Magnetic Helicity, Taylor Relaxation, Mean Field Theory

Contents:

- → Magnetic Helicity — meaning, conservation.
- → Taylor Hypothesis and Relaxation (a relaxation principle!)
- → Mean Field Theory I.)
  - — Realizing Taylor
- → Mean Field Theory II.)
  - — Intro to Mean Field Electrodynamics

Basics of Helicity

→ Magnetic Helicity → Constraint in Relaxation

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x\, \underline{A}\cdot\underline{B}$$

V is taken to be the volume of a 'flux tube'.

- what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x\, A\cdot\underline{\nabla}\times\underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

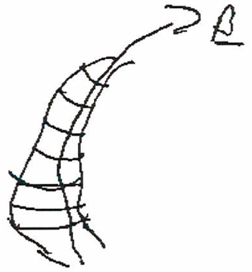
→ K is a pseudo-scalar ∴ has orientation or "handedness"...

Proceed via:
- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

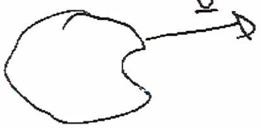
$$K \to K + \int_V d^3x\, \underline{\nabla}\chi \cdot \underline{B}$$



$$= K + \int_V d^3x\, \underline{\nabla}\cdot(\underline{B}\,\chi)$$

$= 0$, to surface term. $\left\{ \begin{array}{l} \underline{B}\cdot\hat{n} = 0 \text{ on surface of} \\ \text{tube} \end{array} \right.$

Now, consider a blob of MHD fluid in motion



can show $\dfrac{dK}{dt} =$

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta\,\underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \underline{\nabla}\phi$$

$\left\{ \begin{array}{l} \text{Field in} \\ \text{blob} \end{array} \right.$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{v}\times\underline{\nabla}\times\underline{A} - c\,\nabla\phi - c\,\eta\,\underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{v}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{v} - \underline{B}\,\underline{\nabla}\cdot\underline{v} + \eta\,\nabla^2\underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt}\int_V d^3x\,(\underline{A}\cdot\underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt}\cdot\underline{B} + \underline{A}\cdot\frac{d\underline{B}}{dt}\right) + \int \underline{A}\cdot\underline{B}\,\frac{d}{dt}\,d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t}\cdot\underline{B} + (\underline{V}\cdot\underline{\nabla}\,\underline{A})\cdot\underline{B} + \underline{A}\cdot\frac{\partial\underline{B}}{\partial t} + \underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B})) + \underline{A}\cdot\underline{B}\;\underline{\nabla}\cdot\underline{V} \right)$$

where $\frac{d}{dt}\, d^3x = \underline{\nabla}\cdot\underline{V}$

i.e. $\frac{d}{dt}\, dV = \frac{d}{dt}\, d\underline{\sigma}\cdot d\underline{\ell} + d\underline{\sigma}\cdot\frac{d}{dt}\, d\underline{\ell}$

$$= -d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma} + (\underline{\nabla}\cdot\underline{V})(d\underline{\sigma}\cdot d\underline{\ell}) + d\underline{\ell}\cdot\underline{\nabla}\,\underline{V}\cdot d\underline{\sigma}$$

$$= \underline{\nabla}\cdot\underline{V}\; d\overset{3}{\cancel{x}}$$

s.t. and $\underline{B}\cdot\hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\big(\underline{B}\cdot\cancel{\underline{V}}\times\underline{B} - c\,\cancel{\underline{B}}\cdot\underline{\nabla}\phi - c\eta\,\underline{J}\cdot\underline{B}\big) + \underline{A}\cdot\big(\underline{\nabla}\times(\underline{V}\times\underline{B})\big) + \underline{\nabla}\cdot\big((\underline{A}\cdot\underline{B})\,\underline{V}\big) + \underline{A}\cdot\eta\,\nabla^2\underline{B} \Big]$$

div flux

where $\underline{A}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{B}) + \underline{B}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{A}) + \underline{A}\cdot\underline{B}\;\nabla\cdot\underline{V} = \nabla\cdot(\underline{V}\,\underline{A}\cdot\underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla\cdot\big((\underline{A}\cdot\underline{B})\,\underline{V}\big) + \underline{\nabla}\cdot\big((\underline{V}\times\underline{B})\times\underline{A}\big) + (\underline{V}\times\cancel{\underline{B}})\cdot(\underline{\nabla}\times\underline{A}) - c\eta\,\underline{J}\cdot\underline{B} - \eta(\underline{A}\cdot\underline{\nabla}\times\underline{J})\,c \Big]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta(\underline{A}\times\underline{J}) \right] - c\eta\, \underline{J}\cdot\underline{B} - c\eta\, \underline{J}\cdot\underline{B} \right\}$$

$$= \int d\underline{S}\cdot\left[(\underline{A}\cdot\underline{B})\underline{v} + (\underline{v}\times\underline{B})\times\underline{A} + c\eta\, \underline{A}\times\underline{J} \right]$$

$$- 2\int d^3x \left[c\eta\, \underline{J}\cdot\underline{B} \right]$$

$$= \int d\underline{S}\cdot\left[\cancel{(\underline{A}\cdot\underline{B})\underline{v}} - \cancel{(\underline{A}\cdot\underline{B})\underline{v}} + \cancelto{0}{(\underline{A}\cdot\underline{v})}\,\underline{B} \right] - c\eta \int d\underline{S}\cdot \underline{J}\times\underline{A}$$

$\underline{B}\cdot\underline{\hat{n}} = 0$, on tube

$$- 2c\eta \int d^3x (\underline{J}\cdot\underline{B})$$

$$= -\int c\eta\, d\underline{S}\cdot \cancelto{0}{\left[\underline{\nabla}\underline{B}\cdot\underline{A} - \underline{A}\cdot\underline{\nabla}\,\underline{B} \right]} - 2c\eta \int d^3x\, \underline{J}\cdot\underline{B}$$

$$= -2c\eta \int d^3x (\underline{J}\cdot\underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta \int d^3x (\underline{J}\cdot\underline{B})}$$

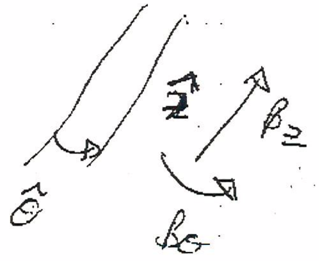
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

∴ Magnetic Helicity is conserved in ideal MHD. Can assign helicity to each flux tube.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $q(r) = \frac{rB_z}{RB_\theta(r)} = \frac{1}{R\mu(r)}$



$$\mu(r) = \frac{B_\theta(r)}{rB_z}$$ → field line pitch (length scale ~~length scale~~ over ~~which, winding varies)~~

cylindrical plasma ⇒ $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r}\int_0^r r B_z \, dr$

$$A_z = -\int_0^r B_\theta \, dr$$

so $$\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\, dr - B_z \int_0^r B_\theta\, dr$$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\, dr - B_z \int_0^r B_\theta\, dr$$

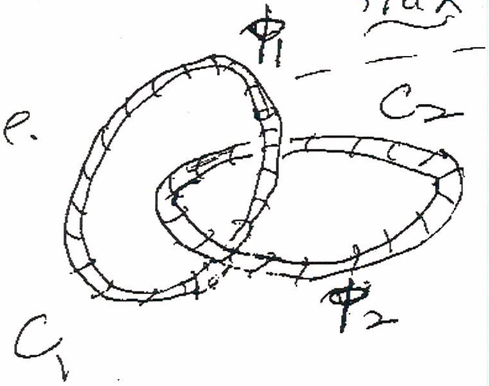
$$\underline{A}\cdot\underline{B} = B_z \left[\mu \int_0^r \frac{B_\theta}{\mu}\, dr - \int_0^r B_\theta\, dr \right]$$

$= 0$ for constant μ

∴ non-zero helicity requires $\mu = \mu(r)$
i.e. – pitch varies with radius
$\Rightarrow \boxed{\text{magnetic shear}}$ twist

– physically → helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux

$$\Phi = \underbrace{\int d\underline{A}}_{\text{x-section area}} \cdot \underline{B} = \underbrace{\phi_1}_{\text{const}}$$

tube 2: $\Phi = \phi_2$

field in loops, only

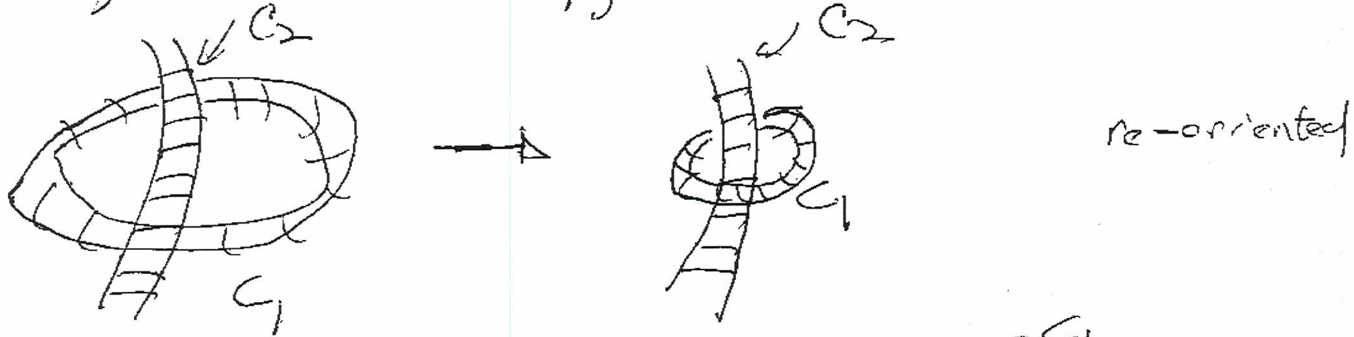
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \oint_{\underbrace{C_1}_{\text{along loop}}} d\ell \int_{\underbrace{A_1}_{\text{X-sect area}}} \underline{dS}\; A\cdot\underline{B}$$

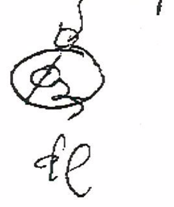
$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\, dA$$

$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



$\Rightarrow$ in x section:



but $\int_{C_1} A\cdot d\ell = \int_{A \text{ enclosed}} B\cdot dS = \Phi_2$

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..

so... $K_1 = \phi_1 \phi_2$ ⇒ product of fluxes

similarly $K_2 = \phi_2 \phi_1$

$\therefore \quad K = 2\phi_1 \phi_2$

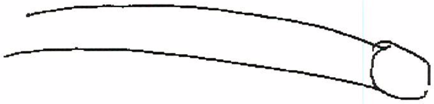
if n windings $K = K_1 + K_2 = \pm 2n\phi_1 \phi_2$.

⇒ helicity is measure of self-linkage of magnetic configuration.

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP



~ toroid

~ toroidal current

well fit by

$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$

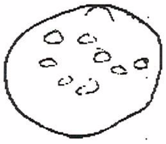
$\Updownarrow$

force free

⇒ why so robust?, especially since RFP's so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to η
- toroidal plasma $\leftrightarrow$ many small tubes



etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\underset{\substack{\downarrow \\ \text{plasma volume}}}{\int} \underline{A}\cdot\underline{B}\, d^3x = K_0 \rightarrow ⊚ \text{ conserved.}$$

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\, \frac{\underline{B}^2}{8\pi} + \lambda \int_V d^3x\, \underline{A}\cdot\underline{B}\right] = 0$$

i.e., minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! - indeed amazingly well - for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay

↗ energy cascades → small scale

↘ helicity cascades → large scale (less dissipation)

- Relevance to driven system?
i.e. in real RFP, transformer on.

$\tau_R \sim L^{3/2}$

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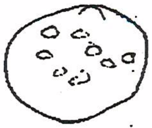
$\underline{\tau_R \sim L^{3/2}}$



etc. $\quad \frac{V}{L} \sim \frac{V_A}{L}/\sqrt{R_m} \sim 1/L^{3/2}$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\int_{V} \underline{A}\cdot\underline{B}\, d^3x \equiv K_0 \rightarrow \textcircled{\sim} \text{ conserved.}$$

plasma volume

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

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→ dynamics? – how does relaxation occur

→ more in discussion of kinks, tearing..

$$\delta\left[\int d^3x\left[\frac{B^2}{8\pi}+\lambda \underline{A}\cdot\underline{B}\right]\right]=$$

$$\frac{\underline{B}}{4\pi}\cdot\delta\underline{B}+\lambda A\cdot\delta B=0$$

$$\frac{\underline{\nabla}\times\underline{A}}{4\pi}+\lambda\underline{A}=0$$

$\underline{\nabla}\times$

$$\underline{J}=\mu\underline{B}\qquad\qquad \underline{\nabla}\times\underline{B}=\mu\underline{B}$$

force free

and

$$\frac{\underline{J}\cdot\underline{B}}{B^2}=\mu \to \text{const}$$

$\nabla J_{\parallel}=0$ → parallel current homogenized

Taylor Relaxation and its Dynamics

No. 10.

Date

see RMP: { J.B, Taylor 1986

→ <u>Taylor</u> <u>Relaxation</u>

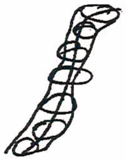
⇝ transition to 'quiescent period' ⇒ "relaxation" → <u>turbulent</u> <u>resistive</u>

→ magnetic energy minimization (P_{0H} only, and $\beta << 1$)

⇒ what constraint?

→ ⓐ in <u>ideal</u> plasma, $\int d^3x\, \underline{A} \cdot \underline{B}$ conserved for all $\int d^3x$

<u>i.e</u> any tube, around line



$$\int_{\text{tube}} d^3x\, A \cdot B = \text{const.}$$

<u>line</u> α, β s/t $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$

→ if $\delta \int_{tube} d^3x \left[\frac{B^2}{8\pi} + \lambda A \cdot B \right] = 0$

$$\boxed{\nabla \times \underline{B} = \lambda(\alpha, \beta)\, \underline{B}} \;;\quad \underline{B} \cdot \underline{\nabla} \lambda = 0$$

force free on micro-tube along line

but $\lambda(\alpha, \beta) \neq \lambda(\alpha', \beta')$

i.e → each tube/line defines conserved helicity

→ ∞ of invariants, due freezing in.

⑤ But, relaxation occurs in resistive, turbulent plasma. $\tau_R \sim \frac{L}{V} \sim \tau_A \sqrt{R_m}$

⇒ small tubes are destroyed by reconnection $\tau_R \sim \ell^{3/2}$

⇒ as t → ∞, only very largest tube survives → global helicity is asymptotic survivor

Could ~~also~~ view from stoch. lines
→ 1 line

motion → turbulence

resistivity → reconnection

i.e. recall, S-P:

$$V \equiv v_A/\sqrt{R_m} \sim \sqrt{\frac{v_A \eta}{L}}$$

$$1/\tau_{R_L} \sim 1/L^{3/2}$$

⇒ smaller scales reconnect faster.

⇒ smaller tubes destroyed first.

∴ 3 arguments for conjecture of global helicity as rugged invariant:

→ enhanced dissipn (above) → largest scales reconnect most slowly

→ stochasticity → if field lines stochastic, then (cf. Fermi-MNR) 1 field line → 1 tube of conserved helicity → Global helicity is only inv.

⇒ RFP has only 1 field line.

→ selective decay → magnetic helicity (inverse cascades) in 3D (MHD)

∴ global (large scale) helicity accumulates.

→ magnetic energy forward cascades.

n.b. compare: energy

heuristic

$$\dot{W} \sim -2\eta \langle (\nabla B)^2 \rangle \quad (\text{if } \nu \to 0)$$

$$K = \int d^3x\, A\cdot B \Rightarrow \dot{K} = -2c\eta\langle J\cdot B\rangle$$

$$\dot{W} \sim -2\eta \frac{\langle B^2\rangle}{L_{eff}^2}$$

$$\dot{K} \sim -\eta \frac{\langle B^2\rangle}{L_{eff}}$$

if $\boxed{L_{eff} \sim \Delta \sim L/\sqrt{Rm} \sim \eta^{1/2}}$

$$\therefore\ \dot{W} \sim \eta^{0} \to \text{finite} \Rightarrow \boxed{\text{indep dissipn aka } \epsilon \text{ in turb}}$$

$$\dot{K} \sim \eta^{1/2} \to 0$$

∞ W diss, K ~ cons!

⇒

Routine calc. variation:

$$\nabla \times B = \mu B$$

$$\bar{J} \cdot B / B^2 \to \text{const} = \mu$$

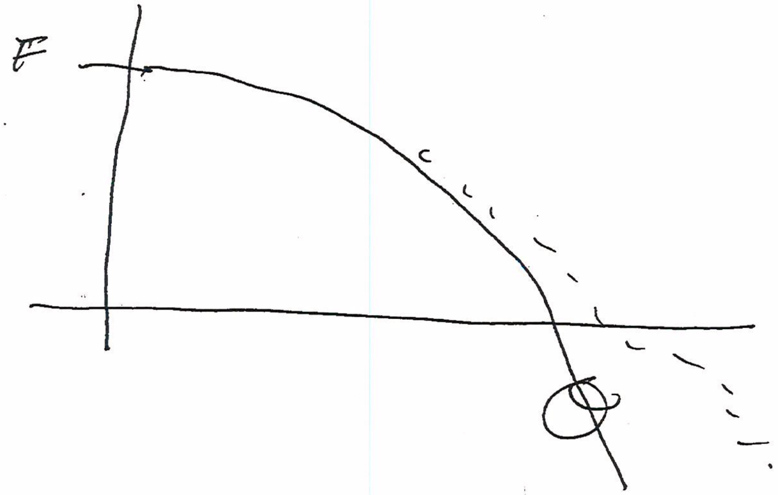
$$\boxed{J_{\parallel}/B \text{ homogenized}}$$

n.b. $\int d^3x\, A \cdot B$ related to volt-seconds in ~~the~~ plasma, via transformer.

No.

Date

Taylor Theory predicts F–Θ curve well



$\Theta = \mu a/2 = 2I/aB_0$

created externally

need $\mu a > 2.4$

$\Theta > 1.2$

$F = B_{z\,wall} / \langle B \rangle$

pretty good . . . —

<u>N.B.</u> An unpleasant reality:

- Relaxation ⟺ stoch/turb.
- stoch/turb → losses.

<u>i.e.</u>

$$\int_{V<n} d^3x\, \eta J^2 = 2\pi r R Q$$

$$Q = ? \quad \text{i.e.} \quad \sim v_T \tilde{b}^2 l_{ac} \; ?$$

$$\sim \frac{\chi_{\parallel}}{L} D_M \; ?$$

$$\begin{tikzcd} \tilde{b}^2 \arrow[r, dash] \arrow[dr] & \underline{\text{relaxation}} \\ & \underline{\text{losses}} \end{tikzcd}$$

∴ heat flux driven dynamo ...

II.) Dynamics of Taylor Relaxation.

ⓐ → How represent dynamics of relaxation?

How does system evolve to Taylor state?

(general)

ⓑ → How does RFP drive poloidal currents which produce reversed toroidal field (specific)

ⓒ → How relate to more general concepts of relaxation, dynamo? — Self-organized criticality...

ⓐ - ⓒ ⇒ Mean Field Electrodynamics

i.e. how calculate $\langle \tilde{v} \times \tilde{B} \rangle$

⇒ goal is turbulence driven EMF

→ akin $\langle E \delta f \rangle$ in QLT

→ issues: structure, symmetry

→ origin of irreversibility

conservation properties

→ topic is fundamental to subject of dynamo theory

→ flow counterpart: ~~Zonal flow generation~~

~~(Monday Lecture)~~

Good resource:

$$\boxed{\text{www.igf.edu.pl/KB/HKM}}$$

items 28, 46

Keith Moffatt pubs.

ⓐ Structural / symmetry Argument
Approach I (Boozer '86)

Write Ohm's Law (Mean Field) in form:

$$\langle \underline{E}\rangle + \underbrace{\cancel{\langle \underline{v}\rangle \times \langle \underline{B}\rangle}}_{\text{hereafter ignore}} = \underbrace{\langle \underline{S}\rangle}_{\substack{\text{un-resolved}\\ \text{EMF} \to \text{"something"}}} + \eta \langle \underline{J}\rangle$$

some unspecified operator.

What is $\langle S\rangle$?

Taylor → i) $\underline{S}$ must $\underline{\underline{\text{not}}}$ dissipate H_M (conserve H_M)

ii) $\underline{S}$ must dissipate E_M.

now,

$$\partial_t \int d^3x \, \langle \underline{A}\rangle \cdot \langle \underline{B}\rangle = \partial_t \int d^3x \left[\langle \underline{A}\rangle \cdot \langle \underline{\nabla} \times \underline{A}\rangle\right]$$

$$= -2c\int d^3x \left[\left(\langle \underline{E}\rangle + \langle \underline{\nabla}\phi\rangle\right) \cdot \langle \underline{B}\rangle\right]$$

$$= -2c\int d^3x \left[\langle \underline{E}\rangle \cdot \langle \underline{B}\rangle\right]$$

$\int \underline{B} \cdot \underline{\nabla}\phi = 0$ to S.T.

now

$$= -2c\int d^3x \left[\langle S\rangle \cdot \langle \underline{B}\rangle + \eta \langle \underline{J}\rangle \cdot \langle \underline{B}\rangle\right]$$

$$\partial_t \int d^3x \, \langle A\rangle \cdot \langle \underline{B} \rangle = -2c\eta \int d^3x \, \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle$$
$$-2c \int d^3x \, \langle \underline{B} \rangle \cdot \langle \underline{\mathcal{E}} \rangle$$

Now, to conserve H_M, 2nd term must integrate to S.T., so:

drop $\langle \ \rangle$

$$\langle \underline{\mathcal{E}} \rangle = \frac{\underline{B}}{B^2} \, \underline{\nabla} \cdot \underline{\Gamma}_H$$

↳ flux, driving helicity evolution

For form $\underline{\Gamma}_H$, consider energy:

$$\partial_t \int d^3x \, \frac{B^2}{8\pi} = \int d^3x \, \frac{\underline{B}}{4\pi} \cdot \partial_t \underline{B}$$
$$= -\int d^3x \, \frac{\underline{B}}{4\pi} \cdot c \, \underline{\nabla} \times \underline{E}$$
$$= -\int d^3x \, \underline{E} \cdot \underline{J}$$
$$= -\int d^3x \left[\eta J^2 + \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right) \underline{\nabla} \cdot \underline{\Gamma}_H \right]$$
$$= -\int d^3x \left[\eta J^2 - \underbrace{\underline{\Gamma}_H}_{\text{flux}} \cdot \underbrace{\underline{\nabla} \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right)}_{\text{force}} \right]$$

i.e. $\underset{\text{(entropy)}}{\dfrac{dS}{dt}} = \alpha \left(-\underline{\nabla} T \cdot \underline{\Gamma}_T \right) = \alpha D_T (\underline{\nabla} T)^2$, general form.

apart η,

$$\partial_t E_M = \int d^3x\, \underline{\Gamma}_H \cdot \nabla(J_\parallel/B)$$

so $\Gamma_H = -\lambda \nabla(J_\parallel/B)$ <u>assures</u>

$$\partial_t E_M = -\int d^3x\, \lambda \left[\nabla(J_\parallel/B)\right]^2$$

and:

$$\boxed{\langle \underline{E} \rangle = \eta \langle \underline{J} \rangle - \frac{\underline{B}}{B^2}\, \underline{\nabla} \cdot \left[+\lambda \nabla\left(\frac{\underline{J} \cdot \underline{B}}{B^2}\right)\right]}$$

simplified form:

$$\boxed{\langle E_\parallel \rangle = \eta J_\parallel - \nabla_\perp \cdot \lambda \nabla_\perp J_\parallel}$$

diffusion of current.

<u>$\lambda \equiv$ 'hyper-resistivity', 'electron viscosity'</u>

structurally:

simpl. Reason.

$$\lambda = \frac{c^2}{\omega_{pe}^2} \underbrace{D_J}_{\text{diffusivity}} \quad , \quad \text{as} \quad \eta \equiv \frac{c^2}{\omega_{pe}^2} \nu_{ei}$$

$\lambda \equiv \mu$.

$D_J \to$ MHD

$\to$ multi-fluid

$\to$ extended stochastic field argument

→ Exercises :

~→ δ-P reconnection, with $E_\parallel = -\mu \nabla_\perp^2 J_\parallel$?

$$V_R/V_A = 1/(S_\mu)^{1/4} \qquad S_\mu = \frac{V_A L^3}{\mu} \qquad (J\,!)$$

$$1/S \to \mu/V_A L^3$$

~→ derive structure of D_J for ensemble stochastic fields

(i.e. shifted electron Maxwellian → $J_\parallel(\underline{x})$ ...).

→ Compare D_J to χ_e for various turbulence models.

In MHD:

– as seek $\langle E_\parallel \rangle$, and concerned with locally strong field

$$\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \mu \underline{J}\right)\cdot \underline{B}/|B|$$

$$\Rightarrow \boxed{-\frac{1}{c}\partial_t A_\parallel - \underline{n}\cdot\nabla\phi - \underline{\nabla A_\parallel}\times\hat{n}\cdot\nabla\phi = \mu J_\parallel}$$

here $\hat{n} = \underline{B}/|B|$ $\qquad \underline{B_0 \nabla_\parallel \phi}$

then for mean field:

$$-\frac{1}{c}\partial_t\langle A\rangle + \partial_r\left[\underset{\uparrow}{\langle v_\perp\tilde{\phi}\tilde{A}_\parallel\rangle}\right] = \eta\langle J_\parallel\rangle$$

flctn. induced EMF

- note naturally in flux form.

- $$\langle v_\perp\tilde{\phi}\tilde{A}_\parallel\rangle \cong \underset{\uparrow}{\langle v_\perp\tilde{\phi}\,\delta A_\parallel\rangle} + \underset{\uparrow}{\langle \tilde{A}_\parallel \nabla_\perp \delta\phi\rangle}$$

iterate Ohm's Law ①

iterate vorticity eqn. ②

i.e.

$$\partial_t \delta A_{\parallel_{\underline{k}}} + \underset{\text{turbulent mixing}}{\Delta\omega_{\underline{k}}\,\delta A_{\parallel_{\underline{k}}}} = \underset{\text{bending}}{ik_\parallel\,\delta\phi_{\underline{k}}} - \underset{\text{resistive dissipn.}}{\eta k_\perp^2\,\delta A_{\parallel_{\underline{k}}}}$$

①

$$\langle v_\perp\tilde{\phi}\,\delta A_\parallel\rangle = \sum_{\underline{k}} k_\perp k_\parallel |\tilde{\phi}_{\underline{k}}|^2 \frac{(\Delta\omega_{\underline{k}} + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2}$$

→ in pure QLT, irreversibility from resistive diffusion, only. → can be slow unless $k_\perp^2$ large

→ if undid normalizations,

$$\langle v_\perp\phi\,\delta A_\parallel\rangle = \alpha\langle B\rangle$$ → alpha effect

α = above formula.

i.e. $\underbrace{k_\perp k_\parallel}$ → motion has handed-ness i.e. $\underline{x} \to -\underline{x} \Rightarrow \alpha \to -\alpha$

$$k_\perp k_\parallel = \frac{k_\perp^2 x}{L_s}$$ ✓

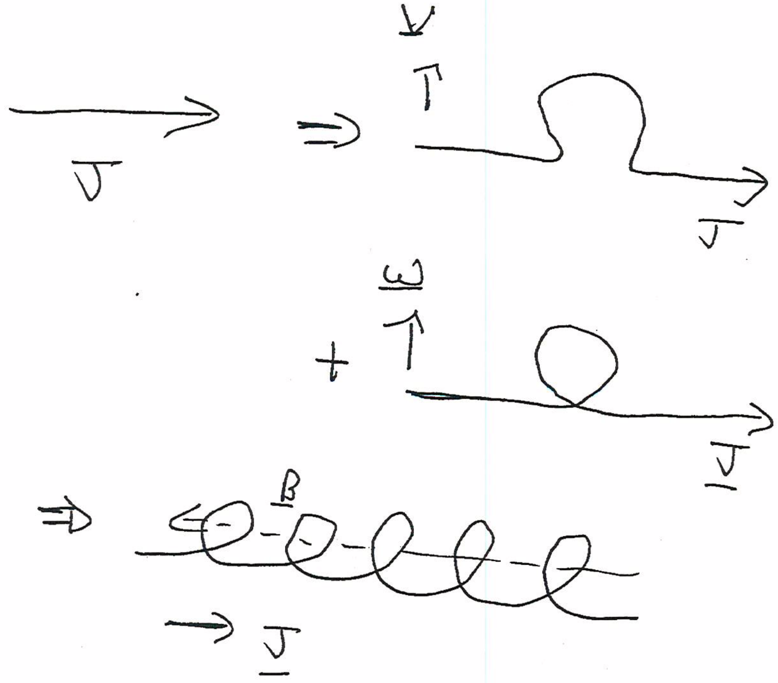
$$\Rightarrow \frac{\partial \langle A_\parallel \rangle}{\partial t} = \alpha \langle B \rangle$$

$$\frac{\partial}{\partial t} \langle B \rangle = \alpha \langle J \rangle$$

i.e. how generate a field parallel/anti-parallel to a current?

aside

(Parker)



need $\langle \tilde{\underline{V}} \cdot \tilde{\underline{\omega}} \rangle \neq 0 \to$ fluctuations have net helicity

Here $\langle \nabla_\perp \tilde{\phi} \nabla_\parallel \tilde{\phi} \rangle$ is magnetized analogue of handedness.

but also ...

$$② = -\langle \nabla \tilde{A}_\parallel \delta\phi \rangle$$

vorticity eqn:

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\tilde{\phi}$$

$$= \frac{\tilde{B}_r}{B_0}\frac{\partial\langle J_\parallel\rangle}{\partial r} + \nabla_\parallel \tilde{J}_\parallel + \underline{\tilde{B}}\cdot\underline{\nabla}\tilde{J} + \mu\nabla^2\nabla^2\phi$$

$$\partial_t\left(-k_\perp^2\tilde{\phi}_{\underline{k}}\right) + \Delta\omega_{\underline{k}}\left(-k_\perp^2\tilde{\phi}_{\underline{k}}\right)$$

$$= \underset{\overset{\downarrow}{b}}{\frac{\tilde{B}_{r\underline{k}}}{B_0}\frac{\partial\langle J_\parallel\rangle}{\partial r}} + \underset{\overset{\downarrow}{a}}{ik_\parallel \hat{A}_{\parallel\underline{k}}\left(-k_\perp^2\right)} + \mu\left(k_\perp^2\right)^2\tilde{\phi}_{\underline{k}}$$

$$\tilde{\phi}_{\underline{k}} = \frac{-\frac{\tilde{B}_{r\underline{k}}}{B_0 k_\perp^2}\frac{\partial\langle J_\parallel\rangle}{\partial r} + ik_\parallel\hat{A}_{\parallel\underline{k}}}{\left(-i\omega + \Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}$$

$$②_{ⓐ} = -\sum_{\underline{k}} \frac{k_\perp k_\parallel |\tilde{A}_{\underline{k}}|^2\left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}{\omega^2 + \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)^2}$$

- magnetic α effect
- opposite in sign to ①

$$②_{b} = \sum_{k} \frac{|\nabla_\perp \tilde{A}_{\parallel k}|^2}{B_0^2 k_\perp^2} \frac{(\Delta\omega_k + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_k + \eta k_\perp^2)^2} - \frac{\partial\langle J_\parallel\rangle}{\partial r}$$

→ clearly corresponds to hyper-η.

i.e.

$$-\frac{1}{c}\frac{\partial\langle A_\parallel\rangle}{\partial t} + \partial_r \langle(\nabla_\perp\tilde{\phi})\tilde{A}_\parallel\rangle = \eta\langle J_\parallel\rangle$$

$$\langle(\nabla_\perp\tilde{\phi})\tilde{A}_\parallel\rangle \equiv \sum_{k} k_\perp k_\parallel \left\{ \overset{\alpha_K}{|\phi_k|^2 L_k^\eta} - \overset{\alpha_M}{|A_{\parallel k}|^2 L_k^\eta} \right\}$$

$$\left[L^\eta = (\Delta\omega_k + \eta k_\perp^2) \Big/ \omega^2 + (\Delta\omega_k + \eta k_\perp^2)^2 \right]$$

$$+ \sum_{k} \left|\frac{\tilde{B}_{r k}}{B_0}\right|^2 \frac{L_k^\eta}{k_\perp^2} \frac{\partial\langle J_\parallel\rangle}{\partial r}$$

hyper-resistivity

N.B.

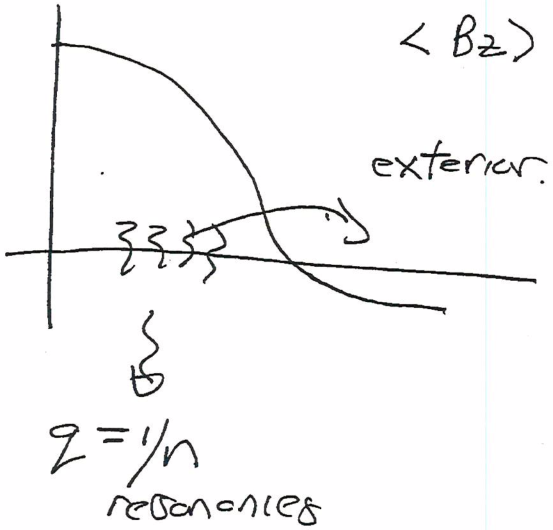
- α's both come from bending
- α_K, α_M opposite sign.
- α's from MHD exterior,

$$\tilde{A}_\parallel \rightarrow \frac{k_\parallel \tilde{\phi}_k}{\omega + i\omega}$$

- hyper-η from @ resonance
  i.e. where vorticity driven.
  $\Rightarrow$ reconnection process site.

- hyper-η tied to basic tearing drive
- α_M + hyper η cancel in exterior
  V_K survive in exterior, vanish near Res. surf
- note total EMF encompasses more than hyper-η ......

ⓑ RFP



$$\left.\begin{array}{l} q < 1 \\ q' < 0 \end{array}\right\} \Rightarrow \text{K-S unstable}$$

$m = 1$ paradise
(global tearing turbulence)

so to compute induced EMF, seek
$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle \hat{\theta}$ in exterior.

$v = \partial_t \tilde{\underline{\xi}}$
↳ displacement

$$\tilde{\underline{B}} = \nabla\times\underline{\xi}\times\langle\underline{B}\rangle$$

$$= -\hat{\xi}\cdot\nabla\langle B\rangle + \langle B\rangle\cdot\nabla\hat{\underline{\xi}} - \langle B\rangle\,\cancel{\underline{\nabla}\cdot\underline{\xi}}$$

field advection irrelevant

kink incompressible

i.e. bending is key.

$$\tilde{\underline{B}} \cong \langle B\rangle\cdot\nabla\tilde{\underline{\xi}}$$

$$\langle\tilde{\underline{v}}\times\tilde{\underline{B}}\rangle = \sum_{\underline{k}}\gamma_{\underline{k}}\langle\tilde{\xi}_{-\underline{k}}\times\tilde{B}_{\underline{k}}\rangle$$

$$= \sum_{\underline{k}}\gamma_{\underline{k}}\,\tilde{\xi}_{-\underline{k}}\times ik_{\parallel}\langle B\rangle_{\theta}\,\tilde{\xi}_{\underline{k}}$$

→ field primarily poloidal near B_z reversal region.

$$\underline{\nabla}\cdot\underline{\xi} = 0 \Rightarrow \frac{\partial_r\tilde{\xi}_r + ik_\theta\tilde{\xi}_\theta}{ik_z} = \tilde{\xi}_z$$

then

$$\langle\tilde{\underline{v}}\times\tilde{\underline{B}}\rangle_\theta = \sum_{\underline{k}}\gamma_{\underline{k}}\,ik_\parallel\langle B_\theta\rangle\left[\tilde{\xi}_z^*\tilde{\xi}_x - \xi_x^*\xi_z\right]$$

$$= \sum_{\underline{k}}\frac{\gamma_{\underline{k}}\,\cancel{i}k_\parallel\langle B_\theta\rangle}{-\cancel{i}k_z}\,(M)$$

$$M = +\left(\partial_r \tilde{\mathcal{E}}_r^* - i k_\theta \tilde{\mathcal{E}}_\theta^*\right)\tilde{\mathcal{E}}_r \quad -$$

$$+ \mathcal{E}_r^*\left(\partial_r \mathcal{E}_r + i k_\theta \tilde{\mathcal{E}}_\theta\right)$$

$$M = + \partial_r |\tilde{\mathcal{E}}_r|^2 + \cancel{i k_\theta\left(\tilde{\mathcal{E}}_\theta^* \mathcal{E}_r - \mathcal{E}_r^* \mathcal{E}_\theta\right)}$$

but $\tilde{\mathcal{E}}_r\Big|_{wall} = 0$

$r_{rev} \sim a \Rightarrow \partial_r \gg k_\theta$

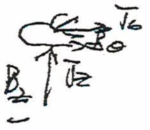
$$\boxed{\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle_\theta = +\sum_{\underline{k}} \gamma_{\underline{k}} \frac{k_\parallel}{k_z} \langle B_\theta \rangle \partial_r |\tilde{\mathcal{E}}_{\underline{k}}|^2}$$

$$\rightarrow k_\parallel / k_z = \left(\frac{m}{r} B_\theta - \frac{n}{R} B_z\right) / B_\theta$$

$$\cong \frac{m}{r} - \frac{n}{r} q(r)$$

$$= 1/r\left(m - n q(r)\right)$$

$$k_z = n/R$$



so $\bar{J}_\theta \parallel B_\theta$ at r_{res}

$$k_\parallel / k_z = (R/r)(m/n) - \frac{R}{r} q(r)$$

$$= (R/r)\left(q_{res} - q(r)\right)$$

$$\langle \tilde{v} \times \tilde{B} \rangle = -\sum_{n} |\gamma_n| \frac{R}{r} \left(q_{res} - q(r)\right) \langle B_\theta \rangle \, \partial_r |\tilde{\xi}_n|^2$$

→ $\partial_r |\tilde{\xi}_n|^2 < 0$

→ γ_n → irreversibility (!)

→ $q_{res} - q(r)$ ⇒ < 0 on axis

> 0 at r_{rev}

so

$$\langle E \rangle + \overset{> 0}{\langle \tilde{v} \times \tilde{B} \rangle} = \eta \langle J_\theta \rangle$$

$$\therefore \quad \langle J_\theta \rangle \cong \frac{1}{\eta} \langle \tilde{v} \times \tilde{B} \rangle_\theta$$

⇒ $\langle B_z \rangle < 0$ → kinks drive reversal.

But what about irreversibility and/or locking in?

S-T-F-R

$1 \curvearrowright 0/1$ | [illegible]

$1/n \quad 1/n+1 \rightarrow \frac{2}{2n+1}$

$\searrow 0/1$

$1/n+1, \; 0/1 \rightarrow 1/n+2$

$m=0$ <u>driven</u> ⇒ reconnection

→ lock in.

Outline

i.) Preamble: → From Reconnection to Relaxation and Self-Organization

- → What 'Self-Organization' means
- → Why Principles are important
- → Examples of turbulent self-organization
- →Preview

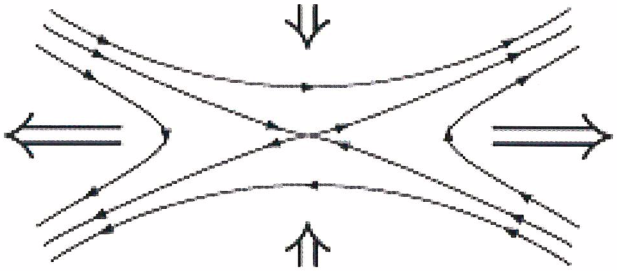
ii.) Focus I: Relaxation in R.F.P. (J.B. Taylor)

- →RFP relaxation, pre-Taylor
- →Taylor Theory
  - - Summary
  - - Physics of helicity constraint + hypothesis
  - - Outcome and Shortcomings
- → Dynamics → Mean Field Theory
  - - Theoretical Perspective
  - - Pinch's Perspective
  - - Some open issues
- → Lessons Learned and Unanswered Questions

I.) Preamble

→ From Reconnection to Relaxation

- Usually envision as localized event involving irreversibility, dissipation etc. at a singularity



S.-P.

$$V = V_A/Rm^{1/2}$$

- ??? - how describe global dynamics of relaxation and self-organization

→

- multiple, interacting/overlapping reconnection events

    → turbulence, stochastic lines, etc

Examples of Self-Organization Principles

→ Turbulent Pipe Flow: (Prandtl → She)

$$\sigma = -\nu_T \frac{\partial \langle v_y \rangle}{\partial x} \qquad \nu_T \sim v_* x \qquad \Rightarrow \langle v_y \rangle \sim v_* \ln x$$

Streamwise Momentum undergoes scale invariant mixing

→ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc)

(Focus 1) Minimize E_M at conserved global H_M $\Rightarrow$ Force-Free RFP profiles

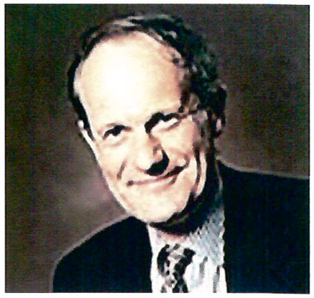
→ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

(Focus 2)
- → PV tends to mix and homogenize
- → Flow structures emergent from selective decay of potential enstrophy relative energy

→ Shakura-Sunyaev Accretion

→ disk accretion enabled by outward viscous angular momentum flux

II.) Focus I - Magnetic Relaxation



(Derek C Robinson)

→ Prototype of RFP's: Zeta (UK: late 50's - early 60's)

- toroidal pinch = vessel + gas + transformer
- initial results → violent macro-instability, short life time
- weak B_T → stabilized pinch ↔ sausage instability eliminated
- $I_p > Ip, crit$ ($\theta > 1+$) → access to "Quiescent Period"

→ Properties of Quiescent Period:

- macrostability - reduced fluctuations
- $\tau_E \sim 1\ msec$ $T_e \sim 150eV$
- $B_T(a) < 0$ → reversal

→ Quiescent Period is origin of RFP

Further Developments

- Fluctuation studies:

turbulence = $m = 1$ kink-tearing → tend toward force-free state

resistive interchange, ...

- Force-Free Bessel Function Model

$$B_\theta = B_0 J_1(\mu r) \qquad B_z = B_0 J_0(\mu r)$$

$$\mathbf{J} = \alpha \mathbf{B}$$

observed to correlate well with observed B structure

- L. Woltjer (1958) : Force-Free Fields at constant α

→ follows from minimized E_M at conserved $\int d^3x \mathbf{A} \cdot \mathbf{B}$

- steady, albeit modest, improvement in RFP performance, operational space

→ Needed: Unifying Principle

Theory of Turbulent Relaxation (J.B. Taylor, 1974)

→ hypothesize that relaxed state minimizes magnetic energy subject to constant global magnetic helicity

i.e. profiles follow from: $\delta \left[\int d^3x \frac{B^2}{8\pi} + \lambda \int d^3x \mathbf{A} \cdot \mathbf{B} \right] = 0$

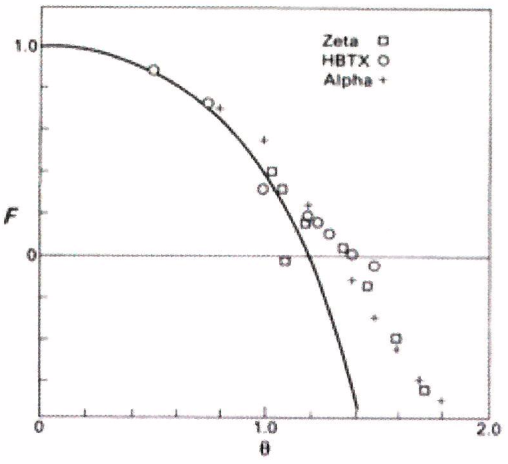
$$\Rightarrow \nabla \times \mathbf{B} = \mu \mathbf{B} \;\; ; \;\; J_{\|}/B = \frac{\mathbf{J} \cdot \mathbf{B}}{B^2} = const$$

Taylor state is:

- force free
- flat/homogenized $J_{\|}/B$
- recovers BFM, with reversal for $\theta = \frac{2I_p}{aB_0} > 1.2$

- Works amazingly well

Result:



$$\theta = \mu a/2 = \frac{2I_p}{aB_0}$$

$$F = B_{z,wall}/\langle B\rangle$$

and numerous other success stories

→ Questions:

- what is magnetic helicity and what does it mean?
- why only global magnetic helicity as constraint?
- Theory predicts end state → what can be said about dynamics?
- What does the pinch say about dynamics?

→ Central Issue: Origin of Irreversibility

Why Global helicity, Only?

- in ideal plasma, helicity conserved for each line, tube

i.e. $\mathbf{J} = \mu(\alpha, \beta)\mathbf{B}$ $\mu(\alpha', \beta') \neq \mu(\alpha, \beta)$

- Turbulent mixing eradicates identity of individual flux tubes, lines!

i.e.

- if turbulence s/t field lines stochastic, then '1field line' fills pinch.

1 line ⟷ 1tube → only global helicity meaningful.

- in turbulent resistive plasma, reconnection occurs on all scales, but: $\tau_R \sim l^\alpha$ $\alpha > 0$

($\alpha = 3/2$ for S-P reconnection)

Thus larger tubes persist longer. Global flux tube most robust

- selective decay: absolute equilibrium stat. mech. suggests possibility of inverse cascade of magnetic helicity (Frisch '75) → large scale helicity most rugged.

Comments and Caveats

→ Taylor's conjecture that global helicity is most rugged invariant remains a conjecture

→ unproven in any rigorous sense

→ many attempts to expand/supplement the Taylor conjecture have had little lasting impact (apologies to some present....)

→ Most plausible argument for global H_M is stochastization of field lines → forces confinement penalty. No free lunch!

→ Bottom Line:

- Taylor theory, simple and successful
- but, no dynamical insight!

Dynamics I:

- The question of Dynamics brings us to mean field theory (c.f. Moffat '78 and an infinity of others - see D. Hughes, Thursday Lecture)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$?

  → how relate to relaxation?

- Caveat: - MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

  - MFT is often very useful, but often fails miserably

- Structural Approach (Boozer): (plasma frame)

$$\langle \mathbf{E} \rangle = \eta \langle \mathbf{J} \rangle + \langle \mathbf{S} \rangle$$

→ something → related to $\langle \tilde{v} \times \tilde{B} \rangle$

$\langle \mathbf{S} \rangle$ conserves H_M

$\langle \mathbf{S} \rangle$ dissipates E_M

Note this is ad-hoc, forcing $\langle \mathbf{S} \rangle$ to fit the conjecture. Not systematic, in sense of perturbation theory

Now

$$\partial_t H_M = -2c\eta \int d^3x \langle \mathbf{J} \cdot \mathbf{B} \rangle - 2c \int d^3x \langle \mathbf{S} \cdot \mathbf{B} \rangle$$

$$\therefore \langle \mathbf{S} \rangle = \frac{\mathbf{B}}{B^2} \nabla \cdot \mathbf{\Gamma}_H$$

Conservation H_M ➔ $\langle S \rangle \sim \nabla \cdot$ (Helicity flux)

$$\partial_t \int d^3x \frac{B^2}{8\pi} = - \int d^3x \left[\eta J^2 - \mathbf{\Gamma}_H \cdot \nabla \frac{\langle \mathbf{J} \rangle \cdot \mathbf{B}}{B^2} \right]$$

so

$\mathbf{\Gamma}_H = -\lambda \nabla (J_{\|}/B)$, to dissipate E_M

→ simplest form consistent with Taylor hypothesis

→ turbulent hyper-resistivity $\lambda = \lambda[\langle \tilde{B}^2 \rangle]$ - can derive from QLT

→ Relaxed state: $\nabla (J_{\|}/B) \to 0$ homogenized current → flux vanishes

Dynamics II: The Pinch's Perspective

- Boozer model not based on fluctuation structure, dynamics

- Aspects of hyper-resistivity do enter, but so do other effects

  → Point: Dominant fluctuations controlling relaxation are m=1 tearing modes resonant in core → global structure

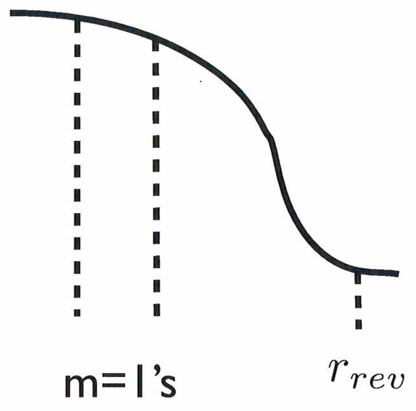
  → Issue: What drives reversal B_z near boundary?

Approach: QL $\langle \tilde{v} \times \tilde{B} \rangle$ in MHD exterior - exercise: derive!

$$\langle \tilde{\mathbf{v}} \times \tilde{\mathbf{B}} \rangle \cong \sum_k |\gamma_k| \frac{R}{r} (q_{res} - q(r)) \langle B_\theta \rangle \partial_r (|\tilde{\xi}_r|_k^2)$$

i.e. $\langle J_\theta \rangle$ driven opposite $\langle B_\theta \rangle$ → drives/sustains reversal

→ What of irreversibility - i.e. how is kink-driven reversal 'locked-in'?



→ drive $J_{\|}/B$ flattening, so higher n's destabilized by relaxation front

→ global scattering → propagating reconnection front

m=1, n   m=1, n+1 → m=0, n=1 → driven current sheet, at r_{rev}

(difference beat)

sum beat $\left\{ \begin{array}{l} \text{m=2,} \\ \text{2n+1} \end{array} \right.$

but then m=1, n+2 driven → tearing activity, and relaxation region, broadens

→ Bottom Line: How Pinch 'Taylors itself' remains unclear, in detail

Summary of Magnetic Relaxation

concept: topology

process: stochastization of fields, turbulent reconnection

constraint released: local helicity

players: tearing modes

Mean Field: EMF = $\langle \tilde{v} \times \tilde{B} \rangle$

Global Constraint: $\int d^3x \mathbf{A} \cdot \mathbf{B}$

NL: Helicity Density Flux

Outcome: B-Profile

Shortcoming: Rates, confinement → turbulent transport

Introduction to Mean Field

Electrodynamics

Mean Field Electrodynamics - A Brief Introduction

→ discussion of relaxation ⇒

$$-\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \frac{\langle \underline{B} \rangle}{\langle B \rangle^2} \, \underline{\nabla} \cdot \vec{\Gamma}_H \, ; \quad \vec{\Gamma}_H = -\chi D \left(\langle J_{\parallel} \rangle / \langle B \rangle \right)$$

for consistency with Taylor hypothesis.

⇒ but, how calculate $\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle$ - i.e. what form does mean field EMF actually have?

→ problem in mean field electrodynamics (i.e. closure, akin Q.L.T.).

→ some simple cases:

- fluid turbulence + weak $\langle \underline{B_0} \rangle$
- $R_m < 1$.

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \langle \underline{v} \times \underset{\uparrow}{\tilde{\underline{B}}^{(1)}} \rangle$$

response of
$\tilde{\underline{B}}$ to $\tilde{\underline{v}}$, in
presence $\langle \underline{B_0} \rangle$

then

$$\partial_t \tilde{\underline{B}} + \overset{\overset{\text{NL} \to \Delta\omega_k}{\downarrow}}{\underline{\nabla} \times \tilde{\underline{v}} \times \tilde{\underline{B}}} - \overset{\overset{\text{resistive diffn}, \eta k^2}{\downarrow}}{\eta \nabla^2 \tilde{\underline{B}}}$$

$$= \langle \underline{B} \rangle \cdot \underline{\nabla} \tilde{\underline{V}} - \tilde{\underline{V}} \cdot \underline{\nabla} \langle B \rangle$$

$$\therefore \quad (-i\omega + \eta k^2) \tilde{B}_{\underline{k},\omega} = \underset{\underset{\text{bending}}{\downarrow}}{i \underline{k} \cdot \langle \underline{B} \rangle} \tilde{V}_{\underline{k},\omega} - \underset{\underset{\text{field advection}}{\downarrow}}{\tilde{\underline{V}}_{\underline{k},\omega} \cdot \nabla \langle B \rangle}$$

$$\tilde{B}_{k,\omega} = \frac{i \underline{k} \cdot \langle \underline{B} \rangle \tilde{V}_{\underline{k},\omega} - \tilde{V}_{\underline{k},\omega} \cdot \underline{\nabla} \langle B \rangle}{-i\omega + \eta k^2}$$

So

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \sum_{\underline{k},\omega} \tilde{V}_{-\underline{k},-\omega} \times \frac{\left[\overset{①}{i \underline{k} \cdot \langle \underline{B} \rangle \tilde{V}_{\underline{k},\omega}} - \overset{②}{\tilde{V}_{\underline{k},\omega} \cdot \nabla \langle B \rangle} \right]}{-i\omega + \eta k^2}$$

② → even in $\underline{k}$

⇒ advection of ⟨B⟩

i.e. turbulent resistivity

① → odd in $\underline{k}$ ↔ bending → kink,
symmetry breaking, for contributing → physics ?
→ ?

N.B.: In both cases, irreversibility provided by resistive diffusion ⇒ otherwise difficulty

For isotropic velocity spectrum:

$$\langle \tilde{v}_i(\underline{k},\omega)\,\tilde{v}_j^*(\underline{k}',\omega')\rangle = \delta(\underline{k}-\underline{k}')\,\delta(\omega-\omega')\,\Phi_{ij}(\underline{k},\omega)$$

$$\Phi_{i,j}(k,\omega) = \overset{①}{\frac{E(k,\omega)}{4\pi k^2}\left(k^2\delta_{ij}-k_ik_j\right)}$$

$$\overset{②}{+\frac{iF(k,\omega)}{8\pi k^2}\,\epsilon_{ijk}\,k_k}$$

① → energy density, even power
→ $\underline{\nabla}\cdot\underline{v}=0$

② Now,

$$F(k,\omega) = i\int \epsilon_{ik\ell}\,k_k\,\Phi_{i,\ell}(k,\omega)\,dS_{\underline{k}}$$

<u>so</u>

$$\langle \underline{v}\cdot\underline{\omega}\rangle = i\epsilon_{i,k,l}\iint k_k\,\Phi_{j,l}(k,\omega)\,d\underline{k},d\omega$$

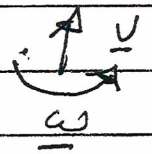
$\Downarrow$

spectral helicity $= \iint dk\,d\omega\,F(k,\omega)$

⇒

② $\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

⇒ turbulence helicity (mean projection $\underline{v}$ on $\underline{\omega}$)



so after some crank (see Moffat: available free, online):

$$\langle \tilde{\underline{v}}\times\tilde{\underline{B}}\rangle = \alpha\langle\underline{B}\rangle - \beta\langle\underline{J}\rangle$$

$$\alpha \equiv -\frac{1}{3}\eta\iint dk\,d\omega\,\frac{k^2F(k,\omega)}{\omega^2+(\eta k^2)^2}$$

⇒ α is weighted integral of <u>helicity spectrum</u>

<u>i.e.</u>

$\sim \langle \underline{v}\cdot\underline{\omega}\rangle$

$$\beta = \frac{2}{3}\eta \iint dk\, d\omega \, \frac{k^2 E(k,\omega)}{\omega^2 + (\eta k^2)^2}$$

⇒ β is weighted integral of energy spectrum

i.e.

$\sim \langle \tilde{v}^2 \rangle$

So

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \underline{\nabla} \times \underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}}$$

$$\underbrace{\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle}_{\text{mean EMF}} = \underbrace{\alpha \langle \underline{B} \rangle}_{\alpha\text{-effect}} - \underbrace{\beta \langle \underline{J} \rangle}_{\beta\text{-effect}}$$

$\therefore$

$$\frac{\partial \langle \underline{B} \rangle}{\partial t} - \eta \nabla^2 \langle \underline{B} \rangle = \alpha \, \underline{\nabla} \times \langle \underline{B} \rangle + \beta \nabla^2 \langle \underline{B} \rangle$$

$\therefore$

→ β as turbulent resistivity ⇒ random advective mixing of $\langle B \rangle$.

→ α ?

Further interesting to note:

→ look for force-free condition fields;

$$\underline{\nabla} \times \langle \underline{B} \rangle = \lambda \langle \underline{B} \rangle$$

$$\partial_t \langle \underline{B} \rangle - (\eta + \beta) \nabla^2 \langle B \rangle = \alpha \lambda \underline{\langle B \rangle}$$

⇒

$$\gamma_B = \alpha \lambda - (\eta + \beta) \lambda^2$$

∴ α can amplify field → depending on λ (scale)

⇒ $\boxed{\text{dynamo}}$ ↔ via α-effect!

→ Physics:

α → helicity ↔ $\langle \underline{\hat{v}} \cdot \underline{\hat{\omega}} \rangle$

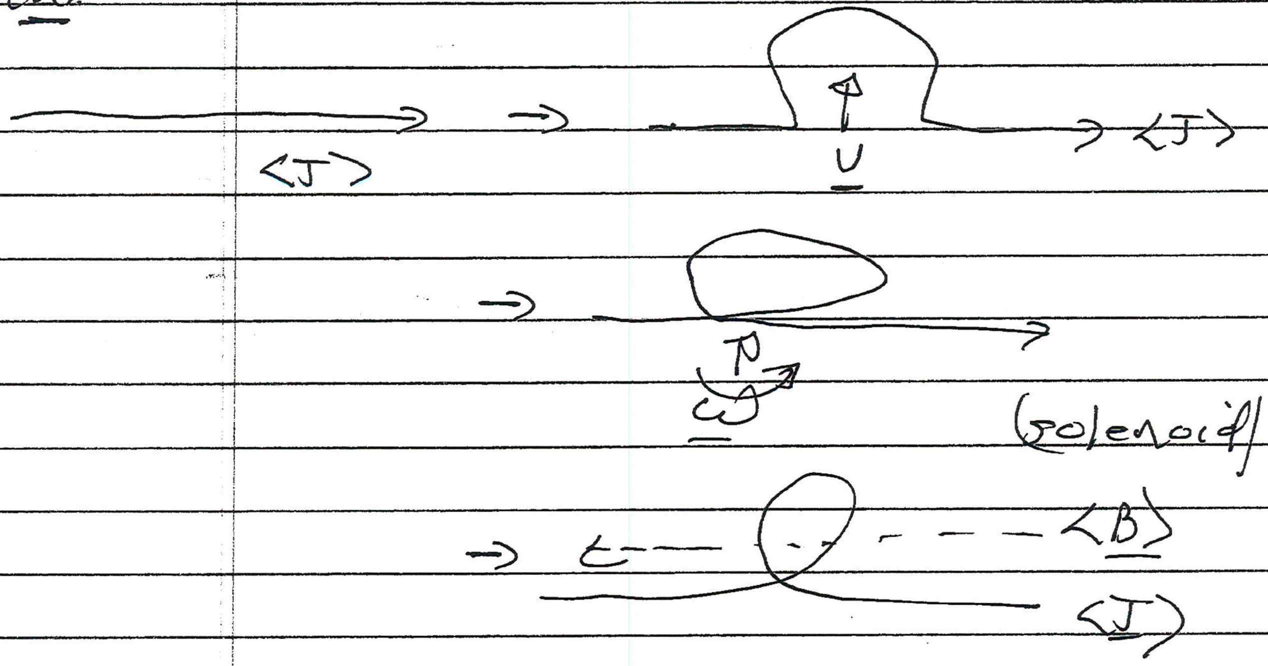
Then:

→ can stetch, twist, fold lines,

to amplify field.

→ α : $\langle J \rangle \rightarrow \langle \underline{B} \rangle$

i.e.



$$\langle \underline{B} \rangle \parallel \langle J \rangle .$$

repeat ⇒ amplify field.

i.e. $\gamma_B = \alpha \lambda - (\eta + \beta) \lambda^2$

$$d\gamma / d\lambda = \alpha - 2(\eta+\beta)\lambda = 0$$

$$\lambda_{\substack{\text{max} \\ \text{growth}}} = \alpha / 2(\eta + \beta)$$

$$\gamma_B\Big|_{max} = \alpha^2 / 4(\eta+\beta)$$

$$\not\uparrow$$

$\sim \alpha^2$ dynamo.

N.B:

→ locking-in (reconnection!) crucial!
  ⇒ role of η as cross-phase.
  ⇒ high Rm → problematic.

→ non-linearity, especially high Rm
  ⇒ a field in itself.

N.B.: Impact/role of magnetic helicity in dynamo control.