Next: Conservation Laws

→ Conservation Laws in MHD

— here discuss: conservation $\begin{cases} \text{momentum} \\ \text{energy} \\ \text{angular momentum} \end{cases}$

and virial theorems

→ Momentum → key: construct evolution of momentum density.

have: $\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \underbrace{\rho\underline{g}}_{\text{body force}}$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{V}) = 0$$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial t}(\rho\underline{V})}_{\substack{\text{momentum}\\\text{density}}} + \nabla\cdot\underbrace{(\rho\underline{V}\,\underline{V})}_{\substack{\text{Reynolds stress}\\\text{tensor}}} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \underbrace{\nabla\cdot\frac{\underline{B}\,\underline{B}}{4\pi}}_{\substack{\text{Maxwell stress}\\\text{tensor}}} + \rho\underline{g}$$

$\underline{T}_R = \rho\underline{V}\,\underline{V}$

$T_B = \frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}$

thus re-write:

$$\frac{\partial}{\partial t}(\rho\underline{V}) = -\nabla\cdot\underline{\underline{T}} + \rho\underline{g}$$

where

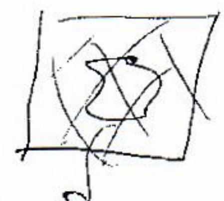
$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho \underline{V}\,\underline{V}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho V_i V_j$$

↳ a la Gaussian surface

Then, if consider a 'blob' of $\left\{ \begin{array}{l} \text{plasma} \\ \text{magnetofluid} \end{array} \right.$:

surface S - crucial



blob enclosed by arbitrary, non-dynamical surface.

$$\underbrace{\frac{\partial}{\partial t}\underline{P}}_{\text{momentum}} = \int d^3x \, \frac{\partial}{\partial t}\overbrace{(\rho \underline{V})}^{\text{momentum density}}$$

$$= -\int d^3x \, \underline{\nabla}\cdot\underline{\underline{T}} + \underbrace{\int d^3x \, \rho \underline{g}}_{\text{net body force}}$$

$$= -\int d\underline{S}\cdot\underline{\underline{T}} + \int d^3x \, \rho \underline{g}$$

So, apart from volume integrated body force,

$$\frac{\partial \underline{P}}{\partial t} = -\int d\underline{S}\cdot\underline{\underline{T}}$$

$\Big\{$ change in momentum set by stress on surface of blob

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho \underline{V}\,\underline{V}$$

Thus, can identify ways momentum is lost by the blob:

- $\underline{\underline{T}}_R \cdot d\underline{S} = +\rho \underline{V}\,\underline{V} \cdot d\underline{s}$ → flux of momentum density thru surface

- $\underline{\underline{T}}_{p_{tot}} \cdot dS = +\left(p + \frac{B^2}{8\pi}\right) \cdot d\underline{S}$ → pressure (total) force on surface, in $-d\underline{S}$ direction

- $\underline{\underline{T}}_{\substack{Mag \\ ten}} \, d\underline{S} = -\left(\frac{\underline{B}}{4\pi}\right)\underline{B} \cdot d\underline{s}$ → magnetic tension force in $+\underline{B}$ direction, piercing surface

$\sim (\underline{B} \cdot d\underline{S}) \, \frac{\underline{B}}{4\pi}$ ↦ tension of $\underline{B}/4\pi$, per line

↓ # of lines threads outward

i.e. 

⇒ Note that magnetic tension is independent of sign of $\underline{B}$ (as it should, tension is, strictly speaking, a dyad $\updownarrow$, not $\uparrow$)

↳ tensor field $\sim \underline{B}\,\underline{B}$

→ can make obvious analogy between strings and field lines

Alfven wave

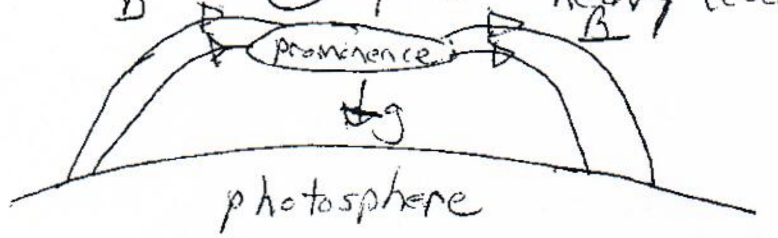
strings/area = B

$\sigma = \rho/B$ → mass per length, of string

$T = B/4\pi$

$V_{ph}^2 = T/\sigma$
$= B^2/4\pi\rho_0$
$= V_A^2$

→ example: {Solar prominence (see cover of Kulsrud)
$\longrightarrow$ heavy (cool) → requires support against gravity
$\underline{B}$



{prominences often associated with radiative condensation

i.e.

$L \Rightarrow$ #lines/area $= \underline{B} \cdot d\underline{S} < 0$ (inward)

force/Line is toward ↓

$\therefore$ $\underline{F}_{L}$ → toward upper left

$R \Rightarrow$ #lines/area $= \underline{B} \cdot d\underline{S} > 0$

f/Line is toward upper right

F_{R} → toward upper right

thus → prominence supported by magnetic tension (a la hammock — string)

→ squashing $\underline{B}$ → support by magnetic pressure, too ....

→ The Skeptic : " what of EM Momentum ? "

$$\underline{P}_{EM} = \underline{E} \times \underline{B} / 4\pi c$$

$$E \sim \frac{VB}{c} \Rightarrow P_{EM} \sim (\rho V)\, B^2 / 4\pi\rho c^2$$

$$\sim \rho V \left(V_A^2 / c^2 \right) \ll 1$$

n.b. obviously important in relativistic and EMHD, for $V_A \ll c$..

→ Angular Momentum → need Kulsrud ---- "virtual surface"

→ Energy

kinetic thermal magnetic gravity

Now energy : $E = E_V + E_p + E_B + E_g$

$$E = \int_V d^3x \left[\frac{1}{2}\rho V^2 + \frac{P}{\gamma - 1} + \frac{B^2}{8\pi} + \frac{\rho\phi}{2} \right]$$

self-gravitating blob

V

where $\begin{cases} \underline{g} = -\underline{\nabla}\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$ i.e. $\underline{g}$ evolves self-consistently (not "constant")

N.B. Problem : Jeans Instability

→ Calculate the growth rate of density perturbations in an un-magnetized, self-gravitating fluid

→ repeat in 1D using Vlasov equation

→ Where does E_p come from?

Consider work to compress plasma/fluid, i.e.

$$dW = -p\,dV$$

$$\Delta E = -\int_0^{\rho_0} P(\rho)\, d(1/\rho) = \int_0^{\rho_0} (\rho/\rho_0)^{\gamma} P_0 \frac{d\rho}{\rho^2}$$

$$= \frac{P_0}{\rho_0(\gamma-1)} \Rightarrow \underset{\substack{\downarrow \\ \text{energy} \\ \text{density}}}{\mathcal{E}} = \rho_0 \Delta E = \frac{P_0}{(\gamma-1)}$$

→ for energy balance, crank it out, using MHD equations ....

$$\frac{dE}{dt} = \underset{①}{\frac{dE_V}{dt}} + \underset{②}{\frac{d}{dt} E_p} + \underset{③}{\frac{d}{dt} E_B} + \underset{④}{\frac{d}{dt} E_g}$$

① $$\frac{d}{dt} E_V = \int d^3x\, \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)$$ use $\underline{V}$

$$= \int d^3x \left[V^2 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underline{V}}{\partial t} \cdot \underline{V} \right] d^3x$$

→ i b p leaves S.T. and cancels 2nd.

$$= \int d^3x \left[-\frac{V^2}{2} \nabla \cdot (\rho \underline{V}) - \underline{V} \cdot \rho (\underline{V} \cdot \nabla \underline{V}) - \underline{V} \cdot \nabla P + \underline{V} \cdot (\underline{J} \times \underline{B}) - \rho \underline{V} \cdot \nabla \phi \right]$$

i.e. $\int -\frac{V^2}{2}\nabla\cdot(\rho \underline{V}) = -\frac{V^2}{2}\rho \underline{V}\Big| + \int(\underline{V}\cdot\nabla\underline{V})\cdot\rho\,\underline{V}$

cancels 2nd term in $\frac{dE_V}{dt}$

② $\frac{d}{dt}E_p = \int \frac{d^3x}{\gamma-1}\frac{\partial p}{\partial t}$

$\frac{d}{dt}(p/\rho^\gamma) = 0$

Now eqn. state $\Rightarrow$ $\frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$

$\left[\frac{1}{(p/\rho^\gamma)}\frac{d}{dt}(p/\rho^\gamma) = 0\right]$

and $\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\underline{V}$

$$\Rightarrow \frac{\partial p}{\partial t} = -\underline{V}\cdot\nabla p - \gamma p \nabla\cdot\underline{V}$$

So $$\frac{d}{dt}E_p = \frac{-1}{(\gamma-1)}\int d^3x\,(\underline{V}\cdot\nabla p + \gamma p\nabla\cdot\underline{V})$$

$$= -\int d^3x\left[\frac{\gamma}{\gamma-1}\nabla\cdot(p\underline{V}) - \underline{V}\cdot\nabla p\right]$$

$\boxed{\text{yields a surface term}}$

cancels $\underline{V}\cdot\nabla p$ term in $\frac{dE_n}{dt}$

expect similar relation between $\underline{J}\times\underline{B}$ and $\frac{\partial}{\partial t}B^2$ .... so...

(3) $$\frac{d}{dt} E_B = \frac{1}{4\pi}\int d^3x \; \underline{B}\cdot\frac{\partial \underline{B}}{\partial t}$$

conduct.

$$= \frac{1}{4\pi}\int d^3x \; \underbrace{\underline{B}}_{a} \cdot \big(\underline{\nabla}\times \underline{v}\times \underbrace{\underline{B}}_{b}\big)$$ by induction eqn.

$$= -\int d^3x \left\{ \underbrace{\underline{\nabla}\cdot\left[\frac{\underline{B}\times(\underline{v}\times\underline{B})}{4\pi}\right]}_{\text{surface term } (\to \text{ Poynting})} - \underbrace{\frac{(\underline{\nabla}\times\underline{B})}{4\pi}\cdot(\underline{v}\times\underline{B})}_{\underline{J}\,\cdot\,\underline{v}\times\underline{B}} \right\}$$

$$\mathcal{J} = \int d^3x \; \underline{J}\cdot(\underline{v}\times\underline{B}) = -\int d^3x \; \underbrace{(\underline{J}\times\underline{B})\cdot\underline{v}}_{\text{cancels } \underline{v}\cdot\underline{J}\times\underline{B} \text{ term in } dE_V/dt}$$

which leaves:

(4) $$\frac{dE_g}{dt} = \frac{1}{2}\int d^3x \left(\phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{1}{2}\int d^3x \; \phi\frac{\partial\rho}{\partial t} + \int d^3x \frac{\nabla^2\phi}{8\pi G}\frac{\partial\phi}{\partial t}$$ ibp →

$$= \frac{1}{2}\int d^3x\,\phi\frac{\partial\rho}{\partial t}\,d^3x + \int\frac{\phi}{8\pi G}\nabla^2\frac{\partial\phi}{\partial t}\,d^3x$$

or
$$\frac{dE_g}{dt} = \frac{1}{2}\int \phi \frac{\partial \rho}{\partial t} d^3x + \frac{1}{2}\int d^3x\, \phi \frac{\partial \rho}{\partial t}$$

$\frac{\partial \rho}{\partial t}$
$$= \int d^3x\, \phi \frac{\partial \rho}{\partial t} = -\int d^3x\, \phi\, \nabla\cdot(\rho \underline{V})$$

$$= +\int d^3x\, \rho \underline{V}\cdot \underline{\nabla}\phi$$

cancels $-\rho \underline{V}\cdot\nabla\phi$ in $\frac{dE_K}{dt}$ $\quad + -\int d\underline{S}\cdot \rho\phi \underline{V}$

Note: $-\underline{V}\cdot\underline{\nabla}P$; $\underline{V}\cdot(\underline{J}\times\underline{B})$; $-\rho\underline{V}\cdot\underline{\nabla}\phi$; $\underline{V}\cdot\rho\underline{V}\cdot\underline{\nabla}\underline{V}$

terms all cancel in dE_V/dt !

Now adding up all 4 pieces $\Rightarrow$

$$\boxed{\frac{d}{dt}E = -\int d\underline{S}\cdot\Big[\underbrace{\rho\underline{V}\frac{V^2}{2}}_{\text{Conv. KE}} + \underbrace{\frac{\gamma}{\gamma-1}\rho\underline{V}}_{\gamma\text{ factor conv. } h} - \underbrace{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}}_{\underline{E}\times\underline{B}} + \underbrace{\rho\underline{V}\phi}_{V\phi}\Big]}$$

ST

i.e. not surprisingly, only survivors are surface terms . . . . $\Rightarrow$ in ideal MHD, only change on energy of blob involves boundary . . .

i.e. have:

$$\frac{dE}{dt} = -\int d\underline{S}\cdot\left[\overset{①}{\rho\underline{V}\frac{\underline{V}^2}{2}} + \overset{②}{\frac{\gamma}{\gamma-1}\rho\underline{V}} - \overset{③}{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}} + \overset{④}{\rho\underline{V}\phi}\right]$$

① → kinetic energy loss via simple kinetic energy flow thru surface.

② → $-\frac{\gamma}{\gamma-1}\underline{V}\cdot d\underline{S}\,p$ ⟷ outward flow of enthalpy

i.e. $-\gamma\frac{p\underline{V}\cdot ds}{\gamma-1} = -\frac{p}{\gamma-1}\underline{V}\cdot \underline{ds} - p\underline{V}\cdot ds$

why the γ ? →

outward flow of thermal energy $\left(\underline{ds}\cdot\underline{V}\frac{p}{\gamma-1}\right)$ thru S

pdV work of blob on exterior

③ as $\underline{E} = -\frac{\underline{V}\times\underline{B}}{c}$ ... →

so ③ $= d\underline{S}\cdot\frac{\underline{E}\times\underline{B}}{4\pi c}$ → $\left\{\begin{array}{l}\text{loss of}\\ \text{energy by}\\ \text{Poynting flux}\end{array}\right.$

④ loss of gravitational potential energy due outflow from blob...

It's all clear!!

this brings us to ....

→ Virial Theorems in MHD

- what is a virial theorem

- why yet another theorem?

→ Virial Theorems are:

- space/time averaged energy theorems

- "lumped parameter" relations for energies in complex, multi-element interacting systems

- useful for 'back-of-envelope' estimates, etc.

- logically extend the moment program:

$$\underbrace{f(\underline{x}, \underline{v}, t)}_{\text{phase space fluid}} \xrightarrow[\underline{v}\text{ moments}]{} \underbrace{n(\underline{x}, t), \underline{v}, T}_{\text{position space fluid}} \xrightarrow[\substack{\text{virial integrals} \\ \text{space integrals}}]{} \underbrace{E_v,\ E_B,\ \text{etc.}}_{\#}$$

Before proceeding:

?? $\boxed{\text{Can an \underline{isolated} blob of MHD plasma \underline{confine} \underline{itself} without self gravity}}$??

Easily answered by Virial Theorem....

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underbrace{2T}_{\text{kinetic energy}} + \underbrace{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}_{\text{via Newton's Law}}$$

Now, if $\boxed{\sum_i \underline{p}_i \cdot \underline{x}_i \text{ bounded}}$, blob

$$\langle \ \rangle = \int_0^T \frac{dt}{T}, \quad T \to \infty$$

$$\left\langle \frac{d}{dt} \sum \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T} \int_0^T dt \frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right)$$

$$\underset{T \to \infty}{\longrightarrow} 0$$

so ...

→ (First) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i}\cdot \underline{x}_i \right\rangle$$

Further, if $U = U(\underline{x}_1, \underline{x}_2, \dots \underline{x}_N)$

where $U(\alpha \underline{x}_1, \alpha \underline{x}_2, \dots \alpha \underline{x}_N) = \alpha^k U(\underline{x}_1, \underline{x}_2, \dots \underline{x}_N)$

homogeneous function

(scaling ⟺ structure of power-law potentials → i.e. h.o. → $k=2$, Coulomb → $k=-1$)

Scaling!

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then $\left(\frac{k}{2}+1\right)\langle U\rangle = E$

$$\boxed{\langle U\rangle = \frac{2}{k+2}E}, \quad \langle T\rangle = \frac{k E}{k+2}$$

check: $k=2$, $\langle U\rangle = (1/2)E$, $\langle T\rangle = (1/2)E$ ✓

$k=-1$, $\langle T\rangle = -E$ ✓ ⇒ (bounded motion only if total energy negative (i.e. bound state)
gravity

($\Rightarrow E<0$)

Aside: $\boxed{\text{Simplest realization of negative specific heat `paradox', i.e.}}$

R $\rightarrow$ consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  $\rightarrow$ $\underline{E \text{ decreases}} \leftrightarrow$ R decreases

$R \downarrow$

$\therefore$ $(-E)$ increases $\Rightarrow$ $\langle T \rangle$ increases

$\hookrightarrow$ kinetic energy

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling $\Rightarrow$ [temperature increase]!

$$\Rightarrow \underbrace{c}_{\text{specific heat}} < 0 \;!?$$

[In the days before the discovery of nuclear fusion, this was thought to be what heated stars. Kelvin, in particular, was a ~~propon~~ proponent.

Now, proceeding to full Virial theorem ...

→ Consider equation of motion

$$\begin{cases} \underbrace{\frac{\partial}{\partial t}(\rho V_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j} \underbrace{T_{ij}}_{\underline{\text{full}}\text{ stress tensor}} \\ T_{ij} = \rho V_i V_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\delta_{ij} \end{cases}$$

Now, recalling relation of virial to $\frac{d}{dt}(\underline{p}\cdot\underline{x})$
⇒ consider:

Moment inertia / distrib.

$$\boxed{I_{ij} = \int d^3x\, \rho\, x_i x_j}$$ ($\sim$ moment of inertia)

↳ Virial theorem is for tensor...

and
$$\frac{d}{dt} I_{ij} = \int d^3x\, \frac{\partial \rho}{\partial t}\, x_i x_j$$

$$= -\int d^3x\, \frac{\partial}{\partial x_t}(\rho V_t)\, x_i x_j$$

integrating by parts assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x\, [\rho x_i v_j + \rho x_j v_i]$$

so

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i\left(\frac{\partial}{\partial t}\rho V_j\right) + x_j \frac{\partial}{\partial t}(\rho V_i)\right]$$

but $$\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming $\left\{ \boxed{\text{compact blob, no external linkage}} \right.$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$ unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as T_{ij} manifestly symmetric $\Rightarrow$

$$\boxed{\begin{aligned} \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} &= +\int d^3x \, T_{ij} \\ T_{ij} &= \rho V_i V_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij} \end{aligned}}$$

— tensor virial theorem. [note unlike simple pt particle example, time dependence remains.]

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{ii} = \mathrm{tr}\, I_{ij}$$

repeated indexes summed

$\mathrm{tr}\,(\text{V.T.}) \Rightarrow$

$$\mathrm{tr}\, \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\, \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr} \int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\,\delta_{ij}\right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$\therefore \quad I \equiv \int d^3x\, \rho x^2/2 \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

$\rightarrow$ Scalar Virial Theorem

Now, first neglect self-gravitation $\Rightarrow$

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$$

Now $\rightarrow$ can an isolated blob of MHD fluid confine itself??

If 'self-confined' $\Rightarrow$ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent $\Rightarrow \dot{I}, \ddot{I} = 0$

stable pulsation $\Rightarrow \ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$

so even if $V^2 = 0$ (no fluid motion in blob) $\Rightarrow$

$p > 0$, $B^2/8\pi > 0$ $\Rightarrow$ $\ddot{I} > 0$!

$\therefore$ No $\rightarrow$ isolated blob can't confine itself.

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x \frac{P}{\gamma-1} = \frac{3}{2}\int d^3x\, P \qquad \text{(gas)}$$

$$E_B = \int d^3x \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2 I}{dt^2} = 2E_V + 2E_P + E_B}$$

simple relation in terms energies.

Aside: $\Rightarrow$ So, isolated blob can't confine itself

$\Rightarrow$ how is $\left\{\begin{array}{l} \text{tokamak} \rightarrow B_T \text{ for } \left\{\begin{array}{l}\text{stability; not} \\ \text{transport}\end{array}\right. \\ \underline{\text{or}} \text{ - better} \quad \text{macro - confinement} \\ \text{RFP} \rightarrow \text{weak external } B_T \text{ guide} \\ \quad\quad\text{(negligible)} \end{array}\right.$

confined ?? Confinement by wall is unacceptable ...

x

Answer: ⇒ toroidal plasma tends to expand toroidally

⇒ held in place by $\left\{ \begin{array}{l} \text{conducting shell (often undesireable) or} \\ \text{" vertical field "} \end{array} \right.$

i.e.

⇒ additional external P_{Mag} to oppose toroidal expansion — vertical field

⇒ image currents in close-in conducting shell can do likewise

JET anecdote re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{gravity} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{gravity}}\delta_{ij}$$

to calculate:

$$\nabla^2\phi = 4\pi G\rho$$

$$\Rightarrow \quad \phi = -G\int d^3x \; \frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}} \delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= + E_{\text{gravitation}} \equiv E_g < 0$$

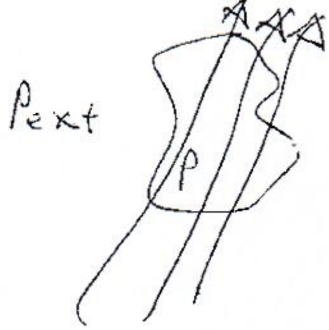
so scalar Virial theorem becomes, with gravity $\Rightarrow$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_v + 2E_p - |E_g| + E_B}$$

so with gravity, can have self-confining blob (no surprise...)

This brings us to another application of virial theorems, namely proto-stellar cloud collapse ....

— now, consider a plasma cloud/blob



- mass M, radius R
- "threaded" by B
- pressure p, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{I} = 0$, $\underline{V} = 0$, must have:

surface terms

$$\left[2E_p - |E_g| + E_B \right] = \int dA\, P_{ext}\, \hat{x} \cdot \hat{n} - \int dA\, \underline{x} \cdot \underline{\underline{T}}_B \cdot \hat{n}$$

external pressure

magnetic stress thru surface (threading fields)

Now, can estimate:

$M = \int \rho\, dV \rightarrow$ total mass $\rightarrow \rho R^3$

$E_p \cong C_s^2 M$

$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \dfrac{GM^2}{R}$

For frozen flux, $\Phi \sim \pi R^2 B$

$$B \sim \frac{\Phi}{R^2}$$
$$R^3 B^2 \sim \Phi^2 / R$$

so $\quad E_B + \int dA \, \underline{x} \cdot \underline{\underline{T}}_B \cdot \hat{n} \sim \beta \Phi^2 / R$

$\Rightarrow$ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{G M^2}{R} + \frac{3}{2} c_s^2 M \right)}$$

$\rightarrow$ scalar virial theorem for cloud....

Now: $\quad P_{ext} \sim \left(\frac{\beta \Phi^2}{R^3} - \alpha \frac{G M^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2} \right)$

$\rightarrow$ if Φ, $G \rightarrow 0$ $\rightarrow$ need $P_{int} = P_{ext}$ for confinement.....

$\rightarrow$ if $\Phi = 0$

$$P_{ext} = -\frac{\alpha G M^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2}$$

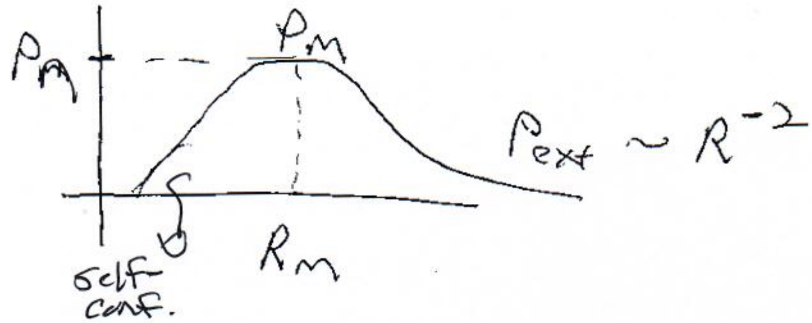
$dP/dR = 0 \Rightarrow \quad 3 \alpha \frac{G M^2}{R^4} = 3 \frac{c_s^2 M}{R^3}$

Radius at which P_{ext} max

$$\boxed{R_{max} = G M \alpha / c_s^2}$$

[Note: $\Rightarrow R_m^2 = (G\rho / c_s^2)^{-1/2} \Rightarrow L_{Jeans}^2$]

so P_{ext} needed



- $P > P_{max} \Rightarrow$ no equilibrium

- $R < R_{max} \Rightarrow P_{ext}$ must decrease to maintain equilibrium $\Rightarrow$ instability to gravitational collapse!

- $\Phi \neq 0$ (magnetic field on...) $\Rightarrow$ [note immediately that magnetic support scales similarly to gravitational attraction

$\Rightarrow$

$$P_{ext} \sim \left[\left(\beta \Phi^2 - \alpha G M^2\right)/R^3 + \frac{3}{2} \frac{C_s^2 M}{R^2} \right]$$

so key point is: $\boxed{\left(\beta \Phi^2 - \alpha G M^2\right) \lessgtr 0 \, ?!}$

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

- $\boxed{M < M_\Phi} \rightarrow$ magnetically subcritical mass for gravitational collapse

$M > M_{\Phi} \Rightarrow$ magnetically super-critical mass for collapse!

i.e. $M < M_{\Phi}$ $(M_{\Phi}^2 - M^2 > 0)$ $\rightarrow$ repulsive effects $\left\{ \begin{matrix} \text{field} \\ \text{thermal} \end{matrix} \right\}$ pressure always win

$\rightarrow$ no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_{\Phi}$ $(M_{\Phi}^2 - M^2 < 0)$ $\rightarrow$ sufficient external pressure/compression can induce gravitational collapse, even if flux frozen in.

[Note: If kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra} \rightarrow$ collapse
  - $M < M_{Chandra} \rightarrow$ no collapse.

$M_{chandra}$ derived for degenerate Fermi gas equation of state $\rightarrow \gamma = 4/3$, instead of $\gamma = 5/3$.

– of flux-freezing $\Rightarrow$ $\Phi \sim B R^2$
$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2} \Rightarrow B \sim \rho^{2/3}$

$\therefore \boxed{B^2 \sim P_{Mag} \sim \rho^{4/3}}$

$\Rightarrow$ <u>if</u> flux frozen, field obeys equation of state like Fermi gas

(i.e. Flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ....

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore \quad n_{\text{Fermion}} \sim N/R^3$

$\therefore \quad$ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$p \sim \hbar/\Delta x \sim \hbar n^{1/3}$ (Heisenberg Uncertainty)

↓
Fermion Momentum

⇒ Fermion energy (per Fermion) : $E_F = pc \sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion) : $E_{Grav} \sim -\dfrac{GMm_b}{R}$

m_b → Baryon mass

$M \sim N \; m_B$

Pressure → electron
Mass → Baryon

$\therefore \quad E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{GNm_B^2}{R}$$

Note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,

$\therefore E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse.

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore M_{Chandra} = N_{max}\, m_B \sim 1.5\, M_\odot$$