n \to n_0 = 1 \end{matrix}\right\} \quad |x| \to \infty$

$$-u n_i + n_0 v = -u \qquad \text{(b.c.)}$$

$$(u - v) n_i = u$$

$$n_i = u/(u-v)$$

likewise:

$$-\frac{q\phi}{m_i} = \frac{v^2}{2} - \frac{2u}{2}v + \frac{u^2}{2} - \frac{u^2}{2}$$

$$\frac{q\phi}{m_i} = -\frac{1}{2}(u-v)^2 + \frac{u^2}{2}$$

b.c. ✓.

so $(u-v) = \left(u^2 - \frac{2q\phi}{m_0}\right)^{1/2}$

$\Rightarrow$

$$\partial_x^2 \phi = -4\pi n_0 q \left(n_i - n_e\right)$$

$$\partial_x^2 \left(\frac{q\phi}{T}\right) = -\frac{1}{\lambda_D^2}\left(\frac{1}{\left(1 - 2q\phi/T)c_s^2/u^2\right)^{1/2}} - \exp\left(\frac{q\phi}{T}\right)\right)$$

$\frac{q\phi}{T} \to \phi$

$$\boxed{\partial_x^2 \phi = -\frac{1}{\lambda_D^2}\left(\frac{1}{\left(1 - 2\phi \frac{c_s^2}{u^2}\right)^{1/2}} - e^{\phi}\right)}$$

NL wave eqn.

$M^2 = u^2/c_s^2$

$\downarrow$

Mach #

$$\phi' \partial_x^2 \phi = -\frac{1}{\lambda_D^2}\frac{\phi'}{\left(1 - \frac{2\phi}{M^2}\right)^{1/2}} - e^{\phi}\phi'$$

$$= -\left(dV(\phi)/d\phi\right)\phi'$$

and integrate:

$$\boxed{V(\phi) = \frac{-1}{\lambda_D^2}\left\{ \frac{M^2}{2}\left(1-\frac{2\phi}{M^2}\right)^{1/2} - e^{\phi}\right\} + C}$$

integration const.

→ sets possibility

⇒

$$\frac{\phi'^2}{2} + V(\phi) = 0$$

$$\phi'' = dV/d\phi$$

⇒ ion acoustic problem reduced to particle orbit.

$$\phi'' = dV/d\phi$$

$$\ddot{x} = -dU/dx$$

⇒ $M^2 > 2\frac{|e|\phi}{T}$ critical velocity for soliton.

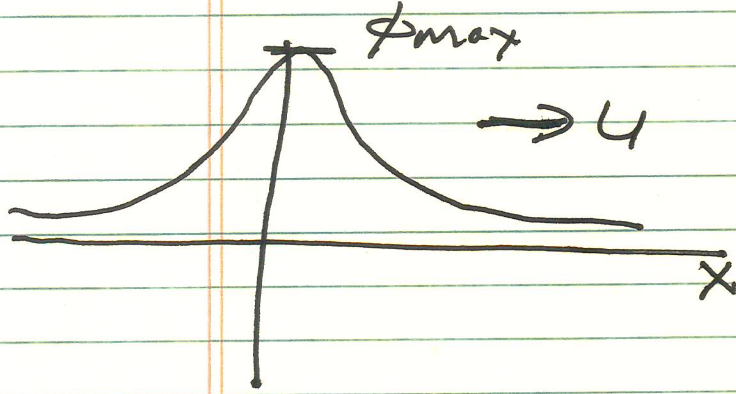
→ small ϕ

$$U(\phi) \simeq \frac{-1}{\lambda_D^2}\left\{ (1+M^2) + \phi - \phi + \frac{1}{2}\phi^2\left(\frac{-1}{M^2}+1\right)\right\}$$

so need $M^2 > 1$.

Now, choose:

$$\phi' \to 0 \quad \text{as} \quad \phi \to 0 \qquad \text{via } G$$



localized pulse.

$$\mathcal{M}^2 = \frac{1}{2}\frac{\left[\exp\phi_{max} - 1\right]^2}{\exp\phi_{max} - 1 - \phi_{max}}$$

low amplitude

$$\phi \approx \frac{3}{2}\left(1 - \frac{1}{\mathcal{M}^2}\right)\operatorname{sech}^2\left\{\sqrt{\frac{\pi n_0 q}{T}}\sqrt{1 - \frac{1}{\mathcal{M}^2}}\, x\right\}$$

need $\mathcal{M}^2 \gtrsim 1.6$.

can consider cases.

→ Need discuss wave-particle interaction to deal with heating, entropy production etc

→ More general discussion (KdV):

Recall, had general:

$$\partial_t \mathcal{E} + (c_s + \mathcal{E})\partial_x \mathcal{E} + \frac{c_s \lambda_D^2}{2}\partial_x^3 \mathcal{E} = 0$$

$$a = \mathcal{E}$$

$$y = x - c_s t$$

$$\boxed{\partial_t a + a\,\partial_y a + \beta \frac{\partial^3 a}{\partial y^3} = 0}$$

simple KdV.

$$\underbrace{l_{soliton} \sim (\beta/a)^{1/2}}_{\text{scale amplitude rel'n.}}$$

Solving (reduced) KdV (integrating)

$$a = a(y - ct)$$

$\Rightarrow$

$$\beta a''' - c a' + a a' = 0$$

invariant
$a \to a + V$
$c \to c + V$

$$\beta a'' - c a + \frac{a^2}{2} = \frac{1}{2} C_1$$

↑
const

$$(2a') * \left(\beta a'' - ca + \frac{a^3}{2} = \frac{c_1}{2}\right)$$

$$2\beta a' a'' - 2caa' + a' a^2 = c_1 a'$$

intr.

$$\beta a'^2 = -\frac{1}{3}a^3 + c_1 a^2 + c_1 a + c_2$$

and can now reduce to quadrature.

→ Convenient to factorize:

$$c, \; c_1, \; c_2 \Rightarrow a_1, \; a_2, \; a_3$$

$$\boxed{\beta a'^2 = -\frac{1}{3}(a-a_1)(a-a_2)(a-a_3)}$$

$$c = \frac{1}{3}(a_1 + a_2 + a_3)$$

For: ⇒ bounded $|a(y)|$

need a_1, a_2, a_3 real

$$\Rightarrow \left[\begin{array}{l} \text{if: } a_1 > a_2 > a_3 \\ a_1 \geq a \geq a_2 \end{array}\right.$$

$a_3 = 0$ no loss generality

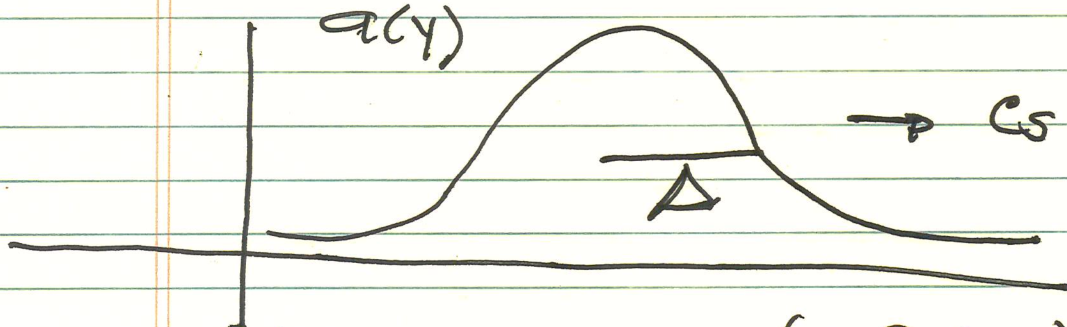
$$\Rightarrow \quad \beta a'^2 = 1/3 \, (a_1 - a)(a - a_2)\, a$$

if $a_2 = 0$

$$a(y) = a_1 \cosh^{-2}\left(\tfrac{1}{2} y \sqrt{a_1/3\beta}\right)$$

$$\rightarrow a_1 \cosh^{-2}\left(\tfrac{1}{2}(x - c_s t)\sqrt{a_1/3\beta}\right)$$

so, have (as before)



$\rightarrow c_s \sqrt{a_1/3\beta}$

$$\Delta \sim (3\beta/a_1)^{1/2}$$

note:

$\rightarrow u \sim c_s \sqrt{a_1/3\beta}$ speed - amplitude

$\rightarrow \Delta \sim (3\beta/a_1)^{1/2}$ width - amplitude

Note:

→ Soliton has finite width

$$\Delta \sim \left(3\beta/a_1\right)^{1/2} \sim \lambda_{De}$$

contrast shock

→ bigger solitons go faster

$$V \sim u_0 \left(a_1/3\beta\right)^{1/2}$$