Collisionless Plasma Waves and Landau Damping I

→ Collective Response / Waves in Vlasov Plasma

- $\omega, kv \gg \nu$

$$f = \langle f \rangle + \delta f$$

δf → treat as collisionless - Vlasov

$\langle f \rangle$ → Collisions, long time ~ Maxwellian

- What of warm plasma wave?

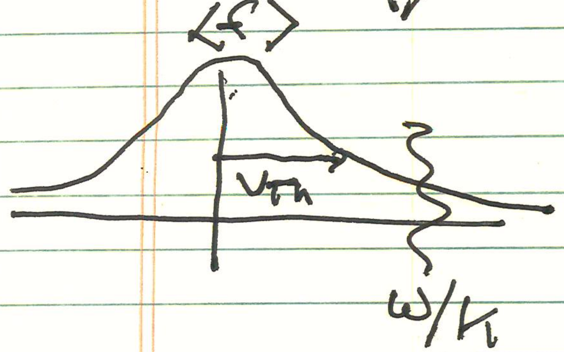
Approach:

{ - direct calculation
- physics - what Landau Damping means?

{ - rigorous calculation
- More on physical interpretation

- Direct Calculation (1D)

$$\langle f \rangle = \left(1/\sqrt{2\pi}\, v_{Th}\right) \exp\left(-v^2/2v_{Th}^2\right)$$



linearizing Vlasov - Poisson:

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m}\tilde{E}\frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \phi = -4\pi n_0 q \int \tilde{f}\, dv$$

$$f = \sum_{k,\omega} f_{k,\omega} e^{i(kx-\omega t)}$$

so

$$-i(\omega - kv)\tilde{f}_{k,\omega} = \frac{q}{m} i k \tilde{\phi}_{k,\omega}\frac{\partial \langle f \rangle}{\partial v}$$

$$k^2 \tilde{\phi}_{k,\omega} = 4\pi n_0 q \int \tilde{f}_{k,\omega}\, dv$$

$\Rightarrow$

$$\tilde{f}_{k,\omega} = -k\frac{q}{m}\frac{\tilde{\phi}_{k,\omega}\,\partial\langle f\rangle/\partial v}{(\omega - kv)}$$

$\hookrightarrow \quad v = \omega/k$?!

so $\quad k^2 \tilde{\phi}_{k,\omega} = -\omega_p^2 k \int dv \frac{\partial\langle f\rangle/\partial v}{(\omega - kv)}\tilde{\phi}_{k,\omega}$

thus

$$\boxed{\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv\, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}}$$

Dielectric Function for Collisionless Plasma

→ What of Pole at $\omega = kv$?

- recall Vlasov eqn. derived for $\nu \to 0$

$$1/\omega - kv = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

- better, causality requires:

$$\phi \sim e^{-i\omega t} \Rightarrow \phi \sim e^{-i(\omega + i\epsilon)t}$$

i.e. $\phi \underset{t \to -\infty}{\to} 0$

so

$$\frac{1}{\omega - kv} = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

$$= \frac{P}{\omega - kv} - i\pi\delta(\omega - kv)$$

(Plemelj Formula)

clearly:

$P \to$ will recover hydrodynamic response

$-i\pi\delta(\omega - kv) \to \epsilon_{IM}$ " $\to$ Wave Energy Dissipation } "

Landau Damping

N.B.:

$$Q_k = \frac{|E_k|^2}{8\pi} \omega \, \mathrm{Im}\, \epsilon \Big|_{\omega_k} \Rightarrow \text{damping of wave energy}$$

- of course for $\partial f_0/\partial v \big|_{res} > 0 \Rightarrow$ can be growth
- damping $\leftrightarrow$ $\epsilon < 0$ $\Rightarrow$ analytic continuation (coming)
- wave energy damps; $\Rightarrow$ macroscopic

  where? $\Rightarrow$ resonant particles

  i.e. particles with $v \sim \omega/k$

  heating ?
- How reconcile with $dS/dt = 0$ for Vlasov Eqn.

Proceed with analysis:

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

$$= \overbrace{1 + \omega_p^2 \int dv \frac{P}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}}^{\epsilon_r(k,\omega)}$$

$$\underbrace{- 2\pi \frac{\omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v} \Big|_{\omega/k}}_{\epsilon_{IM}(k,\omega)}$$

$$\delta(\omega - kv) = \frac{1}{|k|} \delta(v - \omega/k)$$

Now, to deal with P:

$$\frac{\partial \langle f \rangle}{\partial v} = -\frac{v}{v_{th}^2} \langle f \rangle$$

$\omega > k v_{th}$ (hydro limit)

$$\frac{P}{\omega - kv} = \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \cdots\right)$$

so

$$\epsilon_r(k, \omega) = 1 - \frac{\omega_p^2}{k v_{th}^2} \int dv \, \frac{\langle f \rangle}{\omega} \, v \left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \cdots\right)$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - \frac{3 \omega_p^2 v_{th}^2 k^2}{\omega^4}$$

i.e.

$$\langle x^4 \rangle = \int dx \; x^4 \, e^{-x^2/2}$$

$$= 4 \left.\frac{\partial^2}{\partial \alpha^2}\right|_{\alpha=1} \int dx \, e^{-\alpha x^2/2}$$

$$= 4 \left.\frac{\partial^2}{\partial \alpha^2}\right|_{\alpha=1} \left(1/\sqrt{\alpha}\right)$$

$= 3$ (π via normalization)

→ "3" appears from moments of Maxwellian → eqbrm distribution

→ Moments replace / underly eqn. of state

so

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 v_{th}^2}{\omega^2}\right)$$

$\epsilon = \epsilon_r + i\,\epsilon_{Im}$

→ $\epsilon_r = 0$ ⇒ Collective Resonance / Wave

Now:

- should connect to warm plasma wave
- as ϵ derived via $(kv/\omega) \ll 1$ expansion, need determine $\omega(k)$ iteratively.

L.O.:

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{th}^2}{\omega^2}\right)$$

l.o. $\epsilon_r \simeq 1 - \frac{\omega_p^2}{\omega^2}$

$$\omega^{(0)} = \omega_p$$

so

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{th}^2}{\omega_p^2}\right)$$

$$\omega^2 = \omega_p^2 (1 + 3k^2 \lambda_D^2) \rightarrow \left[\text{structure agrees with fluid model}\right.$$

$\gamma \rightarrow 3$, here

N.B.:

- distribution function $\Leftrightarrow$ E.O.S.
- dispersion relation identical to warm fluid model $\Leftrightarrow$ $k v_{th} < \omega$ expansion

→ ϵ_{IM}

$$\epsilon_{IM} = -\frac{\pi \omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}$$

so dissipated wave energy:

$$Q_k = \omega \epsilon_{IM} \frac{|E_k|^2}{8\pi}\bigg|_{\omega_k}$$

$$Q = -\omega_k \frac{\pi \omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\bigg|_{\omega_k/k} |E_k|^2/8\pi$$

<u>and</u>

$$\frac{\partial W_k}{\partial t} + \underline{\nabla} \cdot \underline{S}_k + Q_k = 0$$

$$\Rightarrow \quad \gamma_k = -\frac{Q_k}{W}$$

$$= \frac{-\cancel{\omega_k} \, \pi \frac{\omega_p^2}{k|k|} \frac{\partial f}{\partial v}\Big|_{\omega_k/v}}{\cancel{\omega_k} \frac{\partial \epsilon_r}{\partial \omega}\Big|_{\underline{k}}}$$

$$= -\frac{\epsilon_{IM}(k, \omega_k)}{(\partial \epsilon_r/\partial \omega)\big|_{\omega_k}}$$

<u>or</u>

$$\epsilon = \epsilon_r(k, \omega) + i\,\epsilon_{IM}(k, \omega)$$

$$\omega = \omega_k + i\gamma_k \qquad \gamma_k \ll \omega_k$$

$$\epsilon = \epsilon_r(k, \omega_k + i\gamma_k) + i\,\epsilon_{IM}(k, \omega_k)$$

$$\cong \epsilon_r(k, \omega_k) + i\gamma_k \left.\frac{\partial \epsilon_R}{\partial \omega}\right|_{\omega_k} + i\,\epsilon_{IM}(k, \omega_k)$$

<u>so</u>

$$\gamma_k = -\epsilon_{IM}(k, \omega_k) \Big/ \left.\frac{\partial \epsilon_R}{\partial \omega}\right|_{\omega_k}$$

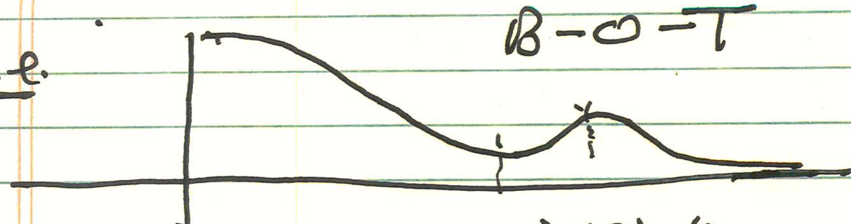
✓ agree

Thus:

- $\partial\langle f\rangle/\partial v < 0$
  $\Rightarrow$ damping

- $\partial\langle f\rangle/\partial v > 0$
  $\Rightarrow$ growth (instability)

<u>i.e.</u>

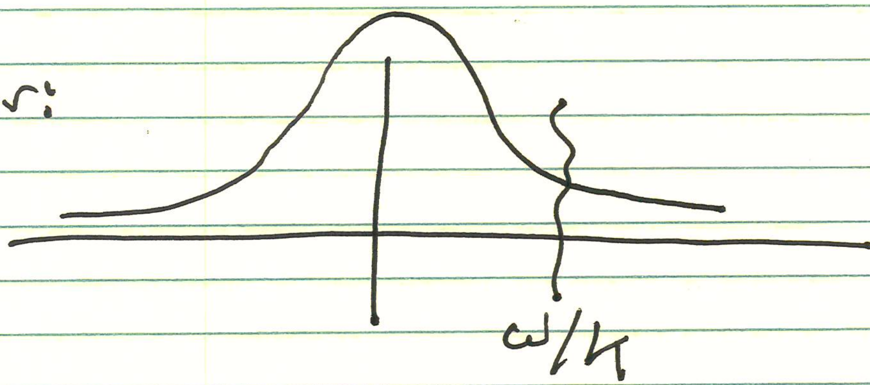


B-O-T
- beam driven
- relate beam + plasma

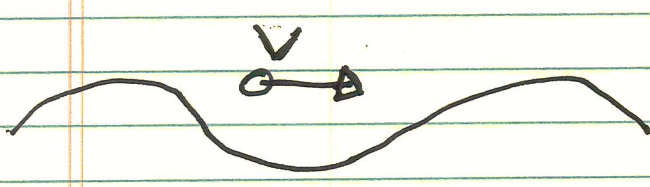
$\partial\langle f\rangle/\partial v > 0$

→ Physics of Landau Damping

Consider:



- Landau damping occurs due to wave-particle resonance at $\omega/k \sim v$
- intuitively, consider wave interaction with ~ resonant particle



$$\omega/k = v_{ph}$$

Particle with $v \sim v_{ph}$ 'sees' ~ DC field

$$\frac{dv}{dt} = \frac{q}{m} E \cos(kx - \omega t)$$

$$= \frac{q}{m} E \cos(k(x - v_{ph} t))$$

if boost to frame at V:

$x' = x - Vt$

$v' = v - V$

$a' = a$

⇒ very heuristic:

$$\frac{dv}{dt} = \frac{q}{m} E \cos\left(k\left(x + (V - v_{ph})t\right)\right)$$

" — secular (in time) interaction at $v \sim v_{ph}$

— $v \lesssim \omega/k$ ⇒ wave does work on particles, loses energy

— $v \gtrsim \omega/k$ ⇒ wave does ⊖ work, gains energy

Q = # losers − # gainers

$\sim \partial \langle f \rangle / \partial v \big|_{\omega/k}$

Now, quantitatively:

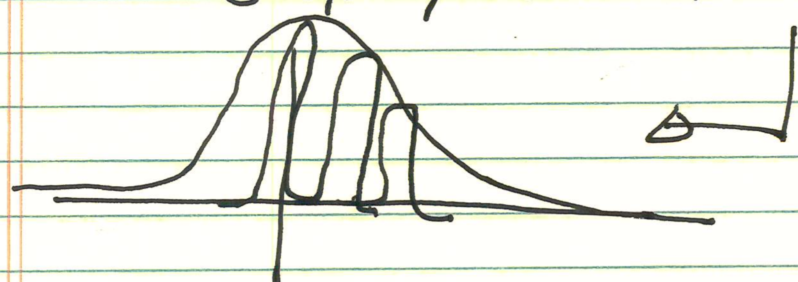
— $Q = \langle E^* \cdot J \rangle$

so for beam at V:

$$\bar{q} = \langle q V E \rangle$$

time avg dissipated power on resonant beam

Now: - view plasma distribution as superposition of beams



then $Q = \int dv\, \bar{q}$

Q: total dissipation

- Now, calculate $\langle q V E \rangle$:

$V = V_0 + \delta V$

$x = x_0 + \delta x$

δV, δx: perturbations induced by wave.

so

$$\frac{d}{dt}\delta V = \frac{q}{m} E \Big|_{x_0, V_0}$$

$$\frac{d}{dt}\delta x = \delta V$$

$$\bar{\mathcal{Q}} = q\langle V E\rangle$$

$$V = V_0 + \delta V$$

$$E = E(t, x = x_0 + \delta x)$$

$$\cong E(t, x_0) + \delta x \, \partial E/\partial x\Big|_{x_0}$$

so, finally: beam power dissipated

$$\bar{\mathcal{Q}} = q\Big\langle \underset{\text{osc}}{(V_0+\delta V)}\,\underset{\text{osc}}{(E(t,x_0)} + \underset{\text{both osc}}{\delta x\,\partial E/\partial x\Big|_{x_0,t}}\,)\Big\rangle$$

so, retaining quadratic beats:

$$\bar{\mathcal{Q}} = qV_0\Big\langle \delta x\,\partial E/\partial x\Big|_{x_0,t}\Big\rangle$$

$$+\, q\,\langle \delta V\, E(t, x_0)\rangle$$

need compute: δx, δV:

$$\frac{d}{dt}\delta V = \frac{q}{m} E(t, x_0)$$

$$\underset{\text{unperturbed orbit}}{x_0 = x_0' + V_0 t}$$

take:

$x_0' = 0$, convenience

$\omega/k = v_{ph}$

$$E(t, x_0) = \frac{q}{m} E_0 \, \cancelto{1}{e^{ikx_0'}} \, e^{ik(v_0 - \omega/k)t} \, e^{\delta t}$$

$$\delta > 0 \quad \text{so} \quad \left\{ \begin{array}{l} \delta v \to 0 \quad t \to -\infty \\ \text{causality} \end{array} \right.$$

$$\frac{d}{dt} \delta v = \frac{q}{m} E_0 \exp\left(ik\left(v_0 - \omega/k - i\delta\right) t \right)$$

$$\Rightarrow \quad \delta v = \frac{q}{m} E_0 \left. \frac{e^{ik(v_0 - \omega/k - i\delta)t}}{i\left(k(v_0 - v_{ph}) - i\delta\right)} \right|_{-\infty}^{t}$$

$$\delta v = \frac{q}{m} E(t, x_0) \Big/ \left(ik(v_0 - v_{ph}) + \delta \right)$$

and obviously:

$$\delta x = \frac{q}{m} E(t, x_0) \Big/ \left(ik(v_0 - v_{ph}) + \delta \right)^2$$

so

$$\bar{Q} = q v_0 \left\langle \delta x \frac{\partial E}{\partial x} \right\rangle + q \left\langle \delta v \, E \right\rangle$$

$$= q v_0 \left\langle -ik\, E^*(t_0 x_0) \frac{q}{m} \frac{E(t_0 x_0)}{\left(ik_1(v_0 - v_{ph}) + \delta\right)^2} \right\rangle$$

$$+ q \left\langle E^*(t_0 x_0) \frac{q}{m} \frac{E(t_0 x_0)}{\left(ik_1(v_0 - v_{ph}) + \delta\right)} \right\rangle$$

as $E^* E$ gives DC beat:

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2 |E|^2}{2m} \frac{|E|^2 v_0}{[ik_1(v_0 - v_{ph}) + \delta]} \right\}$$

$$= \frac{d}{dv_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{-i v_0}{(k_1(v_0 - v_{ph}) - i\delta)} \right\}$$

real part:

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{v_0}{|k_1|} \pi\, \delta(v_0 - v_{ph}) \right\}$$

Then, for total dissipation, average over ensemble of beams, distributed according to $\langle f \rangle$:

norm to 1

$$Q = n\int dv_0\, \underline{q}(v_0)\, \langle f(v_0)\rangle$$

$$= \int dv_0\, \langle f(v_0)\rangle \frac{d}{dv_0}\left\{ \frac{nq^2}{2m} |E|^2 \frac{v_0 \pi}{|k|}\, \delta(v_0 - v_{ph}) \right\}$$

$$= -\pi \frac{\omega_p^2}{|k|} \frac{\omega}{k} \left.\frac{\partial \langle f(v)\rangle}{\partial v}\right|_{\omega/k} \left(|E|^2/8\pi\right)$$

$$\boxed{Q = -\pi \frac{\omega_p^2}{|k|} \frac{\omega}{k} \left.\frac{\partial \langle f\rangle}{\partial v}\right|_{\omega/k} \left(|E|^2/8\pi\right)}$$

- agrees previous.
- establishes Landau damping as $\langle E\cdot J\rangle$ work of wave electric field on resonant particles.

- Fate of energy:

ignoring radiation -

$$\frac{\partial W_k}{\partial t} + \cancel{\nabla\cdot \underline{S}_k} + Q_k = 0$$

$$\partial_t W_{\underline{k}} = -Q_{\underline{k}}$$

but clearly resonant particles heated:

i.e. will show, in QLT:

$$\partial_t \, RPKED + \partial_t W_{k} = 0$$

$\Rightarrow$ Landau damping heats resonant piece of distribution at expense of wave energy.

$\rightarrow$ BUT:

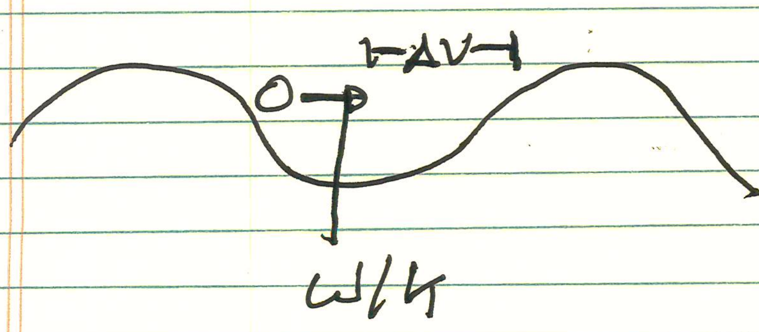
- Landau calculation, and physical argument, are linear $\Rightarrow$

use linearized, free-streaming unperturbed orbits

- Such linearization valid only for: $t < \tau_b$

$\downarrow$

bounce time in wave trough.



i.e. once particle bounces, orbits no longer un-perturbed.

$$\Delta v \sim (q\phi/m)^{1/2}$$

trapping

$$1/\tau_b = k\,\Delta v$$

Then $\gamma_k = \gamma_k^{(0)}$; $t < \tau_b$, only

→ Landau resonance forces/drives a picture of plasma as gas of:

- waves + resonant particles
- collective modes as non-resonant particles and fields.

Remaining:

- How reconcile causality ($\delta > 0$) and damping ($\gamma < 0$)

  → ivp

- Closer look at physics.