Formal Theory of Landau Damping

a) $\Rightarrow$ Crux: Causality $\Leftrightarrow$ Landau Damping

$$\frac{1}{\omega - kv} = \frac{P}{\omega - kv} - i\pi\delta(\omega - kv)$$

from $\omega \to \omega + i\epsilon$ so $\phi(t) \underset{t\to\infty}{\sim} e^{-i\omega t} \to 0$

But: How reconcile with damped modes?
(i.e. "$\epsilon < 0$")

$\Rightarrow$ investigate systematically, as initial value problem.

i.e. $f(t=0) = \langle f(v)\rangle + \tilde{f}(x, v, 0)$

initial pert

ϕ evolution?

then
$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m}\tilde{E}\frac{\partial \langle f\rangle}{\partial v}$$

$$\nabla^2\tilde{\phi} = -4\pi n_0 q\int \tilde{f}\,dv$$

so

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv\tilde{f}_k = ik\hat{\phi}_k \frac{q}{m}\frac{\partial \langle f \rangle}{\partial v}$$

$$k^2\hat{\phi}_k = 4\pi n_0 q \int \hat{f}_k \, dv$$

Now, as cup, Laplace Transform:

$$\hat{\phi}_{k,\omega} = \int_0^\infty e^{i\omega t}\hat{\phi}_k(t)$$

$$\hat{\phi}_k(t) = \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} e^{-i\omega t}\phi_{k,\omega}\frac{d\omega}{2\pi}$$

then,

$$\int_0^\infty e^{i\omega t}\frac{\partial \tilde{f}_k}{\partial t} = -\tilde{f}_k(v,0) - i\omega\int_0^\infty e^{i\omega t}\tilde{f}_k$$

so

$$-\tilde{f}_k(v,0) - i(\omega - kv)\tilde{f}_{k,\omega} = \frac{iq}{m}k\tilde{\phi}_{k,\omega}\frac{\partial \langle f \rangle}{\partial v}$$

$$f_{k,\omega} = \frac{i\tilde{f}_k(v,0)}{\omega - kv} - \frac{q}{m}\frac{k\hat{\phi}_{k,\omega}}{\omega - kv}\frac{\partial \langle f \rangle}{\partial v}$$

$$k^2 \phi_{k,\omega} = 4\pi n_0 q \int dv \left\{ -\frac{q}{m} \frac{k}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v} \hat{\phi}_{k,\omega} + i \frac{\tilde{f}_k(v,0)}{\omega - kv} \right\}$$

$$\epsilon(k,\omega) \hat{\phi}_{k,\omega} = \frac{4\pi n_0 q}{k^2} i \int dv \frac{\tilde{f}(v,0)}{\omega - kv}$$

where, as usual:

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

so:

$$\hat{\phi}_{k,\omega} = \frac{\frac{4\pi n_0 q}{k^2} i \int dv \frac{\tilde{f}(v,0)}{\omega - kv}}{\epsilon(k,\omega)}$$

$$\phi_k(t) = \int_{-\infty + i\sigma}^{+\infty + i\sigma} d\omega \frac{4\pi n_0 q}{k^2 \epsilon(k,\omega)} \left(i \int dv \frac{\tilde{f}(v,0)}{\omega - kv} \right) e^{-i\omega t}$$

– $\Phi_k(t)$ determined by analytic structure of integrand

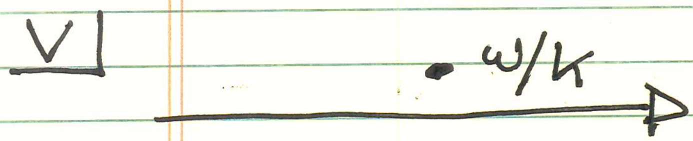
⇒ singularities $\int dv \, \frac{\tilde{f}_k(v,0)}{\omega - kv}$

⇒ $\left\{ \begin{matrix} \text{zeroes} \\ \text{singularities} \end{matrix} \right\}$ $\epsilon(k,\omega)$

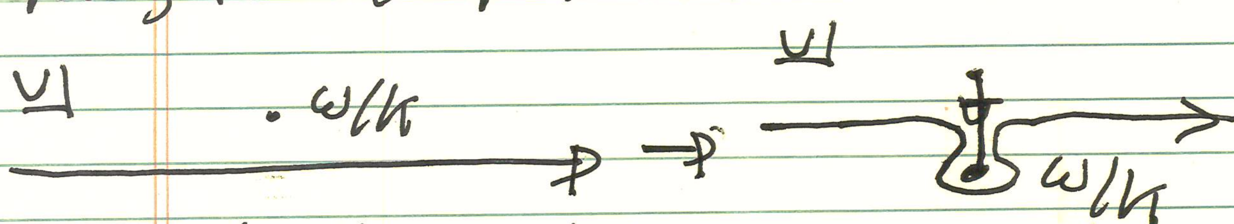
Now, $\omega \Rightarrow \omega + i\epsilon$ $\qquad \frac{1}{\omega - kv + i\epsilon}$

$v = v - i\epsilon$ $\qquad \frac{1}{v - \frac{\omega}{k} - i\epsilon}$

<u>so</u> v integration along contour <u>below</u> pole at ω/k



Now, for damped mode:



analytically continue numerator by deforming v contour so pole stays above it.

so singularities $\int dv \frac{\tilde{f}_{\underline{k}}(v,0)}{\omega - kv}$ only at singularities $\tilde{f}_{\underline{k}}(v,0)$ analytically continued

but:

- $\tilde{f}_{\underline{k}}(v,0)$ entire function
  (no singularity at ~~f~~finite v) and normalizable

so

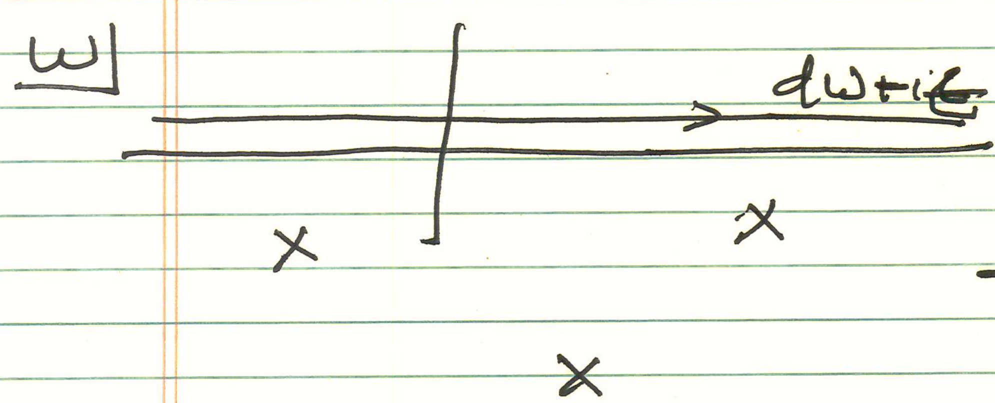
- $\int dv \frac{\tilde{f}_{\underline{k}}(v,0)}{\omega - kv} \rightarrow$ entire function ω

so

- $\epsilon(k,\omega) \rightarrow$ entire function, also.

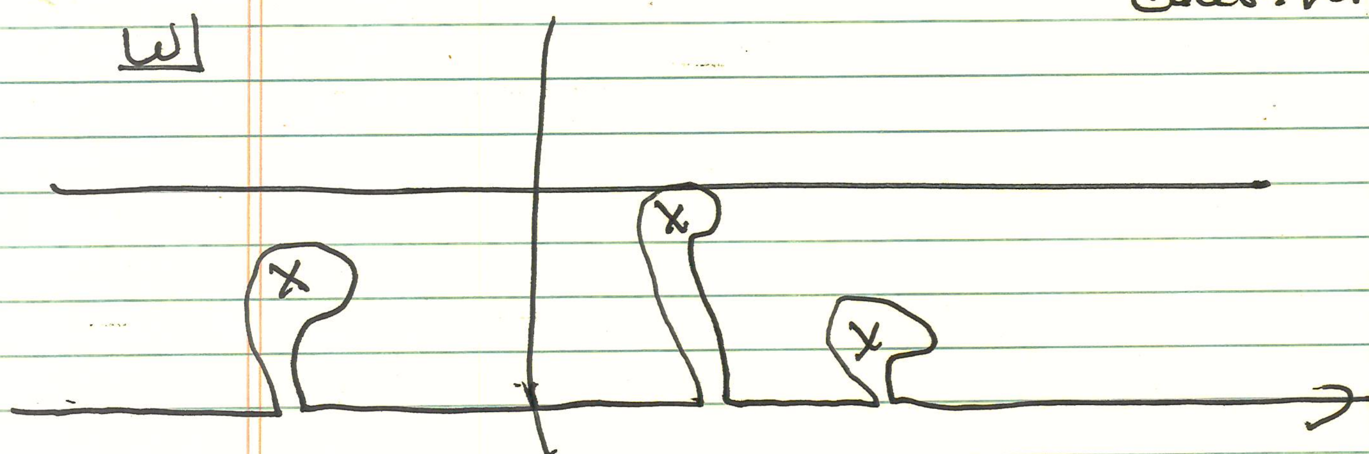
so only singularities of integrand at zeroes $\epsilon(k,\omega)$

Now, for inverse transform:



→ zeroes ε
(damped modes)

treat by deforming contour downward till encircles zeroes ε
(stay above, causality)



then

$$\phi_k(t) = \sum_j \phi_k^j \, e^{-i\omega^j_{k,real} t} \, e^{-i\omega^j_{k,IM} t}$$

ϕ_k^j ↓ residue jth mode

$\epsilon = 0 \rightarrow \omega^j_{k,real} + i\,\omega^j_{k,IM}$

<u>So</u>

→ long time response dominated by least damped mode.

Now

- 'validates' Landau damping
- sufficient to calculate zeroes of $\epsilon(k,\omega)$

b.) Case - Van Kampen Modes (Intro.).

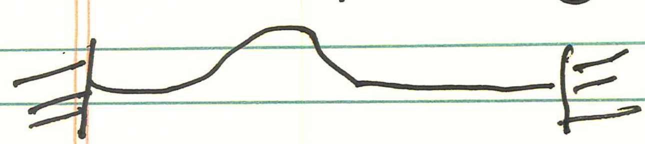
For solution IVP:

→ determine complete set of normal modes of system

then

→ evolution as normal modes with initial value data (sets initial amplitudes) and normal mode evolution.

i.e. Plucked string evolution



treat as:

→ Fourier series with Initial Value Data setting coefficients

or

→ Laplace tranform.

For Vlasov Plasma:

- Complete set of modes is a continuum of singular modes.
- Landau damping as phase mixing of these modes

Singular modes ⟺ Case - Van Kampen Modes.

$$\frac{\partial \tilde{f}_k}{\partial t} + ikv\tilde{f}_k = \frac{iq}{m} k \tilde{\phi}_k \frac{\partial \langle f \rangle}{\partial v}$$

$$k^2 \phi_k = 4\pi n_0 q \int \tilde{f}_k \, dv$$

so

$$\partial_t \tilde{f}_k + ikv\tilde{f}_k = i\frac{\omega_p^2}{k} \frac{\partial \langle f \rangle}{\partial v} \int dv \, \tilde{f}_k(v)$$

re-writing:

$$\left\{ \begin{array}{l} \partial_t f_k + ikvf_k = -ik\eta(v) \int_{-\infty}^{+\infty} dv' \, \tilde{f}_k(v') \\ \\ \eta(v) = -\frac{\omega_p^2}{k^2} \frac{\partial \langle f \rangle}{\partial v} \end{array} \right.$$

integro-diff'tl eqn.

$$f_k = f_{k,\omega} e^{-i\omega t}$$

label modes by $\omega/k \to$ phase velocity

$$(V - \omega/k) f_{\omega/k}(V) = -\eta(V) \int_{-\infty}^{+\infty} dV' f_{\frac{\omega}{k}}(V')$$

$$f = f(V, \omega, k) = f(V, \omega/k)$$
$$= \underset{\substack{\Uparrow \\ v}}{f_{\omega/k}}(V).$$

so

$$\boxed{(V - v) f_v(V) = -\eta(V) \int_{-\infty}^{+\infty} dV' f_v(V')}$$

now:

with normalization $\int_{-\infty}^{+\infty} dV f_v(V) = 1$;

then:

$$f_v(V) = \frac{-P\,\eta(V)}{V - v} + \lambda(v)\,\delta(V - v)$$

singular

Check:

$$(V - v) f_v(V) = -P\,\eta(V) + (v - V)\,\cancel{\lambda(v)\,\delta(V - v)}$$

solves above)

$$1 = \int_{-\infty}^{+\infty} \left[\frac{-P\,\eta(V)}{V - v} + \lambda(v)\,\delta(V - v) \right]$$

so

so $$\boxed{\lambda(v) = 1 + \int_{-\infty}^{+\infty} du \, \frac{P\eta(u)}{u - v}}$$

Thus, normal modes f:

$$\boxed{\begin{aligned} f_v(u) &= \frac{-P\eta(u)}{u-v} + \lambda(v)\,\underbrace{\delta(u-v)}_{\text{singular}} \\ \lambda(v) &= 1 + \int_{-\infty}^{+\infty} du \, \frac{P\eta(u)}{u-v} \\ \eta(u) &= -\frac{\omega_p^2}{k^2} \frac{\partial \langle f \rangle}{\partial u} \end{aligned}}$$

→ Modes (C-VK):

- undamped
- singular

} correspond to ballistic modes

→ particle streams

→ Complete, orthogonal set
(K. Case, Ann Phys. 7, 349 1957)

→ see also Kadomtsev, posted.

N.B.: – Can describe evolution by super-posing C–VK modes

– damping appears by phase mixing

– What is phase mixing?

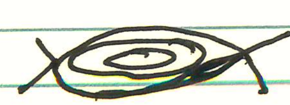
i.e.

$$\underbrace{\int e^{-v^2/v_t^2}}_{\text{distribution of streams}} \underbrace{e^{-ikvt}}_{\text{time evolution of undamped ballistic mode}} dv = \int dv\, e^{-\left(\frac{v}{v_t} + ik\frac{t v_t}{2}\right)^2} \underbrace{e^{-k^2 v_t^2 t^2/4}}_{\text{decay}}$$

Real Problem: Nonlinear Evolution

– Multi-wave: $\begin{cases} \text{Quasilinear Theory } (\tau_{ac} < \tau_T) \\ \text{Resonant heating} \rightarrow \text{plateau.} \end{cases}$

– Single-wave: $\begin{cases} \text{Trapping} \quad \tau_T > \tau_{ac} \\ \text{Phase mixing} \\ \quad + \text{ collns.} \end{cases}$



⇒ homogenization, plateau → restored by collisions