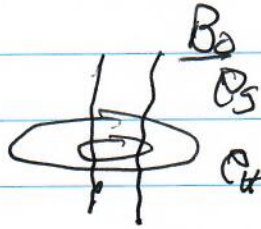


# Lecture I

→ Basic Eqs.

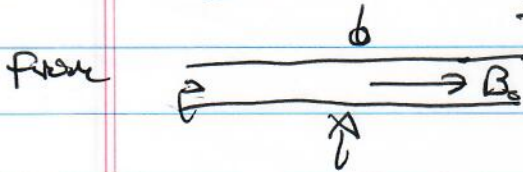


→ Context → SA, time order, tension, etc.; E's

~~→ Consistency~~ → E's, etc.  
- systems - induction, etc.

→ Basic wires

→ Freezing m. → frames



→ Where does it come from?  
- Boltz. → 2 fluid → mtd  
- Ohms Law - many varieties [ → systems

→ Strong Magnetization

→ Reduced mtd  
→ 2D

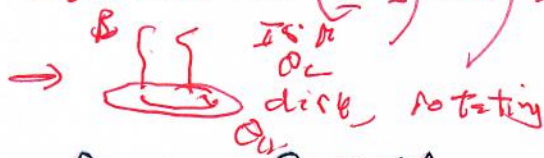
low  $\beta$

drop comparison / fert.  
A / Ferm.

→ Reduced mtd ↔  $\sigma$   
Connectivity

Openers:

→ what is  $(\nabla \cdot \underline{v}) = 0$  MHD?



When slow down.

# Fluid model

1

## Basics of MHD

→ MHD Equations → Eulerian Fluid

{ N.B.: Read  
Kulsrud, Chapt. 3, 4  
1 Fluid  
large scale  
slow  
(Continuity)

①  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$

→ Lorentz,  $\underline{E}$

②  $\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla P + \frac{\underline{J} \times \underline{B}}{c} + \underline{f}_{body}$

(momentum balance)

[frequently  $\underline{f}_{body} = \rho \underline{g}$ ]

③  $\frac{dS}{dt} = \frac{\partial S}{\partial t} + \underline{v} \cdot \nabla S = 0$

{ eqn. of state more general  
(isentropic fluid)

$S = C_v \ln(P/\rho^\gamma)$   
entropy

[frequent form of equation of state]

④  $\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \left( n \underline{J}, \frac{\underline{J}}{\sigma} \right)$

(Ohms Law)

[resistivity  $\eta$  is usually most significant dissipation]

'ideal MHD

$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$

and

①  $\nabla \cdot \underline{B} = 0$

②  $\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$

③  $\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J}$

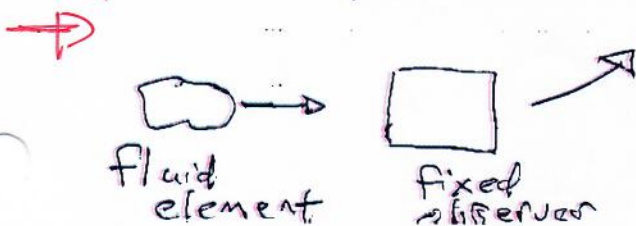
from Maxwell's Eqs.  
neglecting displacement  
current

→ Meaning, Restriction, Validity

- MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

- Variants :
- Reduced MHD → strong  $B_0$  (tokamaks)
  - 2D MHD
  - E MHD → stationary ions (ICF)
  - FLR MHD → MHD + additional effects (MFE, space)
  - Reduced Braginskii
  - hybrid → { bulk - MHD  
hot species - kinetic (i.e.  $\alpha^5$  energetics)

- MHD - Eulerian



"fluid element" ↔ "glue"

here "glue" → collisions  
applies  $L > \lambda_{men}$

collisions  
or  $\gamma_{phos} < \gamma_{fall}$  and so on

( $\omega > \omega_{cr}$ )  
Why MHD ~~often~~  
works at low  
collisional cts

- MHD is:
- 1 fluid - electrons and ions
  - strongly collisional
  - low frequency
  - large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i}, \omega_{pe,i}, \nu_{ei}, \nu_{ci}, \omega_{ce,i}$$

↪ scales relevant: etc.

$$L \gg \lambda_{De,i}, \rho_{e,i}, c/\omega_{pe,i}, l_{mp,e,i}$$

$$l_{mp} < L$$

and

collisions isotropise, equilibrate  $\rho$ .

$$\left( \text{i.e. } \underline{\rho} \sim \int d^3v \tilde{v}_i \tilde{v}_j f(\underline{x}, \underline{v}, t) \right)$$

→ Some Specific Points:

- re: continuity  $\textcircled{1}$ ;

$$\rho = m_i n_i + m_e n_e$$

i.e. (ions control fluid inertia)

total density  
ion dominated

- re: momentum balance ② ;

$$\rightarrow \underline{v} = \left( \int d^3v_i m_i \underline{v}_i f_i + \int d^3v_e m_e \underline{v}_e f_e \right) / \rho$$

i.e. (ions control flow -  $\rho \frac{d\underline{v}}{dt}$ )

→ where has E gone? →  $L \gg \lambda_D$  → quasi-neutrality  
*(fluid eq)* is it consistent?

$$\rho_i \frac{d\underline{v}_i}{dt} = n_i z_i \underline{E} + n_i z_i \frac{\underline{v}_i \times \underline{B}}{c} + \dots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = -n_e z_e \underline{E} - n_e z_e \frac{\underline{v}_e \times \underline{B}}{c} + \dots$$

if add:  $\rightarrow \overset{\circ}{\cancel{n_i = n_e}}$  →  $\frac{\underline{J} \times \underline{B}}{c}$   
 (quasi-neutrality) (Lorentz force term in momentum balance)

Note also:  $\rho_i, \rho_e \rightarrow \rho$

→ re-writing the  $\underline{J} \times \underline{B}$  force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = -\nabla \left( \frac{B^2}{8\pi} \right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

So can write:  $\rho_{tot}$   $\phi$  tension  $\phi$

$$\rho \frac{dv}{dt} = - \nabla \left( \rho + \frac{B^2}{8\pi} \right) + \frac{B \cdot \nabla B}{4\pi}$$

↑ magnetic pressure (field energy density)      ↑ magnetic tension

a) What / Why "Magnetic Tension" ?

$$\underline{B} = B \hat{b} \quad B = |\underline{B}|, \hat{b} = \underline{B} / B$$

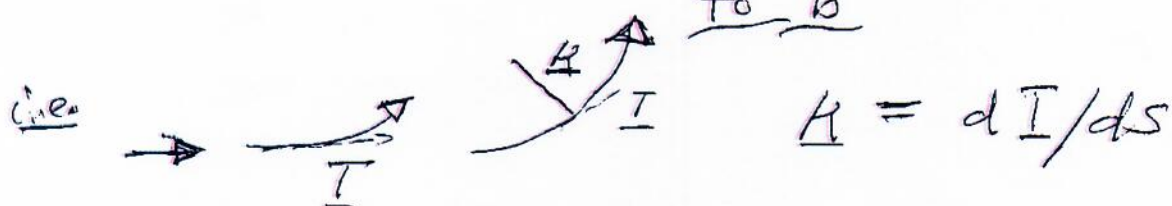
$$\begin{aligned} \underline{B} \cdot \nabla \underline{B} &= B \hat{b} \cdot \nabla (B \hat{b}) \\ &= B^2 \hat{b} \cdot \nabla \hat{b} + \hat{b} \hat{b} \cdot \nabla (B^2) \end{aligned}$$

$\hat{b} \cdot \nabla \hat{b}$  → curvature of  $\hat{b}$  (i.e. rate of change of  $\hat{b}$  along itself) =  $d\hat{b}/ds$

n.b. in general: curve:  $\underline{x}(t)$   
tangent:  $\underline{T} = d\underline{x}/ds$   
( $ds^2 = d\underline{x} \cdot d\underline{x}$ )       $s$  = distance along curve

curvature  $\underline{K} = \frac{d\underline{I}}{ds} = \frac{d\underline{I}/dt}{ds/dt} = \frac{\dot{\underline{I}}}{|\underline{V}|}$

Now:  $\underline{K} = \hat{\underline{b}} \cdot \nabla \hat{\underline{b}} \rightarrow$  points in direction of turning of  $\hat{\underline{b}}$ , orthogonal to  $\hat{\underline{b}}$



$\therefore \underline{K} = + \frac{\hat{\underline{n}}}{R_c}$   $R_c \equiv$  radius of curvature

as curved field line suggests "tension"  $\rightarrow$  "magnetic tension"

b) What about ②?   
 But  $\underline{J} \times \underline{B} \perp \underline{B}$  yet  $\nabla \left( \frac{B^2}{8\pi} \right)$  can have component along  $\underline{B}$  !!!

$\rightarrow$  recombining total  $\underline{J} \times \underline{B}$  gives:

$$\begin{aligned}
 & - \nabla \left( \frac{\beta^2}{8\pi} \right) + \beta^2 \frac{\vec{b} \cdot \nabla \vec{b}}{4\pi} + \vec{b} \vec{b} \cdot \nabla \left( \frac{\beta^2}{8\pi} \right) \\
 = & - \nabla_{\perp} \left( \frac{\beta^2}{8\pi} \right) - \hat{b} \hat{b} \cdot \nabla \left( \frac{\beta^2}{8\pi} \right) + \hat{b} \hat{b} \cdot \nabla \left( \frac{\beta^2}{8\pi} \right) + \beta^2 \frac{\vec{b} \cdot \nabla \vec{b}}{4\pi}
 \end{aligned}$$

$$\Rightarrow \boxed{\frac{\vec{J} \times \vec{B}}{c} = - \nabla_{\perp} \left( \frac{\beta^2}{8\pi} \right) + \beta^2 \frac{\vec{b} \cdot \nabla \vec{b}}{4\pi}} \quad \text{✗}$$

③  $\boxed{dE = \delta Q - PdV}$  (Thermo)

$C_v dT = TdS - PdV$

$v = 1/\rho \quad dV = -d\rho/\rho^2$

$\begin{cases} \delta Q = TdS \\ dE = C_v dT \end{cases}$  (normalized)

$$C_v \frac{dT}{T} = dS + \frac{d\rho}{\rho} \quad |$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln \rho^{1/C_v}$$

$$\boxed{S' = C_v \ln (T/\rho^{1/C_v})}$$

$$p = p(T)$$



$$\Rightarrow S = C_v \ln (P/\rho^{(C_v+1)/C_v})$$

$$= C_v \ln (P/\rho^\gamma)$$

$\gamma = 5/3$ , ideal gas

$(C_v = 3/2$   
(normalized))

$$\frac{dS}{dt} = 0 \Rightarrow \frac{d}{dt} (P/\rho^\gamma) = 0$$

i.e.  $\frac{\partial}{\partial t} (P/\rho^\gamma) + \underline{v} \cdot \underline{\nabla} (P/\rho^\gamma) = 0$

eqn. of state

perfect homogeneity  
stationarity

$$(P/\rho^\gamma = \text{const.})$$

"adiabatic equation of state"

⊕ Ohm's Law - most sensitive part of MHD  
(since controlled by electrons)

MHD variants differ primarily in Ohm's Law

- Hall MHD → Hall term
- E MHD → electron inertia
- Braginskii / drift MHD → DP terms
- ⋮ etc., etc.

Ohm's Law  $\rightarrow$  subtract moments ion, electron  
 equations  $\rightarrow$  electrons  $\left( \underline{J} = n e (\underline{v}_i - \underline{v}_e) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$\sim \nu_{ed} \rightarrow$  momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0 \rightarrow \text{field "frozen" into "fluid"}$$

⑤, ⑥, ⑦: Only 1 approximation:

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

Why drop displacement

$$|\partial \underline{E} / \partial t| \ll |\underline{J}| \rightarrow \text{condition on } \omega! ?$$

$$\rightarrow \omega \frac{v B}{c} \ll \frac{k B}{c^2}$$

$$\Rightarrow |\underline{v}| (\omega/k) / c^2 \ll 1 \quad \text{is condition on } \omega.$$

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible?

consistently

$\rho \frac{d\mathbf{v}}{dt} = n \underline{\Sigma} \underline{E} + \dots$

and  $\Sigma \neq 0$ , as

$n \Sigma = \frac{\nabla \cdot \underline{E}}{4\pi}$   
 $\underline{E} = -\frac{\underline{v} \times \underline{B}}{c}$

so

$n \Sigma \underline{E} = \mp \left(\frac{\underline{v} \times \underline{B}}{c}\right) \cdot \nabla \cdot \left(\frac{\underline{v} \times \underline{B}}{c}\right) \neq 0$  !

but

$\sim \frac{v^2}{c^2} B^2 k$

$\sim \frac{v^2}{c^2} (\underline{J} \times \underline{B}) \rightarrow$  negligible if  $v^2/c^2 \ll 1$ .

Thus, yes indeed it does!

→ Putting it together:

$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = n \underline{J}$  ,  $\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$

⇒ the induction equation, for  $\underline{B}$  evolution ...

$$\frac{\partial \underline{B}}{\partial t} = \underline{v} \times (\underline{v} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

Induction eqn.

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) -  $\underline{v}(\underline{x}, t)$ ,  $\underline{B}(\underline{x}, t)$  evolving simultaneously



- useful and instructive to re-write induction equation

$$\nabla \times \underline{v} \times \underline{B} = -\underline{v} \cdot \nabla \underline{B} + \underline{B} \cdot \nabla \underline{v} - \underline{B} \nabla \cdot \underline{v}$$

so 
$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \nabla \underline{v} - \underline{B} \nabla \cdot \underline{v}$$

This brings us to .....

→ What Does "MHD" mean as a system, really

this is answered most clearly for the case of incompressible MHD ---

$\nabla \cdot \underline{v} = 0$   $\rightarrow$  defines equation of state

$(\omega/k \ll c_s, v_{ms})$   $\rightarrow$  sets  $P_{total}$  field  
sound      magnetosonic

$$\nabla \cdot \left\{ \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{\nabla}{\rho} \left( \rho + \frac{B^2}{8\pi} \right) + \frac{B \cdot \nabla B}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \underline{v} = 0$$

so  $\rho \rightarrow$  constant  $\rho_0$  (can relax to slow variation)

$$\nabla^2 \left[ \left( \rho + \frac{B^2}{8\pi} \right) / \rho_0 \right] = \nabla \cdot \left( \frac{B \cdot \nabla B}{4\pi\rho_0} - \underline{v} \cdot \nabla \underline{v} \right)$$

↑  
total pressure

aka' Poisson's equation:

$$\frac{\rho + B^2}{8\pi} = -\frac{\int d^3x' \left\{ \nabla \cdot \left( \frac{B \cdot \nabla' B}{4\pi\rho_0} - \underline{v} \cdot \nabla' \underline{v} \right) \right\}}{4\pi|x-x'|}$$

solves for:  $\rho_{tot}$  field  $\rightarrow$  eliminates eqn. state.

Basic MHD  $\nabla \cdot \underline{v} = 0$

$$p^* = p_0 +$$

13.

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = - \nabla \left( \frac{p^*}{\rho_0} \right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi \rho_0}$$
$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \nabla \underline{v}$$

with  $\nabla \cdot \underline{v} = 0$ , constitute equations of incompressible MHD.

→ Rather clearly, this system is one of two dynamically coupled, evolving vector fields  $\underline{v}(\underline{x}, t)$ ,  $\underline{B}(\underline{x}, t)$ .

→ Compressible MHD is really a problem in 3 fields, two of which are vectors

i.e.  $\left\{ \begin{array}{l} \underline{v}(\underline{x}, t) \rightarrow \text{fluid velocity} \\ \underline{B}(\underline{x}, t) \rightarrow \text{magnetic field} \\ S(\underline{x}, t) \rightarrow \text{entropy} \Rightarrow \text{energy density} \end{array} \right.$

i.e. scalar equation of state provides 3<sup>rd</sup> field.

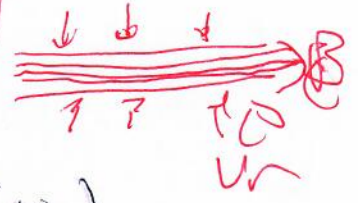
→ Key Question: How closely coupled are  $\underline{v}$ ,  $\underline{B}$   $\left. \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \right\}$

⇒ the key physics element in MHD

⇒ Frozen-in Law, Flux Freezing

2 versions  
 → local  
 → integral

① Frozen-in Law



= consider a (for the moment, passive) vector field:

- frozen into flow  $\underline{v}(\underline{x}, t)$

- consisting of oriented, flexible strands



{ c.i.e. massless rubber strands on flow

How does  $\underline{\Delta l}$  evolve?

$$\text{in } dt, \quad d(\underline{\Delta l}) = (\underline{v}(\underline{l}_0 + \underline{\Delta l}) - \underline{v}(\underline{l}_0)) dt$$

$$= \underline{\Delta l} \cdot \nabla \underline{v} \quad dt$$

$$\therefore \frac{d(\underline{\Delta l})}{dt} = \underline{\Delta l} \cdot \nabla \underline{v}$$

$$\frac{d}{dt} \underline{\Delta l} = \underline{\Delta l} \cdot \underline{D} \underline{V}$$

i.e.  $\frac{d}{dt} \underline{\Delta l} = \underline{\Delta l} \cdot \underline{D}$

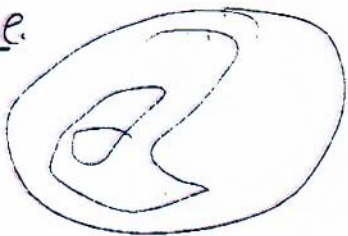
$$\left\{ \frac{d}{dt} (\Delta l)_i = \Delta l_j \cdot D_{ij} \right.$$

$$\left\{ S_{ij} = \frac{1}{2} \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) \rightarrow \text{strain rate tensor} \right.$$

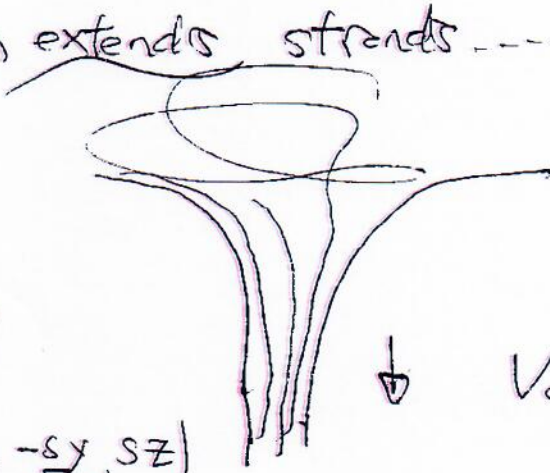
says that  $\rightarrow \underline{\Delta l}$  strands orient along strain

$\rightarrow$  strain extends strands.....

i.e.



$\rightarrow$   
Siphon  
flow



$\downarrow$   $V_0$

$$\underline{V} = V_0 \left( -\frac{s_x}{2}, -\frac{s_y}{2}, s_z \right)$$

plausible to say that  $\underline{\Delta l}$  "frozen into" the flow.

Now, if  $\eta \rightarrow 0$ , ... in MHD.



$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} - \underline{B} \nabla \cdot \underline{v}$$

$$- \nabla \cdot \underline{v} = + \frac{1}{\rho} \frac{d\rho}{dt}$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} = \frac{d\underline{B}}{dt} = \underline{B} \cdot \nabla \underline{v} + \frac{\underline{B}}{\rho} \frac{d\rho}{dt}$$

$$\frac{1}{\rho} \frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2} \frac{d\rho}{dt} = \frac{\underline{B} \cdot \nabla \underline{v}}{\rho}$$

$$\therefore \boxed{\frac{d}{dt} \left( \frac{\underline{B}}{\rho} \right) = \frac{\underline{B}}{\rho} \cdot \nabla \underline{v}}$$

→  $\underline{B}/\rho$  obeys same equation as  $\underline{A}$ !

→  $\underline{B}/\rho$  is frozen into flow field  $\underline{v}(\underline{x}, t)$

Note: →  $\underline{B}/\rho$  is not passive → due  $\underline{J} \times \underline{B}$  force

→  $\underline{B}$  determines flow, while frozen into it!

→ (essence of coupling problem)

For  $\nabla \cdot \underline{v} = 0$ ,  $\underline{B}$  frozen in

→ if  $\eta \neq 0$ , freezing in is broken -----

$$\text{i.e. } \frac{d}{dt} \left( \frac{\underline{B}}{\rho} \right) - \frac{\eta}{\rho} \nabla^2 \underline{B} = \frac{\underline{B}}{\rho} \cdot \nabla \underline{v}$$

↑  
form of frozen  
evolution broken

Vorticity  
connection

→ Observe: → this motivates attention to resistivity  
in MHD above other dissipations  
 $\nu, \chi$ , etc..

→  $\eta \Rightarrow \underline{B}$  diffusion  $\sim \eta \nabla^2$

∴ decoupling of  $\underline{v}, \underline{B}$  occurring on small  
scales

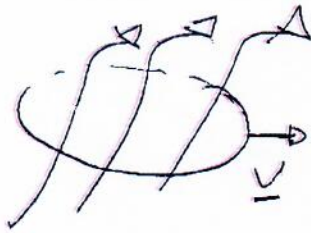
⇒ motivates ('magnetic reconnection') as study of  
singularity dynamics in MHD.

→ A Word to the Wise: In modelling, describing  
complex dynamics in MHD (i.e. MHD  
turbulence, dynamos, etc.) always  
think carefully about frozen-in law...

What is frozen in  
for other systems?

→ Closely Related: Flux Freezing

- consider flux thru surface in flow



i.e. imaginary loop drawn on flow field ...

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times \underline{v} \times \underline{B}$$

$$\underline{\Phi} = \int \underline{B} \cdot d\underline{s}$$

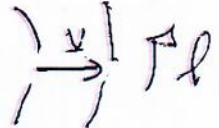


- ① change in B      ② change in ds

$$\frac{d\underline{\Phi}}{dt} = \int d\underline{s} \cdot \frac{\partial \underline{B}}{\partial t} + \int \frac{d\underline{s}}{dt} \cdot \underline{B}$$

change in B

motion of loop ...



$$\begin{aligned} \textcircled{1} &= \int d\underline{s} \cdot \nabla \times (\underline{v} \times \underline{B}) \\ &= \oint d\underline{l} \cdot (\underline{v} \times \underline{B}) \end{aligned}$$

$$d\underline{s} = \underline{v} \Delta t \times d\underline{l}$$

For ②

$$dt \left( \frac{d\underline{\Phi}}{dt} \right) = \int (\underline{v} dt \times d\underline{l}) \cdot \underline{B} = d\underline{\Phi}$$

$$\underline{ds} = \underline{v} dt \times d\underline{l}$$

↳ change in s in dt.

$$\left( \frac{d\underline{\Phi}}{dt} \right) = \int (\underline{v} \times d\underline{l}) \cdot \underline{B} = - \int d\underline{l} \cdot (\underline{v} \times \underline{B})$$

so

$$\frac{d\Phi}{dt} = \textcircled{1} + \textcircled{2}$$

$$= 0 \quad \checkmark$$

so  $\Rightarrow$  magnetic flux invariant  $\leftrightarrow$  cancellation

$\rightarrow$  in absence of resistivity, flux thru surface in flow is invariant, or frozen in

$\rightarrow$  no surprise:  $\underline{B}$  frozen in  $\Rightarrow$   $\Phi$  frozen in

$\rightarrow$  analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \quad \omega = \underline{\nabla} \times \underline{v}$$

In inviscid hydro, ( $\nu \rightarrow 0$ ) circulation  $\Gamma_c$  is conserved.

Exercise : Prove this!  
 Note relation between  $\underline{\omega}$  equation and  $\underline{B}$  eqn. Assume  $\rho = \text{const.}$ ,  $\underline{g} = 0$ .

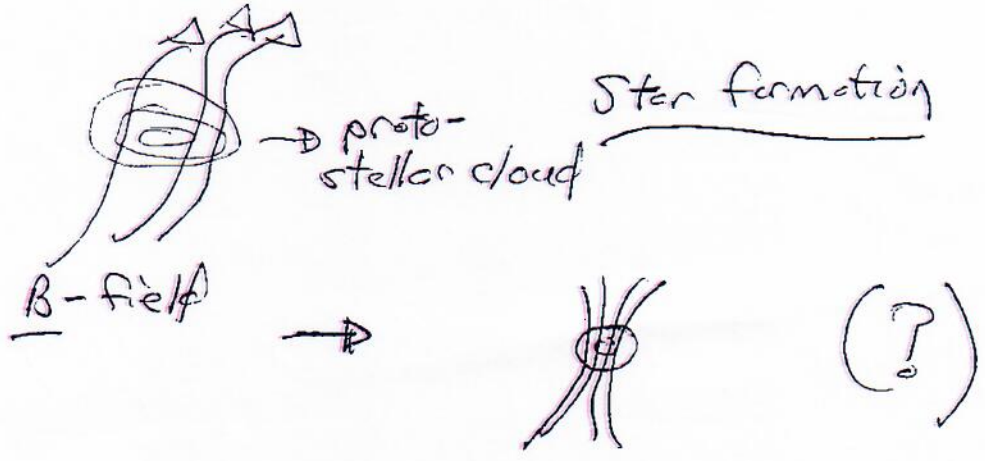
Extra Credit: ① Discuss the extension to the case where  $\rho \neq \text{const.}$

② what is 'frozen in' for viscus plasma?

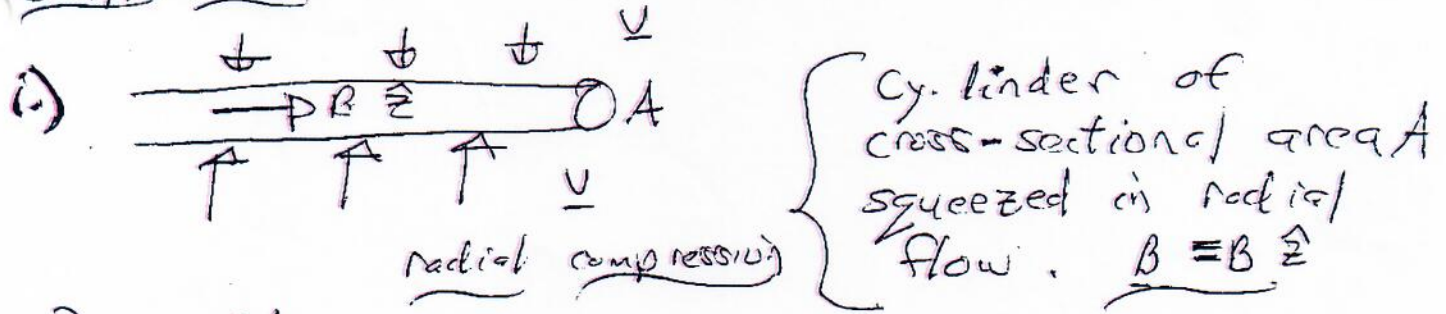
→ What Does "Freezing" Mean?

→ can relate field evolution in a flow to density evolution, since  $B/\rho$  is "frozen in"

Application:



Simple Cases → How does  $B$  change in a flow?



2 ways:

$$\frac{d(B \hat{z} / \rho)}{dt} = \frac{B \hat{z} \cdot \nabla \mathbf{v}}{\rho} \Rightarrow \mathbf{v} = v \hat{r} \Rightarrow \mathbf{v} \perp \mathbf{B}$$

$$= 0$$

$$\text{so } \underline{B}/\rho = \text{const}$$

$$\text{Now: } \rho AL = \text{const} \quad \text{so } \underline{B} \sim A^{-1}$$

$$\rho \sim A^{-1} \quad (L \text{ const.})$$

or

$$\text{Flux Frozen: } BA = \Phi = \text{const.}$$

$$\rho AL = \text{const} = M$$

$$L \text{ const.}$$

$$BA \sim \Phi_a, \quad B \sim A^{-1}$$

$$\rho A \sim M_a, \quad \rho \sim A^{-1}$$

$$\text{so } B \sim \rho^{(1)} \Rightarrow \underline{B}/\rho \sim \text{const!}$$

$$V = V(z)\hat{z} \text{ - compressible!}$$

(ii.)

$$\underline{\underline{\rightarrow B\hat{z}}} \quad \text{i.e. stretch, } \underline{1D}$$

here  $\frac{\underline{B} \cdot \underline{\nabla} \underline{V}}{\rho} \neq 0$ , but easier to work with  $\underline{B}$  than  $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{B} = \underline{B} \cdot \underline{\nabla} \underline{V} - \underline{B} \underline{\nabla} \cdot \underline{V}$$

$$= B \frac{\partial V(z)}{\partial z} - B \frac{\partial V(z)}{\partial z}$$

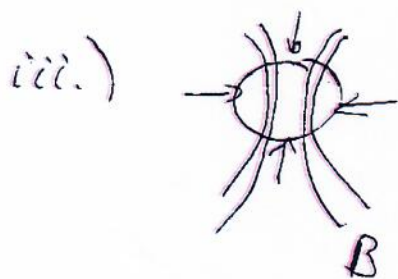
$$= 0 \quad !$$

For  $\rho$ ,  $\frac{d\rho}{dt} = -\rho \nabla \cdot \underline{v} = -\rho \frac{\partial v_z}{\partial z}$

here B invariant,  $\rho$  changes

i.e.  $B \sim \rho^{(0)}$

$\frac{d}{dt} \left( \frac{B}{\rho} \right) = \frac{B}{\rho} \nabla \cdot \underline{v}$   
 freezing in  $\Rightarrow$  const

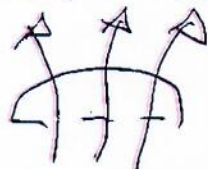


collapsing sphere:  $\underline{v} = v \hat{r}$



(i.e.  $\Phi = 0$  total sphere)

consider hemispherical surface (i.e. mushroom cap)



$$\Phi \sim B R^2 \sim \text{const}$$

$$M \sim \rho R^3 \sim \text{const.}$$

$$\Rightarrow B \sim r^{-2}$$


$$\rho \sim r^{-3}$$

$$\Rightarrow \underline{B/\rho^{2/3} \sim \text{const.}}$$

why the scaling  $\int \leftrightarrow$  why of interest  $\int$

→ <sup>"implosion"</sup>  $\left. \begin{array}{l} \text{gravitational collapse} \\ \text{equation of state} \end{array} \right\}$  problems sensitive to material collapsing

$$\underline{\text{IF}}: \rho \Rightarrow \rho_{\text{tot}} = \rho + \frac{B^2}{8\pi}$$

 collapse threaded by magnetic field

$$P = P_0 \left( \rho / \rho_0 \right)^\gamma$$

then natural to ask: Can one write  $B^2 = B^2(\rho)$  and thus extend equation of state to encompass magnetic pressure contribution?

⇒ proceed via flux-freezing!

$$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$$

⇒  $P_{B^2}$  has " $\gamma_{\text{eff}} = 4/3$ ". This resembles equation of state for degenerate gas (see Handouts I).

⇒ More on this in discussion of flux freezing and Virial theorems . . . .

→ Pragmatic Question: Is flux 'frozen' during star formation?  $\leftrightarrow$  Does resistivity matter?



$$\dot{M} \sim \frac{4 \times 10^6 \text{ cm}^2/\text{sec}}{T_{\text{ev}}^{3/2}} \quad (\text{Spitzer})$$

start  $\rightarrow$  collapse  $\rightarrow$  protostar

$$n \sim 1 \text{ atom/cm}^3$$

$$\rho \sim 1 \text{ g/cm}^3$$

$$n \sim 10^{24} \text{ /cm}^3$$

(related  $N_A$ )

but

$$B/\rho^{2/3} \sim \text{const}$$

$$\Rightarrow B/B_0 \sim (10^{24})^{2/3} \sim 10^{16} \quad \text{! huge amplification}$$

so  $B_0 \sim 10^{-6} \text{ G}$ , characteristic of ISM

$$\Rightarrow B \sim 10^{10} \text{ G in protostar}$$

$$\therefore P_{B^2} \sim 10^{19} \text{ erg/cm}^3 \quad (P_{B^2} \sim B^2/8\pi)$$

but  $P_{\text{Th}}$  for normal star  $\sim 10^{14} \text{ erg/cm}^3$

$P_{B^2} \gg P_{\text{Th}}$   $\Rightarrow$  clearly flux-freezing is bad assumption.

→ In terms of time scales:

$$\frac{\partial \underline{B}}{\partial t} = \underline{v} \times (\underline{v} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

①
②
③

$$\frac{1}{T_{\text{collapse}}} \sim \frac{1}{T_{\text{dynamic}}} + \frac{\eta}{L^2}$$

$\frac{\eta}{L^2} \sim \frac{1}{T_{\text{diff}}}$

3 scales,  
2 balance  
i.e. ① & ② ③  
negligible  
① & ③ ②  
negligible.

if  $T_{\text{collapse}} \ll T_{\text{diff}} \rightarrow$  flux frozen, OK

$T_{\text{collapse}} \gg T_{\text{diff}} \rightarrow$  must consider diffusion  
freezing invalid

N.B.: In star formation,  $T_{\text{coll.}} \ll T_{\text{diff}}$

but ISM has large neutral component.  
Plasma-neutral drag sets dissipation  
→ Ambipolar diffusion.