

# Basis of Nonlinear Plasma Theory

→ Introduction to Fluid Turbulence (NSE)

- ① → homogeneous - Kolmogorov → cascade → } Local Interaction
- ② → pipe flow - Prandtl → mixing -  $\chi$

→ Introduction to Wave Kinetics and Langmuir Turbulence

① → wave kinetics → Whitham variational formulation (phase symmetry)  
→ Boltzmann analogy

- ② → Eikonal Theory of Langmuir Turbulence } Non-Local Interaction
  - Zakharov ESNL + NLS
  - self-focusing + collapse

→ NL Wave-Particle Interaction - Intro

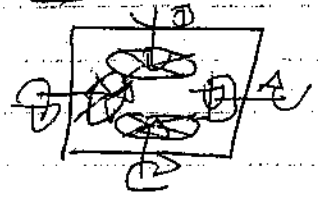
- ① → Rev QIT/Kubo # + validity
- ② → Higher order QIT
- ③ → Renormalization in Vlasov Turbulence

# Turbulence Theory

## - An Introduction

P. Diamond  
i.e.

### I.) Basics of Fluid Turbulence (30)



### Characteristics of Fluid Turbulence:

"turbulence" vs "noise" → energy flux

- broad range of spatio-temporal scales excited  
contrast TAM Ref. U. Frisch
- decay of large scale energy → need input/stirring to maintain stationarity - "The legacy of A. N. Kolmogorov"
- energy input dissipated as heat (to maintain stationarity) → viscosity  
 → irreversibility
- irreversible mixing occurs → i.e. passive tracer
- intermittency manifested  
 i.e. spatial → coherent structures (i.e. vortices)  
temporal → bursts un/pur/pur probe trace
- self-similarity / scale-similarity :  
 turbulence looks the same on all scales, except the very largest (stirring) and the very smallest (dissipation)  
 Caveat: Intermittency - memory of large scales on small.

2.2) Navier Stokes Equation - Describes Fluid

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{v}$$

$\rho = 1$   
here after

$\frac{\partial \underline{v}}{\partial t}$  → advection / straining → nonlinearity  
 $\frac{\nabla p}{\rho}$  → pressure  
 $\nu \nabla^2 \underline{v}$  → viscous diffusion of momentum

$$\nabla \cdot \underline{v} = 0 \quad \text{incompressibility}$$

Note: Pressure determined from incompressibility

c.e.

$$\nabla \cdot \left[ \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right] = -\nabla^2 p + \nu \nabla^2 (\nabla \cdot \underline{v})$$

$$\nabla^2 p = -\nabla \cdot \nabla \underline{v} \cdot \underline{v}$$

$$p = -\nabla^{-2} [\nabla \cdot \underline{v} \cdot \underline{v}]$$

$$= \frac{1}{4\pi D} \int \frac{d^3 x'}{|x-x'|} [\nabla' \cdot \underline{v}(x') \cdot \nabla' \underline{v}(x')]$$

More generally, can eliminate  $p$

$$\partial_t v_i + (\partial_{ij} - \partial_{ie} \nabla^{-2}) \partial_j (v_j v_e) = \nu \nabla^2 v_i$$

Key Parameter: Reynolds #

$$Re = |\underline{v} \cdot \underline{\nabla} \underline{v}| / |\nu \nabla^2 \underline{v}|$$

$$\sim \frac{V(L)L}{\nu}$$

$\sim$  nonlinearity  
collisional diffusion  
↓  
measure of strength  
of NL.

- Re usually referenced to largest scale

$L = L_{max}$   
 $V(L) =$  large-scale velocity

- Re always referenced to a particular scale

$$L_{max}, \lambda = \left[ \frac{\langle (\partial_i v_j)^2 \rangle}{\langle v_i^2 \rangle} \right]^{-1/2}, \quad \lambda_{disspn.}, \quad (Re=1)$$

(Taylor Scale)

-  $Re \gg 1$  in turbulent { pipe flow  
atmosphere  
etc

$$Re \sim 10^6 - 10^8, \text{ etc.}$$

i.e. planetary boundary layer:  $h_{out} \sim 1 \text{ km}$   
 $\sim 10^5 \text{ cm}$   
 $h_{diss} \sim 1 \text{ cm}$

$\Rightarrow$  6 decades!

- Re: measure of ratio of inertial  
mixing of momentum to collisional  
mixing...

④

## ii) Experimental 'Laws' of Fully Developed Turbulence

⇒ Much / most of turbulence theory is empirically motivated. Experimental info / results preceded sophisticated theoretical analyses....

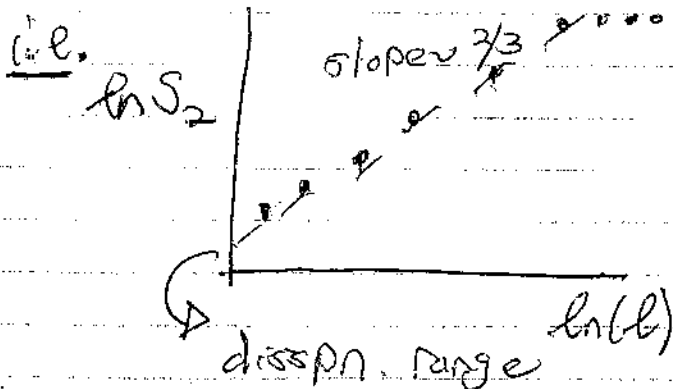
The experimental facts:

### 1.) 2/3 Law (Mundane)

In a turbulent flow with  $Re \gg 1$ ,  
 $\langle \delta v(l)^2 \rangle$  (mean square velocity increment  
 between two scales) separated by distance  
 $l$ ) scales as  $l^{2/3}$ .

i.e.  $\delta v(l) = |v(x+l) - v(x)| \Rightarrow$  {a difference!}

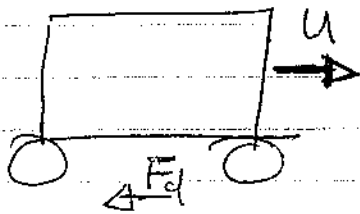
$S_2(l) = \langle \delta v(l)^2 \rangle \sim l^{2/3} \rightarrow$  Fundamental scaling relation  
 2<sup>nd</sup> order structure function



## B) Law of Finite Energy Dissipation (Profound)

If in an experiment on turbulent flow, all the control parameters are kept the same, except the viscosity, which is lowered as much as possible, the energy dissipation per unit mass  $dE/dt$  behaves in a way consistent with a finite limit.

- What means 'Energy Dissipation Rate'?



Consider a car, experiencing atmospheric drag

$$F_d = \frac{1}{2} C_D \rho S U^2$$

↓  
face surface area

c.e.  $\frac{\rho S U^2}{2}$

$$p = (\rho S U^2) U$$

→ momentum in air of slug:

$$M = \rho S U^2 \rightarrow \text{mass}$$

if assume air momentum completely transferred to car  $V=U$

$$\frac{dp_{\text{car}}}{dt} = F_d = \rho S U^2$$

$\frac{C_D}{2}(Re) \equiv$  drag coefficient (slowly varying function of  $Re$ , depends on shape, etc.)

$$\therefore \bar{F}_d = \frac{C_D}{2} \rho S U^2$$

Now, power dissipated by drag force

$$P_d = \bar{F}_d U$$

$$\Rightarrow P_d = \frac{C_D}{2} \rho S U^3$$

Energy dissipation rate  $E = P_d / \text{Mass}$   
(per volume)

$$= \frac{C_D}{2} \frac{U^3}{L}$$

also NS  $\Rightarrow \partial_t \langle v^2 \rangle \sim - \langle \underline{v} \cdot \nabla v^2 \rangle \sim \frac{U^3}{L}$

→ Why should we care?

Note, energy budget:

$$\frac{\partial v_i}{\partial t} + v_j \partial_j v_i - \nu \nabla^2 v_i = -\partial_i p$$

$$\partial_t \frac{V_i^2}{2} + \partial_j V_j \frac{V_i^2}{2} - \nu V_i \partial^2 V_i = -V_i \partial_i p$$

$\langle \rangle \equiv$  ensemble (fast space-time avg.)

$$\partial_t \left\langle \frac{V_i^2}{2} \right\rangle + \left\langle \partial_j V_j \frac{V_i^2}{2} \right\rangle - \nu \left\langle V_i \partial^2 V_i \right\rangle$$

surface terms  $= \left\langle \partial_i V_i p \right\rangle$   
upon IBP

$$\Rightarrow \partial_t \left\langle \frac{V_i^2}{2} \right\rangle = -\nu \left\langle |\partial V|^2 \right\rangle$$

but  $\epsilon = -\partial_t \left\langle \frac{V_i^2}{2} \right\rangle$  ! (- dissipation rate)

$$\epsilon = \nu \left\langle |\partial V|^2 \right\rangle$$

$\Rightarrow$  experiments suggest that  $\epsilon \rightarrow$  finite as  $\nu \rightarrow 0$  ! ! !  $\Leftrightarrow$  re-markable

$\Rightarrow$  suggests that extremely large  $\partial V$  forms as  $\nu \rightarrow 0$ , singular vortex sheets

$\Rightarrow$  singular velocity gradients, formed in limit of weak viscosity ! !

Heart of turbulence problem is grappling with singularity (especially its degree) of velocity gradients



No. B. : { Dissipation Law  
Singularity formation is at the heart of why turbulence is a "hard" problem.

Re: Dissipation Law:

$$\epsilon \sim \frac{U^3}{L} \sim U^2 / (L/U)$$

$$\sim \frac{\text{K.E. per Mass}}{\text{circulation Time}}$$

i.e.  $\rightarrow$  in 1 macro circulation time, a finite fraction of (macro) kinetic energy is dissipated by viscosity.

$\rightarrow$  dissipation time scale is  $(L/U)$ .

v.) Kolmogorov's Hypotheses and their Predictions / Implications.  $\rightarrow$  K41 Theory of Turbulence

1: In the limit of  $Re \rightarrow \infty$ , all possible symmetries of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.

lose memory

What means?

- "small scales":  $l \ll l_0$

integral scale  $\rightarrow$  characteristic of production

- symmetries

First, symmetries of Navier-Stokes Eqn.!

a.) space translations  $\Omega \rightarrow \Omega + \underline{a}$   
(no explicit  $\Omega$  dep.)

b.) time translation  $t \rightarrow t + \tau$   
(no  $t$  dep.)

\* c.) Galilean boosts  $\left\{ \begin{array}{l} \Omega \rightarrow \Omega + \underline{u}t \\ \underline{v} \rightarrow \underline{v} + \underline{u} \end{array} \right. \quad \underline{v} = \underline{u} + \underline{v}(\Omega - \underline{u}t)$   
(no frame dep.)

i.e.  $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$

insert  $\Rightarrow$

$$= \cancel{\underline{u} \cdot \nabla \underline{v}} + \cancel{\underline{u} \cdot \nabla \underline{v}} + \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nu \nabla^2 \underline{v}$$

d.) Parity (no preferred direction) left-right  $\underline{x} \rightarrow -\underline{x}, \underline{v} \rightarrow -\underline{v}$

e.) Rotation (no preferred direction)  $\left\{ \begin{array}{l} \Omega \rightarrow R \Omega \\ \underline{v} \rightarrow R \underline{v} \end{array} \right.$

\* e.) Scaling (for  $r \rightarrow 0$ )  $\Rightarrow$  critical: scale elimination; band passing

$$\underline{r}, \underline{v}, t \Rightarrow \lambda \underline{r}, \lambda^a \underline{v}, \lambda^b t$$

i.e.  $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p$

$$\underline{v} \Rightarrow \lambda^a \underline{v}$$

$$t \Rightarrow \lambda^b t$$

$$\frac{\lambda^a \partial \underline{v}}{\lambda^b \partial t} + \frac{\lambda^{2a}}{\lambda} \underline{v} \cdot \nabla \underline{v} = -\nabla p$$

$$\hookrightarrow \text{From } \nabla \cdot \underline{v} = 0$$

$$\lambda^{2a-1} = \lambda^{a-b}$$

$$b = 1-a$$

$$\Rightarrow \lambda^{-(a-1)} = \lambda^b$$

$\therefore$  scalings  $\underline{r} \Rightarrow \lambda \underline{r}, \underline{v} \Rightarrow \lambda^a \underline{v}, t \Rightarrow \lambda^b t$

Now, turbulence onset  $\Rightarrow$  symmetry breaking!

i.e. ① KH: shear breaks  $\left\{ \begin{array}{l} \text{translational} \\ \text{rotational} \end{array} \right.$  invariance.

②



rigid body boundary flow, etc

③ Flushing toilet  $\left\{ \begin{array}{l} \text{space} \\ \text{time} \end{array} \right.$

etc.

\* However, fully developed turbulence tends to restore symmetry, except near boundaries, on small scale.

b.c.  $\leftrightarrow$  boundary conditions

i.e. if  $dV(\underline{r}, \underline{e}) = \underline{v}(\underline{r} + \underline{e}) - \underline{v}(\underline{r})$   
 $\Rightarrow$

$$dV(\underline{r} + \underline{e}) = dV(\underline{r})$$

similarly: isotropy, parity ...

(Facilitates scaling approach)

H2 For  $Re \rightarrow \infty$  turbulence, at small scales and away from boundaries, the flow is self-similar at small scales

i.e. possesses a unique scaling exponent  $h$  s/t

$$dV(\underline{r}, \lambda \underline{e}) \rightarrow \lambda^h dV(\underline{r}, \underline{e})$$

( $\rightarrow$  addresses 2/3 Law)

H3 With assumptions similar to H1, the turbulent flow has a finite, nonvanishing mean rate of dissipation  $\epsilon$  per unit mass.

$Re \rightarrow \infty \Rightarrow \nu \rightarrow 0$  with  $V_0 = v_{rms}$  to fixed

$$\epsilon = V_0^3 / l_0$$

Alternative (not necessary): Kolmogorov's 'Second' Universality Assumption: In the limit of infinite Reynolds number, all the small-scale statistical properties are uniquely and universally determined by the scale  $l$  and the mean energy dissipation rate  $\epsilon$ .

(First:  $-\frac{4}{5}\epsilon l = \langle u^3 \rangle$ )

i.e. Implications:

$$-\langle \partial v(\ell)^2 \rangle = S_2 \quad ?$$

$$S_2 \sim L^2/T^2, \text{ dimensionally}$$

$$\text{Now } \epsilon \sim L^2/T^3$$

$$\Rightarrow \langle \partial v(\ell)^2 \rangle \sim \epsilon^{2/3} l^{2/3} \Rightarrow \text{recovers } 2/3 \text{ Law.}$$

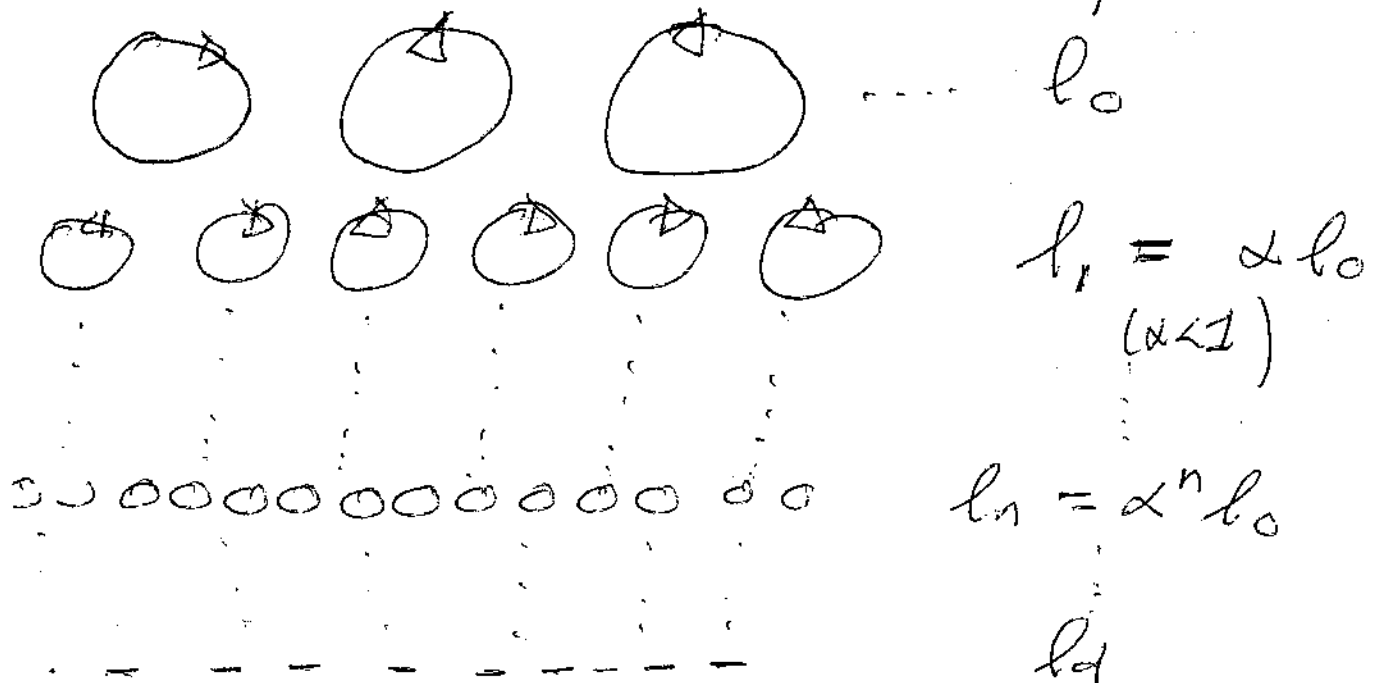
also, implies  $h = 1/3 \Leftrightarrow$  scaling exponent, etc.

$H_1, H_2, H_3$  (2<sup>nd</sup> Universality Assumption)  $\Rightarrow$

K41 phenomenology.

# K41 Phenomenology

Picture: (Richardson) Cascade / Eddy Mitosis



Key Idea: <sup>①</sup> → Flux of energy in 'scale space',  
 from  $l_0$  (integral scale) to  $l_d$ ,  
 (dissipation scale)

② → energy flux is self-similar

③ symmetry restoration.

Flux  $\rightarrow$  <sup>④</sup> energy dissipation  $\rightarrow$  finite limit as  $\nu \rightarrow 0$ .  
 (i.e. end-point re-adjustment)

self-similarity  $\rightarrow$  2/3 Law

$$S_2 = C (l/l_0)^{2/3}$$

$$l_0 \rightarrow \alpha l_0, \quad C \rightarrow C \alpha^2$$

etc.

Ingredients in K41 Phenomenology:

→  $l$  : scale parameter : eddy scale

→  $v(l)$  :  $v(l) \sim \langle \delta v_{||}(l)^2 \rangle^{1/2}$

↓  
eddy  
velocity

$$\delta v_{||} \sim (v(l+\underline{l}) - v(\underline{l})) \cdot \frac{l}{l}$$

≡ longitudinal velocity increment

→  $v_0$  : rms velocity fluctuation  
(large scale dominates)

$$v(l_0) \sim v_0$$

→  $\tau(l)$  : eddy lifetime / turn-over rate  
↓  
characteristic rate of transfer thru  
scale  $l$ .

self-similarity : energy thru-put rate is  
scale  $l$  energy scale invariant

⇒

$$\epsilon = v(l)^2 / \tau(l)$$

↓  
dissipation  
rate

energy balance / thru-put  
eqn.

↳ scale  $l$   
life-time

, ow,  $\tau(l)$  ?

compare  $\tau(l)$  with  $\tau_{in}(l)$

1962

$\tau(l) \rightarrow$  'lifetime' of structure of scale  $l$

$\rightarrow$  i.e. time for structure to be distorted out of existence

scales  $l' \gg l$  :

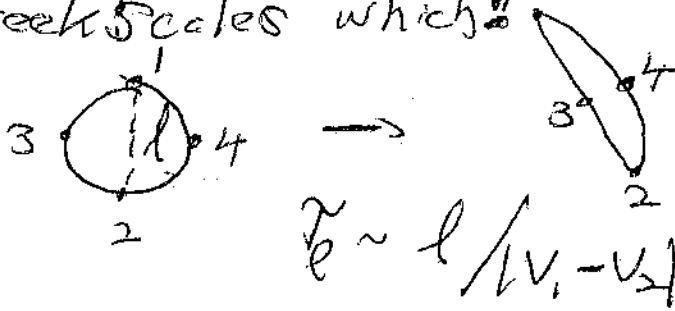
$\rightarrow$  advect eddy  $\rightarrow$  apply Galilean boost, but don't affect life-time.

irrelevant?  $\rightarrow$  symmetry under random Galilean transformations  
 $\rightarrow$  would also violate symmetry restoration.

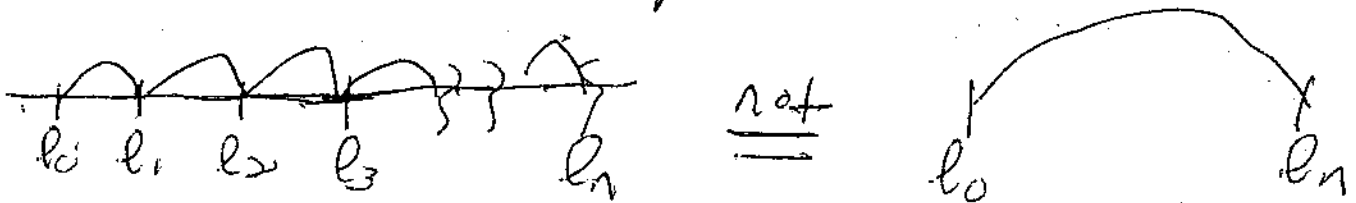
scales  $l \ll l'$  :

$\rightarrow$  irrelevant as very little energy/shear in such eddys/scales

seek scales which



$\rightarrow \tau(l) \sim \frac{l}{\tau(l)}$  : cascade local in scale space.





$$\epsilon = \frac{v(l)^3}{l}$$

$$\Rightarrow v(l) \sim (\epsilon l)^{1/3} \quad ; \quad \text{K41 scaling relation}$$

$$v(l)^3 \sim \epsilon^{2/3} l^{2/3}$$

- verifies  $2/3$  Law

- for spectrum:

$$\text{if } E(k) = \dots \quad |v(k)|^2$$

$$\text{s/t } E = \int dk E(k)$$

{ i.e. absorbs density of states

$$\text{then } v(l) = \int_{k_l}^{k_{l+1}} dk E(k)$$

$$v(l)^2 \sim \epsilon^{2/3} l^{2/3} = \epsilon^{2/3} k_l^{-2/3}$$

$$\Rightarrow \boxed{E(k) = \epsilon^{2/3} k^{-5/3}} \quad ; \quad \text{Kolmogorov Spectrum}$$

at  $l_0$ :

$$v_0 \sim \epsilon^{1/3} l_0 \Rightarrow \frac{v_0^3}{l_0} = \epsilon$$

For dissipation scale:

$l_d$  occurs in  $l$ -space when cascade terminated  
 over viscosity asserts itself  $\rightarrow Re(l) \rightarrow 1$

$$1/Re(l) \sim 1/Re = \nu/l^2$$

$$\Rightarrow \epsilon^{1/3} l^{-2/3} = \frac{\nu}{l^2}$$

$$l^{4/3} = \nu/\epsilon^{1/3}$$

 $\Rightarrow$ 

$$l_d = \nu^{3/4} / \epsilon^{1/4}$$

$$l_d \equiv \eta, \text{ in Frisch}$$

Recall:  $\epsilon = \nu \langle (\nabla v)^2 \rangle$

$$\Rightarrow \nu \rightarrow 0 \Rightarrow \langle (\nabla v)^2 \rangle \text{ divergent}$$

$$\langle (\nabla v)^2 \rangle = \int_{k_0}^{k_{ed}} dk k^2 \epsilon^{2/3} k^{-5/3}$$

$$= \int_{k_0}^{k_{ed}} dk k^{1/3} \epsilon^{2/3}$$

$$= k_{ed}^{4/3} \epsilon^{2/3}$$

$$= \frac{\epsilon^{1/3}}{\nu} \epsilon^{2/3} = \epsilon/\nu$$



$\langle (\nabla v)^2 \rangle$  divergent  
 as  $\nu \rightarrow 0$

## Counting Degrees of Freedom

How big is the inertial range?

$$\begin{aligned} n &\sim \frac{l_0}{l_d} \sim \frac{l_0}{(v^3/\epsilon)^{1/4}} \\ \# \text{ eddies} &\sim \frac{l_0}{v^{3/4}} \left( \frac{v_0}{l_0} \right)^{1/4} \sim \left( \frac{v_0 l_0}{v} \right)^{3/4} \\ &\sim Re^{3/4} \end{aligned}$$

∴ number of degrees of freedom for 3D turbulence is;

$$N \sim Re^{9/4} \quad : \quad \text{would be (minimum) \# grid points to resolve range of scales in numerical simulation}$$

Now, i.e. atmospheric boundary layer:

$$l_0 \sim 1 \text{ km}$$

$$l_d \sim 1 \text{ mm}$$

$$n \sim 10^6 \Rightarrow \begin{cases} N \sim 10^{18} \\ Re \sim 10^8 - 10^9 \end{cases} \quad \begin{matrix} | \\ 0 \end{matrix}$$

⇒ subgrid scale modelling, ...

B. : Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.

Exercises:

→ Consider passive scalar, with concentration  $C$ :

$$\frac{\partial C}{\partial t} + \underline{v} \cdot \underline{\nabla} C - K \nabla^2 C = \tilde{f}_C$$

$C$  dissipation rate in  $k41$  turbulence  $\equiv \alpha$

i.e.  $\alpha = \tilde{\epsilon}_0^2 \frac{v_0}{l_0}$

- ⇒ a.) Calculate  $k41$  spectrum for  $C$ .  
 Discuss  
 b.) What if  $k \ll \nu$  ?  
 $\gg$  ?

→ Consider incompressible turbulence with  $M \equiv \frac{v_0}{c_s} \ll 1$ .

Show:  $\underline{l_d} \sim M^{-1} Re^{1/4}$   
 $l_{ms}$

⇒ validity of continuum hydrodynamics gets better at high  $Re$ .

Particle Separation / Richardson Law

Consider 2 particles (test) in K41 turbulence.  
Rate of separation?



→ larger eddys advect both

→ smaller eddys do @ nothing

⇒ divergence controlled by eddys of scale  $l \sim |\underline{x}_1 - \underline{x}_2|$ .

∴ if  $\lambda \equiv |\underline{x}_1 - \underline{x}_2|$

$$\Rightarrow \frac{d\lambda}{dt} = v(\lambda) = \epsilon^{1/3} \lambda^{1/3}$$

$$\lambda^{2/3} = \epsilon^{1/3} t$$

$$\therefore \lambda \sim \epsilon^{1/2} t^{3/2}$$

Richardson's 3/2 Law

N.B.: Non-diffusive!

$$\lambda^2 \sim \epsilon t^3 \Rightarrow \left\{ \tau_{sep} \sim \lambda^{2/3} / \epsilon^{1/3} \right\}$$

N.B.:

→ process is self-accelerating ⇒ large eddys move faster

• non-diffusive.

→ [Momentum] Flux Driven  
Turbulence

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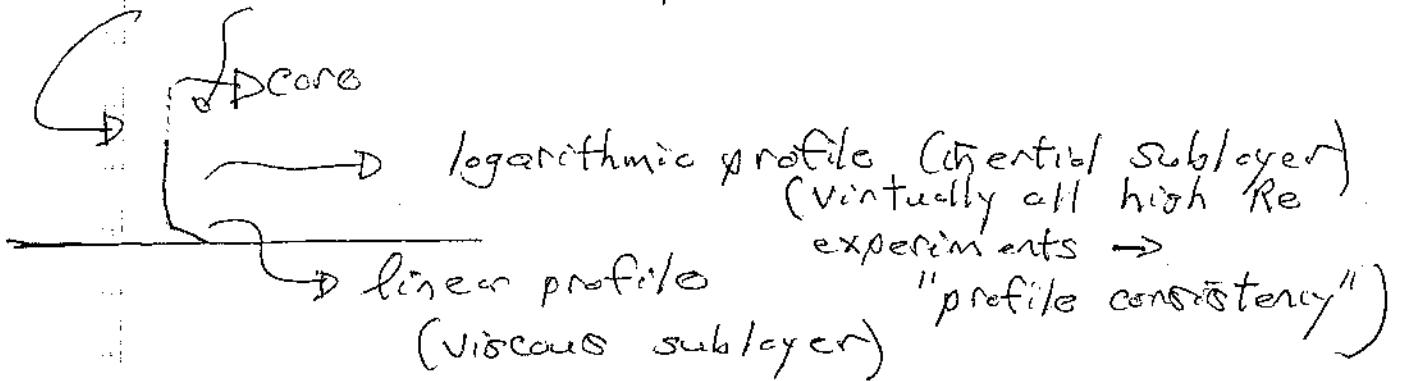
## ↳ Turbulent Pipe Flow

(cf. Landau, Lifshitz "Fluid Mechanics")

Till now → homogeneous flow in a periodic box  
→ cascade in scale space (Kolmogorov)

Now → inhomogeneous flow in a pipe  
→ momentum transport in a turbulent boundary layer (Prandtl)

Consider turbulent pipe flow:



Common features of pipe flow:

- linear → logarithmic  $U(x)$  profile
- logarithmic profile persists over a broad range of  $Re$

$$\left( Re = 2Ua / \nu \right)$$

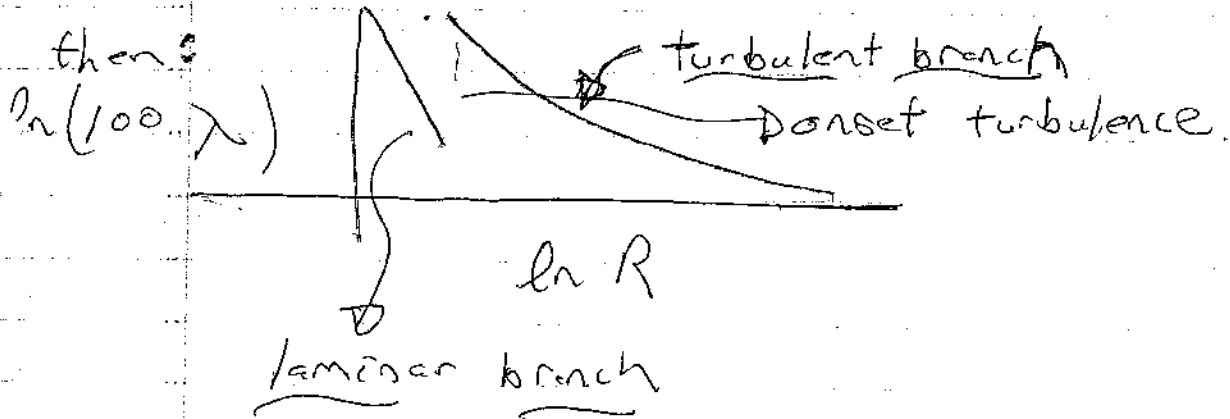
• logarithmic profile "universal" (Prandtl "Law of the Wall")

- resistance <sup>increases</sup> with increasing  $Re$ ,  
discontinuously  $\rightarrow$  pressure drop/length

$$\lambda = 2a \Delta p / l$$

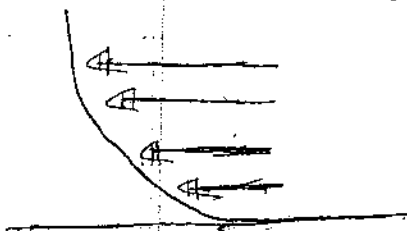
$$\frac{1}{2} \rho U^2$$

$\rightarrow$  mean flow energy



- turbulent resistance curve universal.

• What is going on?



no slip boundary condition  
 $U = U(x) \rightarrow 0$   
 $x \rightarrow 0$

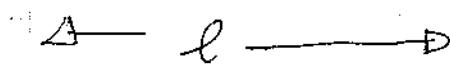
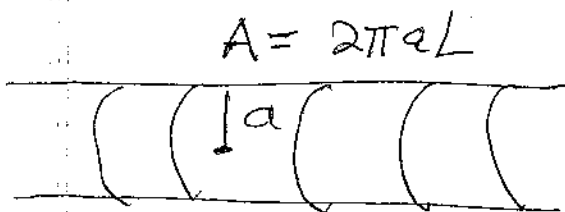
$\therefore U = U(x) \Rightarrow$  { momentum flux to wall }

→ Momentum flux to wall  $\Rightarrow$  stress on the wall

→ Wall stress must balance pressure drop, for steady flow

so Wall stress  $\approx \rho U_*^2$       i.e.  
 $U_* \equiv$  friction velocity       $\frac{F_{\text{vis}}}{F_{\text{SA}}} \sim \frac{F_{\text{vis}}}{F_{\text{PA}}} \sim \frac{\Delta p}{\rho}$   
 $\Rightarrow U_*^2 l R \sim \Delta p \pi R^2$

$$\rho U_*^2 2\pi a l = \Delta p \pi a^2$$



pressure drop

Force on wall  $\approx$   
 $\rho U_*^2 A_{\text{wall}}$

(Pressure Drop)  $A_{\text{flow}}$

$\approx$  Force on Fluid

friction only  $\Rightarrow \rho U_*^2 (2\pi a l) = (\Delta p) \pi a^2$

$$U_* = \left[ \frac{\Delta p}{2\rho} \left( \frac{a}{2l} \right) \right]^{1/2}$$

Friction Velocity



$U_x \equiv$  friction velocity  
 $\equiv$  "typical" velocity of turbulence in  
 turbulent pipe

Deriving the inertial sublayer profile:

i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters  $\rho$ ,  $\tau_w$ ,  $x$   
 $\rho$  density  $\tau_w$  wall stress  $x$  distance from wall  
 $U_x$

key point: ~~Assumption~~  
 of scale invariance

on scale  $l_{vs} = \frac{\nu}{U_x} < x < a$

$\rightarrow$  universality of logarithmic profile motivated  
 scale invariance assumption

now, seek velocity gradient  $dU/dx$ ,

$\frac{dU}{dx} : U_x, x, \rho$

so simplest form for  $du/dx$  is:

$$\frac{du}{dx} = \frac{u_*}{x}$$

$$\Rightarrow \left\{ \begin{aligned} u &= \frac{u_*}{K} \ln(x/x_0) \\ &= \frac{u_*}{K} \ln x + \text{const.} \end{aligned} \right.$$

→ logarithmic profile (consequence of scale invariance in pipe flow)

→  $K \approx 4$  universal constant → von-Karman constant

→  $x_0$  ↔ width of viscous sublayer  $\sim \nu/u_*$

(c.) Physical Reasoning

stationary flow  $\Rightarrow$

momentum flux to wall = pressure drop

$$\langle \tilde{v}_x \tilde{v}_z \rangle = U_*^2$$

Reynolds stress

$$\rho \langle \tilde{v}_x \tilde{v}_z \rangle = \tau_p$$

↳ momentum flux

$$\tau_p / \rho = U_*^2$$

Now, to calculate

$$\langle \tilde{v}_x \tilde{v}_z \rangle :$$

→ take velocity fluctuation as generated by mixing of  $U(x)$ , so

$$\tilde{v}_z \sim l \frac{\partial U}{\partial x}$$

↳ "mixing length"

analogous to Chapman-Enskog expansion, i.e.

$$l \leftrightarrow l_{mfp}$$

$$\tilde{v}_x \leftrightarrow v_{th}$$

here, scale invariance ~~is~~  $l \sim x$

mixing length set by  
distance from wall

$$\begin{aligned} \text{so } \langle \tilde{v}_x \tilde{v}_z \rangle &= \langle v_x l \rangle \frac{\partial U}{\partial x} \\ &\approx u_* x \frac{\partial U}{\partial x} \end{aligned}$$

$\tau_T = u_* x \rightarrow$  "eddy viscosity"  
"turbulent viscosity"  $\rightarrow$  key concept.

$\Rightarrow$  rate of turbulent transport  
of momentum

then momentum balance  $\Rightarrow$

$$u_* x \frac{\partial U}{\partial x} = u_*^2$$

$$\Rightarrow U = \frac{u_*}{K} \ln(x/x_0) \rightarrow \text{Logarithmic Profile}$$

$\rightarrow$  Law of the Wall

Some comments:

→ as in k41, clear phenomenology critical to guiding the approximations → scale invariance

30 "Mixing length theory always works... provided you know the mixing length..."  
- P. D.

⇒ why a single value of velocity, i.e.  $U_x$ ?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

$$\Rightarrow \tilde{v} \sim l \frac{\partial U}{\partial x} \sim x \frac{\partial U}{\partial x}$$

$$\sim \cancel{x} \frac{U_x}{\cancel{x}}$$

absence of preferred scale.

consistent.  $\therefore$  Assumptions consistent with:  
- logarithmic profile  
- scale invariance.

→ viscous sublayer / cut-off of inertial layer

∴ when:  $\nu_T < \nu$

{ molecular viscosity dominates mixing

$$\Rightarrow u_* x \lesssim \nu$$

$$x \lesssim \nu / u_* \equiv x_0$$

{  
viscous sublayer  
scale.

In viscous sublayer, flow linear:

$$\nu \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore u = \frac{u_*^2}{\nu} x$$

⇒ note effect of turbulence is to:

- flatten profile - { higher transport at fixed wall stress
- reduce central velocity
- limit  $Q$  (quality factor)

- matching, for const:

$$x_0 = \nu / U_* \quad \text{so}$$

$$U = \frac{U_*}{K} \ln \left( \frac{U_* y}{\nu} \right)$$

Note: Flow in viscous sublayer is turbulent, but mixing there affected by dissipation range scales  $\Rightarrow$  linear profile

Now - turbulent dissipation

Consider NSE:

$$\frac{\partial \hat{v}}{\partial t} + \hat{v} \cdot \nabla \hat{v} + \langle \frac{v_z}{z} \rangle \frac{\partial \hat{v}}{\partial z} + \hat{v}_x \frac{\partial \langle v_z \rangle}{\partial x} = -\nabla \hat{p} + \nu \nabla^2 \hat{v}$$

$\hat{v}$  and avg  $\Rightarrow$

$$\frac{\partial \langle \hat{v}^2 \rangle}{\partial t} + \langle \hat{v} \cdot \hat{v} \cdot \nabla \hat{v} \rangle + \langle v_z \rangle \langle \hat{v} \cdot \frac{\partial \hat{v}}{\partial z} \rangle + \langle \hat{v}_x \hat{v}_z \rangle \frac{\partial \langle v_z \rangle}{\partial x} = - \underbrace{\langle \hat{v} \cdot \nabla \hat{p} \rangle}_{i.b.A} - \nu \langle \nabla^2 \hat{v}^2 \rangle$$

odd

input  $\rightarrow$  mean flow mixing 276.

obviously:  $\nu \langle (\nabla \cdot \mathbf{v})^2 \rangle = \nu_T \left( \frac{\partial U}{\partial x} \right)^2$

small scale  
dissipation

and

$$E = (U_* X) \left( \frac{U_*}{X} \right)^2 \quad (\text{ignoring } R)$$
$$= \frac{U_*^3}{X}$$

$\rightarrow$  sets dissipation rate

i.e.  $E = \frac{V_0^3}{l} \quad \begin{matrix} V_0 \leftrightarrow U_* \\ l \leftrightarrow X \end{matrix}$

$\rightarrow E$  finite as  $\nu \rightarrow 0$  (i.e. viscous sublayer gradient diverges then)

Additional References:

- S.B. Pope, "Turbulent Flows"
- H. Tennekes and J. Lumley, "A First Course in Turbulence"



For net energy budget:

$$\partial_t \mathcal{E} = - \underbrace{\langle \tilde{U}_x \tilde{V}_z \rangle}_{\substack{\downarrow \\ \text{input to fluctuations} \\ \text{by relaxation of} \\ \text{mean shear flow} \\ \text{(Reynolds work)}}} \frac{\partial \langle V_z \rangle}{\partial x} - \underbrace{\nu \langle (\nabla \tilde{U})^2 \rangle}_{\substack{\downarrow \\ \text{dissipation} \\ \text{of fluctuations} \\ \text{energy by viscosity}}}$$

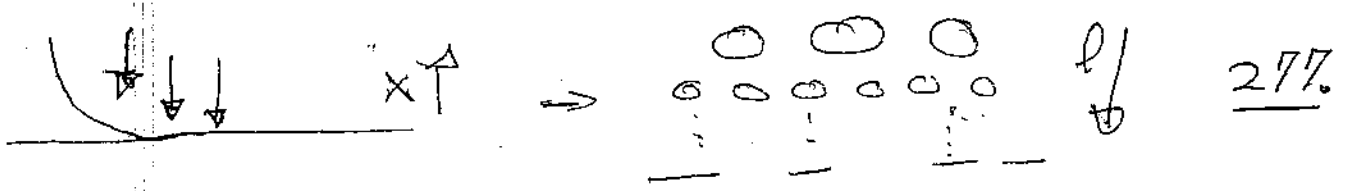
∴ can define:

$$\underbrace{\mathcal{E}}_{\substack{\downarrow \\ \text{turbulent} \\ \text{dissipation} \\ \text{rate}}} = \langle \tilde{U}_x \tilde{V}_z \rangle \frac{\partial U}{\partial x}$$

and using mixing length theory:

$$\langle \tilde{U}_x \tilde{V}_z \rangle = u_* x \frac{\partial U}{\partial x}$$

$$\Rightarrow \mathcal{E} = (u_* x) \left( \frac{\partial U}{\partial x} \right)^2 = \underbrace{\gamma_T}_{\substack{\downarrow \\ \text{rate of "heating" by} \\ \text{turbulent relaxation} \\ \text{of mean flow}}} \left( \frac{\partial U}{\partial x} \right)^2$$



→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl)	K41 (Kolmogorov)
scales: $a, x, \nu/u_*$	$l_0, l_n, l_d$
invariance: $x \rightarrow$ real space	$l \rightarrow$ scale space
inertial sublayer viscous sublayer	inertial range dissipation range
balance: $u_*^2 = \nu_T \frac{\partial u}{\partial x}$	$\epsilon = \frac{\nu(l)^2}{T(l)}$
dynamics: eddy viscosity $\nu_T = u_* x$	turn-over rate $1/T(l) = \frac{\nu(l)}{l}$
result: $u = \frac{u_*}{R} h(x)$	$\nu(l) = \epsilon^{1/3} l^{1/3}$
universal profile	universal spectral scaling
dissipation: $\nu = \nu_T$ $x_0 = \nu/u_*$	$\nu(l)/l = \nu/l^2$ $l_d = \nu^{3/4} / \epsilon^{1/4}$

## → Practical Issues

Resistance Law  $\leftrightarrow$  Pipe Flows.

have:  $\frac{v}{u_*} < x \leq a$   
 $\downarrow$   
 radius

can push to  $x \approx a$ , with logarithmic accuracy

$$U \approx \frac{u_*}{R} \ln \left( \frac{u_* a}{v} \right)$$

but

$$v_* = u_* = \left( \frac{g \Delta P}{l \cdot 2\rho} \right)^{1/2}$$

$\Rightarrow$  can re-write:

$$U = \left( \frac{a \Delta P}{2\rho l k^2} \right)^{1/2} \ln \left( a \left( \frac{a \Delta P}{2\rho l} \right)^{1/2} / v \right)$$

Convenient to define:

$$\lambda = \frac{2a \Delta P / l}{\frac{1}{2} \rho U^2}$$

$\hookrightarrow$  flow KE

$\rightarrow$  friction / Resistance  
Factor / Coefficient

⇒ taking  $Re = 2aU/v$

can rewrite friction law as:

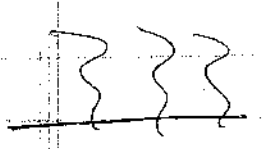
$$\left\{ \begin{array}{l} 1/\sqrt{\lambda} = .88 \ln(Re\sqrt{\lambda}) - .80 \\ Re = 2aU/v \end{array} \right. \quad \begin{array}{l} \downarrow \\ \text{phenom.} \end{array}$$

$$\left\{ \begin{array}{l} \lambda = \frac{2a \Delta P / l}{\frac{1}{2} \rho U^2} \end{array} \right.$$

→ good fit to pipe flow data.

Problems:

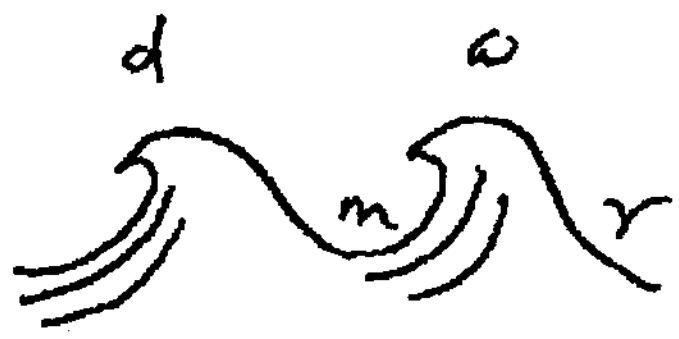
1a) A very strong explosion, with energy released  $\Delta E$ , creates a spherized blast wave in an atmosphere of pressure  $P_0$ , density  $\rho$ . Use dimensional analysis to derive the radius of the blast front as a function of time, i.e.  $r(t)$ ? When does this scaling fail?

b)  A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate, assuming the hot plate drives a surface heat flux  $Q$ . (See Chapter 5; Landau).

## Why Wave "kinetics"?

" A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions "

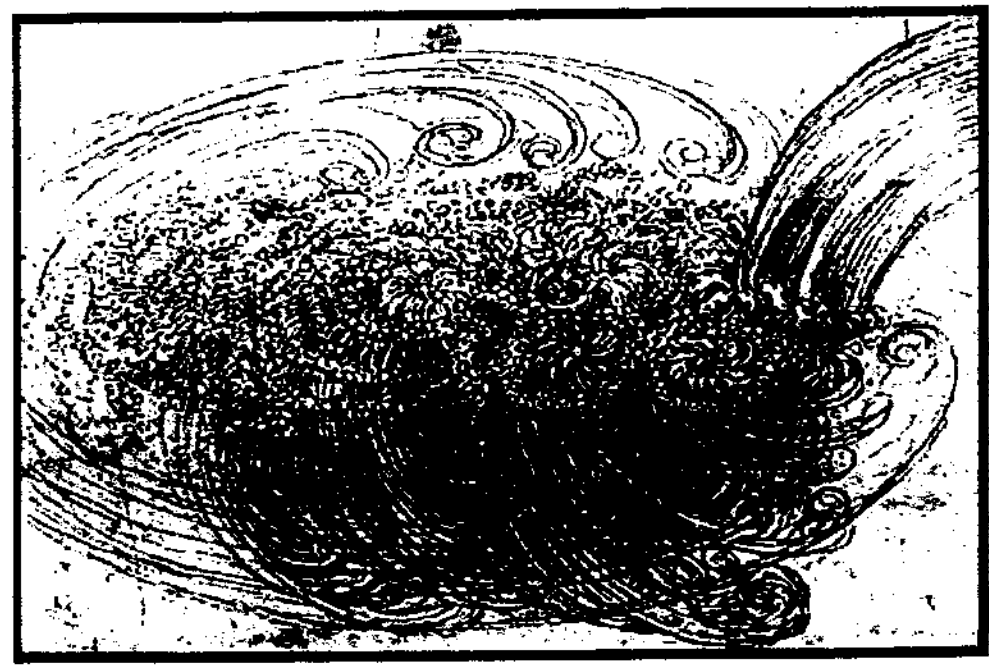
- Leonardo da Vinci  
Codice Atlantico, c.1500.



a b n o l l i a m i o g u l o p o d e m  
 g m r a l l o p s r i m b u n r o m  
 d a m s d . g n . m i o r



[ Leonardo  
 on waves ... ]




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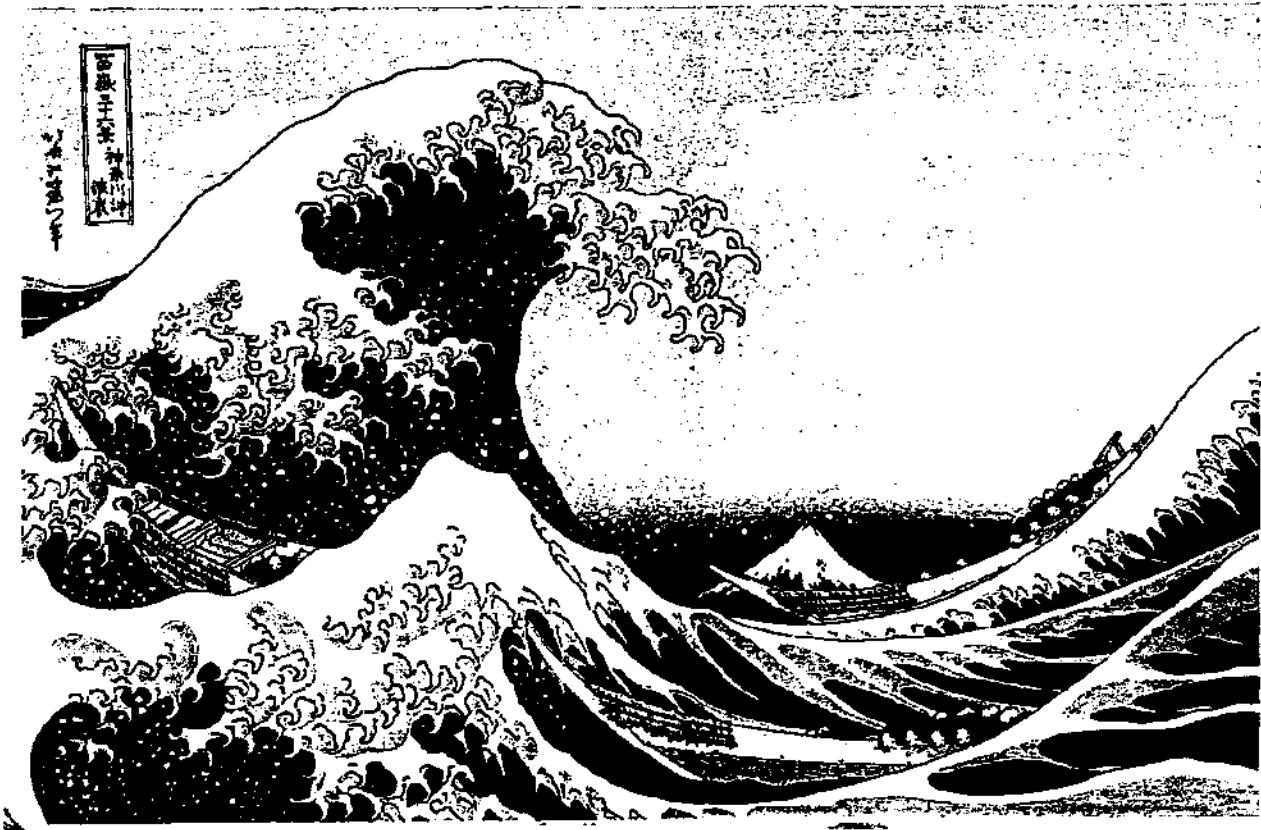
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From Asian art...

The great wave at Kanagawa

Hokusai





## Outline

### i.) Introduction and Basic Ideas

→ what is wave kinetics?

→ basic structure and limitations?!

### ii.) Why Bother? - Utility of Wave Kinetics

→ particle ⊕ quasi-particle picture

applications:

- energetics → conservation laws

- "radiation hydrodynamics" → wave forces

- structure formation

→ Langmuir turbulence

→ zonal flow formation

- non-locality → turbulence spreading

### iii.) Basic Theory

- intuitive arguments
- outline of Whitham derivation (see supplement I)
- full wave kinetics
- special items: drift waves, 2D fluids, etc.

### iv.) Applications I → Conservation Relations

- Energetics: simple and not-so-simple . . . .
- Parallel Momentum

### v.) Applications II → Structure Formation (Introduction)

- Langmuir Turbulence (see supplement II)
- Zonal Flow Formation

w) Introduction and Basic Ideas

→ Wave kinetics or Quasi-Particle Picture

- Boltzmann-like equation for wave population

$\frac{dN}{dt} = C(N)$  ;  $N(\underline{k}, \underline{x}, t) \equiv \left\{ \begin{array}{l} \text{population} \\ \text{density} \end{array} \right.$

$\frac{\partial N}{\partial t} + \underbrace{(\underline{v}_g + \underline{v}) \cdot \nabla}_{\text{LHS}} N = \underbrace{-\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{v}) \cdot \nabla_{\underline{k}}}_{\text{RHS}} N$  ala' Boltzmann

LHS:  $\frac{dN}{dt} \Big|_{\text{rays}}$

characteristics

$\frac{dx}{dt} = \underline{v}_g + \underline{v}$

$\frac{dk}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{v})$

} eikonal equations  
⇒ ray trajectories for waves

for LHS (cont'd):   
short  $\lambda$ , high  $\omega$   
wave  
long  $\lambda$  low  $\omega$   
modulator

eikonal theory  $\rightarrow$  scale separation

$\tilde{\omega}, \tilde{v} \rightarrow$  modulation field with scales  $\rightarrow$   $\begin{cases} \underline{q} \rightarrow \text{wave vector} \\ \Omega \rightarrow \text{frequency} \end{cases}$

$\therefore$  LHS  $\downarrow$  disparate scale interactions   
 $\omega_n > \Omega, \underline{q} \cdot \underline{v}, |\underline{v}'|; \text{etc.}$   
 $|\underline{k}| > |\underline{q}|; \text{etc.}$

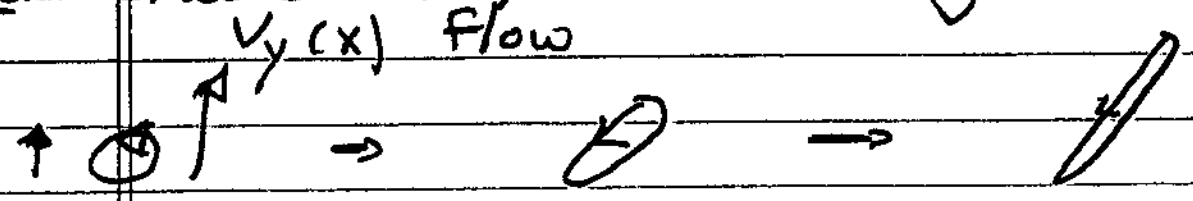
if  $C(N) \rightarrow 0$  and  $\omega > \Omega, \text{etc.}$    
 $\rightarrow$  adiabatic invariance....

$\frac{dN}{dt} = 0 \Rightarrow N \equiv$  wave action density   
 $= \epsilon_n / \omega_n$

$\downarrow$    
energy density  
wave population density conserved along rays.

$\Rightarrow$  tells how wave population responds to modulation field.

ie. classic example  $\rightarrow$  shearing (coherent)



$k_x$  increases  $\Rightarrow$  (refraction)

$$\frac{dk_x}{dt} = -\frac{\partial}{\partial x} (\omega + V_y(x) k_y)$$

$$\approx -k_y \frac{\partial V_y(x)}{\partial x}$$

how does energy change?

$N = \Sigma / \omega_H$  ,  $V_y' < \omega_H$

$\Rightarrow N \approx N_0$  - const  
adiabatic invariant

$$\frac{dE}{dt} = \frac{d}{dt} (N \omega_H) = N_0 \frac{\partial \omega}{\partial k_x} \cdot \frac{dk_x}{dt}$$

$$= N_0 v_{grx} \frac{dk_x}{dt}$$

$$= N_0 v_{grx} \left( -k_y \frac{\partial V_y(x)}{\partial x} \right)$$

$\frac{dE}{dt} > 0 \rightarrow$  sign:  $-v_{gr} k_y V_y'$

$$\sim \frac{k_y^2 k_x V_y'}{(1+k_x^2 \lambda^2)^2} \omega_H^2 v_{gr}$$

stochastic  $\rightarrow$  zonal flows (later.)

RHS  $\rightarrow$  non Action-conserving wave interactions

i.e.

$$C(N) = \sum_{\substack{k', k'' \\ \oplus = k}} \left\{ C_1(k', k'') N_{k'} N_{k''} - C_2(k, k') N_k N_{k'} \right\} * \delta(\omega_{k''} - \omega_k - \omega_{k'})$$

effective collision operator  
 (non-adiabatic wave interactions)  $\rightarrow$  {topic for another lecture}

so obvious analogy:

$$\left\{ \frac{\partial N}{\partial t} + (\underline{v}_g + \underline{v}) \cdot \nabla N - \frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{v}) \cdot \nabla_k N \right\} = C(N)$$

LHS  $\rightarrow$  N conserving

$\delta$   
 wave-wave interaction

$$\left\{ \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} \right\} = C(f)$$

LHS  $\rightarrow$  f conserving

$\delta$   
 particle collisions

$\therefore$  N  $\equiv$  quasi-particle density

so : wave kinetics :

→ Boltzmann-like equation for wave population density (action density → usually)  $N$

→ LHS → wave-mean interactions (slow)

RHS → wave-wave interaction (break adiabatic invariance)

→ useful when : scale separation between mean, waves

→ useful because : can apply tricks from particle kinetics to wave population.

i.e. if  $C(N) = -\nu_{eff} (N - \langle N \rangle)$

how calculate flux of wave momentum if  $\langle N \rangle = \langle N(x) \rangle$  ?  
" (radiation pressure) "

→ "radiation hydrodynamics" :

see: [Michelas and Michelas  
Zeldovich and Raizer

→ useful because: natural way to  
formulate conservation laws of  
quasi-linear theory

→ useful because: natural way to formulate  
structure formation →

- Langmuir turbulence
- zonal flow formation



## Supplement I:

### Variational Theory of Wave Kinetics

(Whitham)

→ Where does  $N \leftrightarrow$  Action emerge from?

→ adiabatic invariance  $\leftrightarrow$  conservation law

Noether's Thm }  $\rightarrow$  symmetry!

{ Phase symmetry of wave train  
underlies action conservation  
and wave kinetics

→ derivation of wave kinetics via variational principle.

## ▷ Variational Theory (Whitham)

Now, consider system, like ideal MHD, which can be described in terms of a displacement  $\underline{\xi}$  such that

$$\underline{\xi} = \text{Re} \left\{ A e^{i\phi} + A^* e^{-i\phi} \right\}$$

then relevant wave equation can be derived from:

$$\delta S = \delta \int dt \int d\underline{x} \mathcal{L}(\underline{\xi}) = 0$$

↳ Lagrangian density

Now, if write Lagrangian density in terms phase  $\phi$  and amplitude  $a$ , have:

$$S = \int dt \int d\underline{x} \mathcal{L}(-\phi_t, \phi_x, a)$$

where  $\omega = -\phi_t = \partial\phi/\partial t$

$$\underline{k} = \phi_x = \underline{\nabla}\phi$$

→ this neglects all corrections to eikonal theory (WKB) i.e. all corrections to  $\underline{k}$ ,  $\omega$ , amplitude, etc.

→  $L$ , above, corresponds to period averaged Lagrangian -  $\phi$  indeterminate to constant

$$\Rightarrow \delta S = \delta \int dt \int dx \mathcal{L}(-\phi, \phi_x, a)$$

so have 2 variational equations:

$$1) \delta S / \delta a = 0$$

$$2) \delta S / \delta \phi = 0$$

Now, within scope of linear theory

$$\mathcal{L} = G(\omega, k) a^2$$

c.e. for MHD, can write:

$$\mathcal{L} = \frac{1}{2} \rho \dot{\xi}^2 - \frac{1}{2} \rho [D(\xi, x, t)]^2 \xi^2$$

and if  $\xi = \underline{A} e^{+i\phi} + \underline{A}^* e^{-i\phi}$   $\left\{ \begin{array}{l} \text{eikonal form of} \\ \text{potential energy} \\ \text{(from stiffness matrix)} \end{array} \right.$

$$\mathcal{L} = \frac{1}{2} \rho \left( \frac{\partial \phi}{\partial t} \right)^2 |A|^2 - \frac{1}{2} [D(\partial \phi, x, t)]^2 |A|^2$$

is concrete form of avg. Lagrangian.

$$\Rightarrow G(\omega, k) = \frac{1}{2} \rho \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - \left[ \rho(\nabla \phi, \underline{x}, t) \right] \right] |\Lambda|^2$$

$$\text{Now, } 1) \Rightarrow \frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \boxed{G(\omega, k) = 0}$$

$$\text{but: } G = \omega^2 - \left[ \rho(k, \underline{x}, t) \right]^2 = 0$$

is just dispersion relation!

$$2) \Rightarrow dS/d\phi = 0$$

$$\frac{\partial S}{\partial \phi} = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial(-\dot{\phi})} \delta(-\dot{\phi}) + \frac{\partial \mathcal{L}}{\partial(\phi_x)} \delta(\phi_x) \right\}$$

$$= \int dt \int dx \left\{ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial(-\dot{\phi})} \right) - \frac{\partial}{\partial x} \cdot \left( \frac{\partial \mathcal{L}}{\partial \phi_x} \right) \right\} \delta \phi$$

$$\Rightarrow dS = 0 \Rightarrow$$

$$\boxed{\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \omega} \right) - \underline{\nabla} \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0}$$

so have:  $\mathcal{L} = \epsilon(\omega, \underline{k}) a^2$

$$\Rightarrow \epsilon(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Now  $\epsilon(\omega, \underline{k}) = 0$

$$\Rightarrow \frac{\partial \epsilon}{\partial \omega} d\omega + \frac{\partial \epsilon}{\partial \underline{k}} d\underline{k} = 0$$

$$\therefore \underline{v}_{gr} = \frac{d\omega}{d\underline{k}} = - \frac{\partial \epsilon / \partial \underline{k}}{\partial \epsilon / \partial \omega}$$

$$\left( \text{d.e. } \epsilon(\underline{k}, \omega) = 0 \right. \\ \left. d\epsilon = 0 = \frac{\partial \epsilon}{\partial \omega} d\omega + \frac{\partial \epsilon}{\partial \underline{k}} \cdot d\underline{k} \right)$$

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\partial \epsilon a^2}{\partial \omega} \right) + \nabla \cdot \left[ \frac{-\frac{\partial \epsilon / \partial \underline{k}}{\partial \epsilon / \partial \omega} \frac{\partial \epsilon}{\partial \omega} a^2}{\frac{\partial \epsilon}{\partial \omega}} \right] = 0$$

and so  $N \equiv \frac{\partial \epsilon}{\partial \omega} a^2$

$$\frac{\partial N}{\partial t} + \nabla \cdot (\underline{v}_{gr} N) = 0$$

(N not yet action ...)

- Now, can further note for  $G$  invariant to time trans.

$\Rightarrow$  energy is conserved.

so,  $\exists$  energy conservation equation (from Noether's Thm  $\leftrightarrow$  symmetry).

Now, note have:

can proceed by working with:

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \quad \Rightarrow \quad G(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

$$\frac{\partial \underline{k}}{\partial t} = - \frac{\partial \omega}{\partial \underline{x}} \quad \left( \frac{\partial}{\partial t} \frac{\partial \phi}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}} \frac{\partial \phi}{\partial t} \right)$$

$$\nabla \times \underline{k} = 0 \quad (\underline{k} = \nabla \phi)$$

$$\Rightarrow \text{have } \frac{\partial}{\partial t} (\omega \mathcal{L}_\omega - \mathcal{L}) + \nabla \cdot \left( -\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

$$\text{check: } \omega \frac{\partial \mathcal{L}_\omega}{\partial t} - \frac{\partial \mathcal{L}}{\partial t} + \nabla \cdot \left( -\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0 + \frac{\partial \mathcal{L}}{\partial \omega} \frac{\partial \omega}{\partial t}$$

$\frac{\partial \mathcal{L}}{\partial \omega} = 0$

$$\begin{aligned}
&= \frac{\partial \mathcal{L}}{\partial \omega} \frac{\partial \omega}{\partial t} + \omega \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) - \nabla \cdot \left( \omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) - \frac{\partial \mathcal{L}}{\partial t} \\
&= \omega \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) - \omega \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) - \frac{\partial \mathcal{L}}{\partial \underline{k}} \cdot \nabla \omega - \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega} \frac{\partial \omega}{\partial t} \\
&= + \frac{\partial \mathcal{L}}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega} \frac{\partial \omega}{\partial t} - \frac{\partial \mathcal{L}}{\partial t} \\
&= 0 \quad \checkmark
\end{aligned}$$

constructed form of energy  $\mathcal{E}$ :

$$\frac{\partial}{\partial t} \left( \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right) + \nabla \cdot \left( -\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

but  $\mathcal{L} = 0$ , so energy density of wave is  
 $(\mathcal{E}(\omega, \underline{k}) = 0)$

$$\mathcal{E} = \omega \frac{\partial \mathcal{L}}{\partial \omega}$$

above states energy conservation

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \omega} = \frac{\mathcal{E}}{\omega} \equiv \text{Action density!}$$

$\Rightarrow$  now  $N$  is action density!

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{G}}{\partial \omega} a^2 \right) + \underline{\nabla} \cdot \left[ \frac{-\partial \mathcal{G} / \partial \hbar}{\partial \mathcal{G} / \partial \omega} \frac{\partial \mathcal{G}}{\partial \omega} a^2 \right] = 0$$

$$\Rightarrow \frac{\partial N}{\partial t} + \underline{\nabla} \cdot [v_{gr} N] = 0$$

$$\omega N = \Sigma$$

Now note if write Vlasov-like equation:

$$\frac{\partial N}{\partial t} + v_{gr} \cdot \underline{\nabla} N - \frac{\partial \omega}{\partial x} \cdot \frac{\partial N}{\partial \hbar} = 0$$

Liouville  $\Rightarrow$

$$\frac{\partial N(\underline{h}, x, t)}{\partial t} + \underline{\nabla} \cdot [v_{gr} N] + \frac{\partial}{\partial \hbar} \cdot \left[ \frac{-\partial \omega}{\partial x} N \right] = 0$$

and  $\int_{\underline{h}} d\underline{h}$ , with assumption of narrow spread in  $\underline{h}$   
for  $N$ ,

$\Rightarrow$

$$\frac{\partial N(x, t)}{\partial t} + \underline{\nabla} \cdot [v_{gr} N] = 0$$



Recall:

→ Hamiltonian structure of eikonal theory, etc.  $\Rightarrow$   

$$\frac{\partial \rho(\underline{k}, \underline{x}, t)}{\partial t} + \underline{v}_g \cdot \nabla \rho(\underline{k}, \underline{x}, t) - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} \rho(\underline{k}, \underline{x}, t) = 0$$

→ Physical arguments suggest  $\rho = \frac{\underline{\epsilon}}{\omega} = \frac{N}{\omega}$   
 wave action density

→ Variational Approach

$$S = \int dt \int d^3x \mathcal{L}$$

$$\mathcal{L} = G(\omega, \underline{k}) a^2$$

$$\omega = -\partial \phi / \partial t = -\phi_t$$

$$\underline{k} = \nabla \phi = \phi_{\underline{x}}$$

$$\delta S = 0$$

but two parameters varied  $\begin{cases} a \\ \phi \end{cases}$

$$\delta S / \delta a = 0 \Rightarrow G(\omega, \underline{k}) = 0 \rightarrow \text{dispersion relation}$$

$$\delta S / \delta \phi = 0 \Rightarrow \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \omega} \right) - \frac{\partial}{\partial \underline{x}} \cdot \left( \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\partial G a^2}{\partial \omega} \right) - \frac{\partial}{\partial \underline{x}} \cdot \left( \frac{\partial G a^2}{\partial \underline{k}} \right) = 0$$

and time translation symmetry and  $G=0 \Rightarrow$

$$\underline{\underline{\epsilon}} = \omega \frac{\partial \underline{\underline{G}}}{\partial \omega} a^2 \quad \Rightarrow \quad N = \frac{\underline{\underline{\epsilon}}}{\omega} = \frac{\partial \underline{\underline{G}}}{\partial \omega} a^2$$

and  $\frac{\partial \underline{\underline{G}}}{\partial \underline{\underline{k}}} a^2 = \underline{\underline{v}}_{go} \cdot N$

→ Helpful Reminder:

Recall, for electrostatic plasma waves

if  $\epsilon(\omega, \underline{\underline{k}}) = 0 \quad \Rightarrow$  dispersion relation

then  $\Sigma_{\underline{\underline{k}}} = \frac{\partial (\omega \epsilon)}{\partial \omega} \bigg|_{\omega_{\underline{\underline{k}}}} \frac{|E_{\underline{\underline{k}}}|^2}{8\pi}$

$$= \omega_{\underline{\underline{k}}} \frac{\partial \epsilon}{\partial \omega} \bigg|_{\omega_{\underline{\underline{k}}}} \frac{|E_{\underline{\underline{k}}}|^2}{8\pi} \quad \rightarrow \text{wave energy density}$$

∴  $N_{\underline{\underline{k}}} = \frac{\partial \epsilon}{\partial \omega} \bigg|_{\omega_{\underline{\underline{k}}}} \frac{|E_{\underline{\underline{k}}}|^2}{8\pi}$

and  $\underline{\underline{P}}_{\underline{\underline{k}}} = - \frac{\partial \epsilon}{\partial \underline{\underline{u}}} \bigg|_{\omega_{\underline{\underline{k}}}} \frac{|E_{\underline{\underline{k}}}|^2}{8\pi} \quad \rightarrow \text{wave energy density flux}$

$$= \underline{\underline{v}}_{go} \cdot N_{\underline{\underline{k}}}$$

since  $G(h, \omega) = 0$ , so along rays

$$dG = d\omega \frac{\partial G}{\partial \omega} + dh \cdot \frac{\partial G}{\partial h} = 0$$

$$d\omega/dh = - (\partial G/\partial h) / (\partial G/\partial \omega)$$

etc.

we have Vlasov-like eqn. in  $x, k$  phase space

$$\frac{\partial N}{\partial t} + \underline{v}_g \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and continuity-type eqn. in  $x$  space:

$$\frac{\partial N}{\partial t} + \nabla \cdot [\underline{v}_g N] = 0$$

observe:

→ order of derivatives matters, but Liouville helps

→ continuity-type eqn. for packets

→ useful to note that total derivative of  $\underline{k}$ , following packet

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_g \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

$$= - \left( \frac{\partial \omega}{\partial \underline{x}} \right) + \underline{v}_g \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

if  $\omega = \omega(\underline{u}, \underline{x}, t)$   
from  $\epsilon = 0$

$$= - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} + \underline{v}_g \cdot \frac{\partial \underline{k}}{\partial \underline{x}} = - \frac{\partial \omega}{\partial \underline{x}}$$

$$= - \left( \frac{\partial \omega}{\partial \underline{x}} \right) \cdot \underline{k} \quad \text{no conflict with } \frac{d\underline{k}}{dt} = - \frac{\partial \omega}{\partial \underline{x}}$$

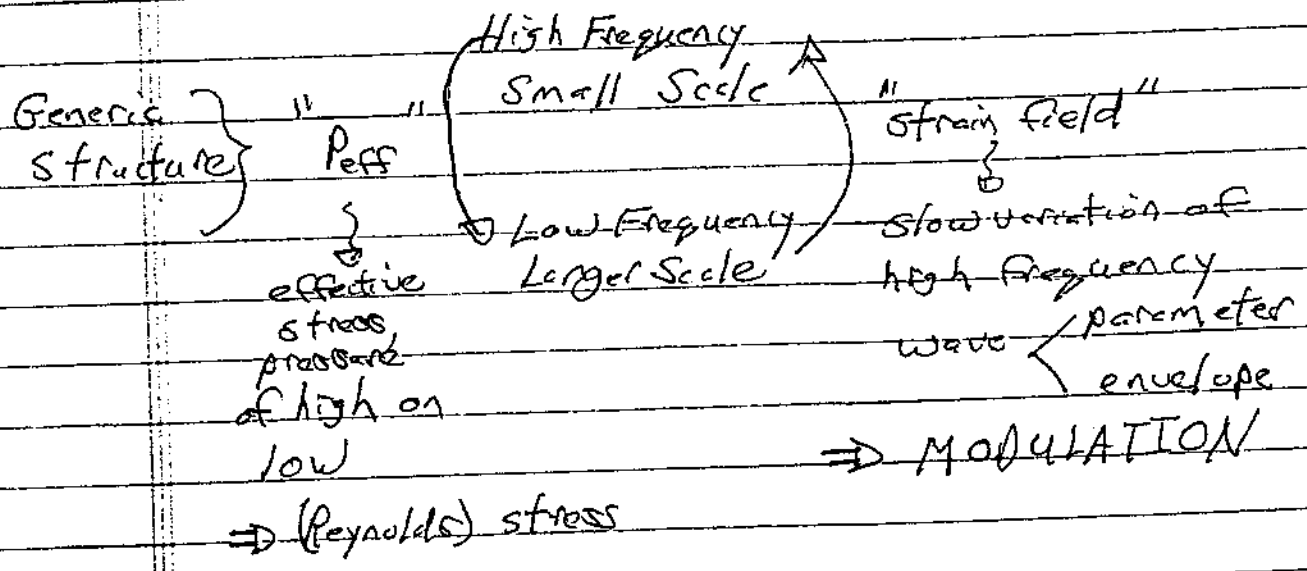
# Langmuir Turbulence and Disparate Scale Interaction

## Contents:

- 1.) Overview
- \* 2.) Langmuir Turbulence - { Basics  
Eikonal Analysis
- 3.) Adiabatic Response vs. Adiabatic Elimination (Zwanzig - Mori)
- \* 4.) Langmuir Collapse  $\Rightarrow$  Self-Focusing

## i.) Overview - Non-local Energy Transfer

the problem of two interacting mode/structure populations is generic.



→ partial list of examples ⇒  $\left\{ \begin{array}{l} \text{structure formation and} \\ \text{maintenance} \end{array} \right.$

### \* ① Langmuir Turbulence - CLASSIC EXAMPLE

high frequency → electron plasma wave

low frequency → ion-acoustic waves

interactions:  $h \rightarrow L$ : plasmon pressure / ponderomotive force  
 $L \rightarrow h$ : refraction  $dk/dt = -\frac{\partial \omega}{\partial x}$

### ② Disparate Scale IW Interaction - Induced Diffusion

high frequency → internal waves

low frequency → long wave length internal/IW's, "currents"

$h \rightarrow L$ : Wave Reynolds stresses, transport  
 $\langle \tilde{v}_z \tilde{v}_H \rangle, \langle \tilde{v}_z \tilde{\rho} \rangle$

$L \rightarrow H$ : random (i.e. hence 'diffusion') refraction  
 $\langle \delta k^2 \rangle$  in RPA theory

### ③ Drift Wave - Convective Cell / Zonal Flow

N.B.  $\omega_{low} \rightarrow 0$

high frequency → drift wave - HM

low frequency → convective cell

$$\nabla_{\perp}^2 \tilde{\phi} + \nabla \phi \times \hat{z} \cdot \nabla \nabla_{\perp}^2 \tilde{\phi} + \gamma \nabla \nabla_{\perp}^2 \tilde{\phi} = 0$$

$k_{y, cell} \rightarrow 0 \Rightarrow$  zonal flow

$h \rightarrow l$ : drift wave Reynolds stress - stochastic  
 (see for zonal flow,  $\langle \tilde{v}_x \tilde{v}_y \rangle$ ) radiation stress

parametric modulation - coherent  
 instability

$L \rightarrow H$ : induced diffusion in  $k_r \Rightarrow$  "random shearing"

$$\frac{dk_r}{dt} = -\frac{\partial}{\partial x} (\psi + k_r \tilde{v}_E)$$

$$k_r = k_r^{(0)} - k_r \tilde{v}_E' t, \quad \langle \delta k_r^2 \rangle = k_0^2 \langle \tilde{v}_E'^2 \rangle T_G$$

$$\odot \rightarrow \otimes$$

④ Acoustic Wave + Vortex (i.e. flow-induced noise)

high frequency  $\rightarrow$  sound wave  
 low frequency  $\rightarrow$  vortex

$h \rightarrow L$ : ponderomotive acoustic stresses, pressure

$L \rightarrow H$ : refraction of acoustic wave ray in vertical flow field.

and (not really) last and by no means least...

⑤ turbulent dynamo problem, i.e.  $\langle \underline{B} \rangle$  from  $\langle \underline{v} \times \underline{\tilde{B}} \rangle$

'high frequency'  $\rightarrow$  fluid turbulence / waves

'low frequency'  $\rightarrow$  mean  $\langle \underline{B} \rangle$  field

question: is, in some sense, a pre-existing bath/gas of turbulence (MHD) "unstable" to amplification of  $\langle \underline{B} \rangle$ .

high freq  $\rightarrow$  low freq: mean EMF  $\langle \underline{v} \times \underline{\tilde{B}} \rangle$ , calculated  
 ② perturbatively

$\mathcal{R} \rightarrow \mathcal{h}$ : evolving concept

i.e.  $\langle \underline{B} \rangle \rightarrow \langle \underline{\tilde{B}}^2 \rangle \rightarrow$  back reaction on EMF  
 $\langle \underline{B} \rangle \rightarrow$  bending-induced stabilization of drive

N.B. 'Classic dynamo' is 'generic fluid turbulence' driven but can have wave-driven dynamo.

Note: ①  $\rightarrow$  ⑤ have very similar theoretical structure

Here, will focus on Langmuir turbulence.



## (ii.) Basics of Langmuir Turbulence

### → Zakharov Equations

Basic Physics: - acoustic wave density perturbation  
 refracts plasma waves  
 - plasma waves form pressure field for  
 acoustic wave

plasma wave:  $\underline{E} = \underbrace{\mathcal{E}(x,t)}_{\substack{\delta \\ \text{envelope} \\ (\text{slow})}} \underbrace{e_0 \exp[i(\underline{k} \cdot \underline{x} - \omega t)]}_{\substack{\delta \\ \text{carrier}}}$

$$\omega^2 = \omega_{pe}^2 + \gamma_T k^2 v_{th}^2$$

$$\therefore -\frac{\partial^2 \underline{E}}{\partial t^2} = \omega_p^2 (1 + \delta n) \underline{E} - \gamma_T v_T^2 \nabla^2 \underline{E}$$

$\downarrow$   
 refracting perturbation

$$\Rightarrow \underbrace{\omega_p^2}_{\delta n} \underline{E} + i \omega_p \frac{\partial \delta n}{\partial t} \underline{E} - \frac{\partial^2 \underline{E}}{\partial t^2} \stackrel{h.o.}{=} \omega_p^2 \underline{E} + \delta n \omega_p^2 \underline{E} - \gamma_T v_T^2 \nabla^2 \underline{E}$$

$$\therefore \left\{ \begin{array}{l} i \frac{\partial \underline{E}}{\partial t} + \lambda_D^2 \nabla^2 \underline{E} = \delta n \underline{E} \\ \downarrow \qquad \qquad \downarrow \\ \text{diffraction} \qquad \text{refraction} \end{array} \right.$$

For acoustic waves:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P \quad P = P_{\text{Thermal}} + P_{\text{Plasma Wave}}$$

$$\tilde{P} = \underbrace{c_s^2}_{\text{thermal}} \tilde{\rho} + \underbrace{\frac{|\tilde{\mathbf{E}}|^2}_{4\pi\rho_0}}_{\text{radiation}} = P_{\text{Th}} + \frac{\partial (W_G)}{\partial \omega} \bigg|_{\omega_0} \frac{|\tilde{\mathbf{E}}|^2}{8\pi}$$

→ energy density of plasma waves  
→ also via 'ponderomotive force' approach

$$\frac{\partial \tilde{\rho}}{\partial t} = \delta n$$

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta n = \frac{\nabla^2 |\tilde{\mathbf{E}}|^2}{4\pi n_0 m_i}$$

acoustic wave eqn.

Further, let:  $\omega_p t \rightarrow t$   
 $z/\lambda_D \rightarrow x$

$$|\tilde{\mathbf{E}}|^2/4\pi \rightarrow |\tilde{\mathbf{E}}|^2/4\pi n_0 T_0$$

yield dimensionless Zakharov Equations :

envelope Eqs.

$$\left( i \frac{\partial}{\partial t} + \nabla^2 \right) \varepsilon - \delta n \varepsilon = 0$$

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta n - c_s^2 \nabla^2 |\varepsilon|^2 = 0$$

$$c_s^2 = m_e / m_i$$

- coupled envelope equations for

plasma wave envelope:  $\varepsilon(x, t)$

density perturbation:  $\delta n$

- (1) with  $\delta n \rightarrow 0 \Rightarrow$  Schrödinger eqn. for free particle

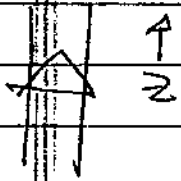
(2) with  $|\varepsilon|^2 \rightarrow 0 \Rightarrow$  acoustic wave equation

- space-time scale separation is crucial to both derivations.

- can estimate c/s! dimensional analysis (in acoustic equation)



- subsonic Langmuir turbulence problem maps to optical self-focusing



light beam

$$n^2 = \left( 1 + \frac{\Delta n}{n} \frac{|E|^2}{E_0^2} \right)$$

$\downarrow$  index  $\downarrow$

intensity dependent  
perturbation in refractive  
dependence } index

$$\nabla^2 E + \frac{\omega^2 n^2}{c^2} E = 0$$

$$E = \Sigma(z, \underline{x}_\perp) e^{ikz} e^{-i\omega t} \quad \omega^2 = c^2 k^2$$

$\Rightarrow$

$$2ik \frac{\partial \Sigma}{\partial z} + \nabla_\perp^2 \Sigma + k^2 \frac{\Delta n}{n} |E|^2 \Sigma = 0$$

- NLS

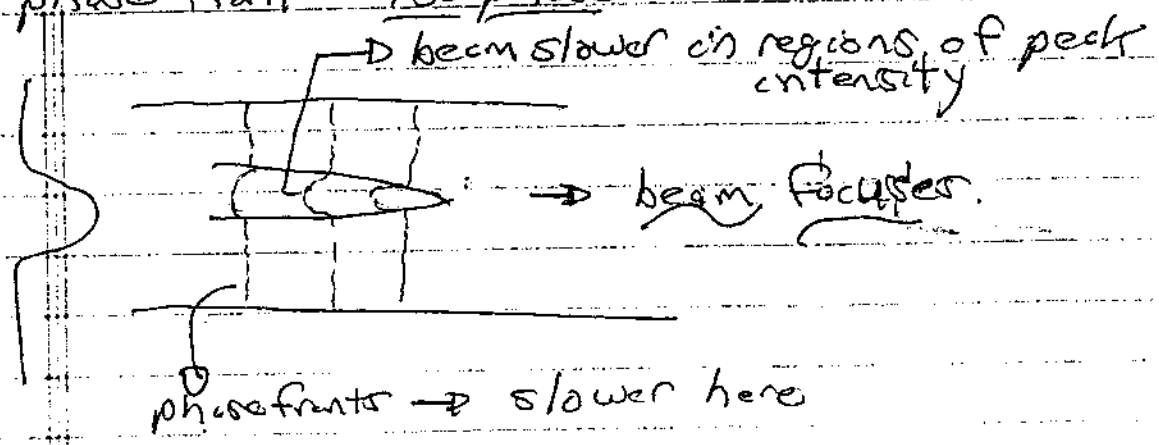
Physics of self-focusing:

$$v_{\text{phase}}^2 = c^2/n^2 = c^2 / \left( 1 + \frac{\Delta n}{n} |E|^2 \right)$$

$\therefore$  phase speed lower in regions of high  
intensity

c.e. consider progression of phase fronts in beam of finite width (rays & phase fronts)

n.b. phase front = iso-phase surface



focus  $\Rightarrow$  singularity formation!

N.B. Aside - Langmuir turbulence of interest both as disparate scale interaction problem and as route to and understandable example of singularity formation in finite time - recall finite time singularity is essence of turbulence problem

more on this in collapse discussion

$E$  independent  $r$  in 3D NST  $\Rightarrow$   
 $\langle \nabla v^2 \rangle \sim E/r$   
 singular velocity gradient

- Linear Theory of  $\left\{ \begin{array}{l} \text{NLS} \\ \text{Self-Focusing} \end{array} \right.$

$$i \partial_t \epsilon + \nabla^2 \epsilon + |\epsilon|^2 \epsilon = 0$$

$$\epsilon = A e^{i\phi} \rightarrow \begin{array}{l} \text{phase} \\ \text{amplitude} \end{array}$$

- 2 fields  $\rightarrow$  NLS is complex equation
- similar to treatment of cycles in CGL

$\Rightarrow$

$$\frac{\partial \phi}{\partial t} + (\nabla \phi)^2 - \left( \frac{\nabla^2 A}{A} + |A|^2 \right) = 0$$

$$\frac{1}{A} \frac{\partial A}{\partial t} + \nabla^2 \phi + 2 \frac{\nabla \phi \cdot \nabla A}{A} = 0$$

so linearizing:

$$A_0 \frac{\partial \tilde{\phi}}{\partial t} + \nabla^2 \tilde{A} + |A_0|^2 \tilde{A} = 0$$

$$\frac{\partial \tilde{A}}{\partial t} + A_0 \nabla^2 \tilde{\phi} = 0$$

$$\begin{Bmatrix} A \\ \phi \end{Bmatrix} = \frac{F}{2\Omega} e^{i(\underline{q} \cdot \underline{x} - \Omega t)}$$

$$\Rightarrow \Omega^2 = -g^2 (|A_0|^2 - g^2)$$

$$\gamma_g^2 = g^2 (|A_0|^2 - g^2)$$

$\uparrow$  self-focusing (destabilizing)       $\uparrow$  diffraction (spreading  $\rightarrow$  stabilization)

NLS tends to amplify  $|E_0|^2$ ,  $\delta n$

$\rightarrow$  instability for  $\frac{|E_0|^2}{4\pi n_0} > k^2 \lambda_0^2$   
 $\uparrow$  self-attraction       $\downarrow$  diffraction

$\rightarrow$  linear marginality at:

$$k^2 \sim \frac{1}{\lambda_0^2} |E_0|^2 / 4\pi n_0 \epsilon_0$$

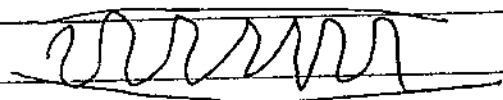
$\rightarrow$  nonlinear evolution / ultimate fate ?



→ Langmuir Turbulence ?

- Zakharov equations, etc. is envelope analysis

- assumes narrow band envelope on carrier



- what of turbulence → multiple structures

low, can write:

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta n = \frac{c_s^2 \nabla^2 |E|^2}{4\pi n_0 T}$$

but - exploit scale separation to note adiabatic invariant

- invariant is wave action density of plasma waves

$$N = \Sigma_{\mathbf{k}} / \omega_{\mathbf{k}}$$

$$\Sigma_{\mathbf{k}} = \frac{\partial (\omega \epsilon)}{\partial \omega} \bigg|_{\omega_{\mathbf{k}}} \frac{|E_{\mathbf{k}}|^2}{8\pi}$$

$$\omega_{\mathbf{k}}^2 = \omega_p^2 (1 + \gamma_{\mathbf{k}} k^2 v_T^2)$$

- view  $N(k, x, t)$  as population density of waves/  
excitons

N.B. Interesting questions occur re: just what  
is  $N$ ?  $\rightarrow$  what is "the correct"  $N$ ?

① de  
de for DWT,

$$N = \epsilon / (\omega + i k_x^2 \nu^2)$$

action  
density

$$= (+k_x^2 \nu^2)^2 |e^{i\mu/T}|^2 / \omega +$$

$$= \Omega / \omega +$$

potential enstrophy

$$\left\{ \begin{array}{l} \frac{d\omega_{pts}}{dt} = 0 \text{ is} \\ H-M \text{ equation} \end{array} \right.$$

action density  $\equiv$  # of waves  
density

potential enstrophy  $\equiv$  # of rotors/vortices  
density

for  $dk_y/dt = 0$  (zonal flow shear field  
i.e.  $\underline{v} = v_y(x) \hat{y}$ )

$N = \Omega$  up to constant factor  $\Rightarrow$  ambiguity  
dissolved (but not resolved)

What of  $\frac{dk_y}{dt} \neq 0$ ?  $\rightarrow$  technical.  
see Smolyakov, P.O. PoP 2000

② related: are "waves" necessary for wave kinetic picture

→ No! see B. Dubrulle and S. Nazarenko, Physics D 1997

ie. considered evolution of 2D turbulence in 2D large scale slow strain field

in 2D,  $\frac{d}{dt} \omega = 0$        $\omega = \nabla^2 \phi$

↪ wavelet field (windowed F.T.)

showed:  $N = |\omega(k, x, t)|^2$       → enstrophy

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \underline{v} \cdot \nabla N - \frac{\partial}{\partial x} (k \cdot v) \cdot \frac{\partial N}{\partial k} = 0$$

$v \equiv$  strain field.

key:  $N$  tied to intensity of quantity conserved along particle trajectories

Suggests in 2D MHD, "magnetic"  $N = |A(k, x, t)|^2$ .

→ investigate  
→ spectral transfer

↑  
wavelet field of magnetic potential

Now, can write:

$$P_{EM} = \omega_p N, \text{ so}$$

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla \cdot \tilde{\mathbf{v}}$$

$$\rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -C_s^2 \nabla \tilde{\rho} - \nabla \int dk \omega_k \tilde{N}(k, x, t)$$

①  
acoustic wave with radiation pressure

if:

$$\frac{dN}{dt} + (v_{gr} \cdot \nabla) N - \frac{\partial}{\partial x} (\omega + k \cdot v) \cdot \frac{\partial N}{\partial k} = 0$$

②  
plasma wave population density

$$\Rightarrow \frac{\partial \tilde{N}}{\partial t} + v_{gr} \cdot \nabla \tilde{N} = + \frac{\partial \tilde{N}}{\partial x} \cdot \frac{\partial \langle N \rangle}{\partial k}$$

where took  $\langle N \rangle = \langle N(k) \rangle$  homogeneous  
( $\Rightarrow$  what is  $\langle N \rangle$ ?, what sets it?)

so

$\rightarrow$  ①, ② together "close loop" if disparate scale feed back.

→  $N_{\text{eff}}$  is effectively a "subgrid scale" model for effect of plasma waves (unresolved) on resolved ion-acoustic wave

reduced degree of freedom  $\left\{ \begin{array}{l} \tilde{N} \leftrightarrow \tilde{N} \\ \downarrow \\ \text{perturbations} \\ \text{of plasma wave} \\ \text{field} \end{array} \right. \rightarrow$  somewhat related to k-E model idea

→ what happens?

$$N = \frac{\Sigma}{\omega} = N_0, \text{ const.} \quad \left( \begin{array}{l} \# \text{ plasmons} \\ \text{conserved} \end{array} \right)$$

$$\Sigma = N \omega$$

$$\frac{d\Sigma}{dt} = N' \frac{\partial \omega}{\partial k} \frac{dk}{dt} = N \frac{d}{dt} \left( \frac{\omega}{\omega_0} \right)$$

For multiple modulation field (i.e. Langmuir turbulence)  $\Rightarrow$

$$dn \Rightarrow d\tilde{n} \\ N \rightarrow \langle N \rangle + \tilde{N} \\ \text{induced by } d\tilde{n}$$

$$\frac{d\epsilon}{dt} = \left\langle (\langle N \rangle + \tilde{N}) \frac{\partial \omega}{\partial k} \left( - \frac{\partial \omega_0}{\partial x} dn \right) \right\rangle$$

$$= - \frac{\partial \omega}{\partial k} \cdot \left\langle \frac{\partial \omega_0}{\partial x} dn \tilde{N} \right\rangle$$

need calculate this correlator.

Proceed via QLT:

$$\langle \rangle = \sum_{\Omega} -i q \omega_0 dn \tilde{N}_{\Omega}$$

$$\tilde{N}_{\Omega} = \frac{\omega_0 i q \tilde{n}_{\Omega} \cdot \partial \langle N \rangle / \partial k}{-i(\Omega - 2 \cdot vgr)}$$

$$\langle \rangle = \sum_{\Omega} \frac{i \omega_0^2 dn_{-\Omega} dn_{\Omega} q \cdot \partial \langle N \rangle / \partial k}{(\Omega - 2 \cdot vgr)}$$

$$= \sum_{\Omega} \omega_0^2 q^2 |dn_{\Omega}|^2 \frac{\partial \langle N \rangle}{\partial k} \cdot \frac{\partial \langle N \rangle}{\partial k}$$

so

$$\frac{dE}{dt} = -v_{gr}(k) \cdot \sum_{\underline{q}, \Omega} \omega_p^2 \underline{q} \cdot \underline{q} |dn_{\underline{q}}|^2 \pi^2 \delta(\Omega - \underline{q} \cdot \underline{v}_{gr}) \cdot \frac{\partial \langle N \rangle}{\partial \underline{q}}$$

now:

$$-v_{gr} = \gamma v_T^2 k / \omega_H > 0 \quad (y \parallel k)$$

so

$$\Rightarrow \frac{dE}{dt} < 0 \quad \Rightarrow \quad \frac{\partial \langle N \rangle}{\partial \underline{q}} > 0$$

$$\text{but } \frac{dE_{\text{plasma wave}}}{dt} < 0 \quad \Rightarrow \quad \frac{dE_{\text{ion acoustic wave}}}{dt} > 0$$

(will prove energy conservation later, but obvious from means of  $dE/dt$ )

so growth of ion-acoustic mode at expense of plasmon gas iff

plasmon population inversion!

→ for this type of "inverse cascade"

— What does the calculation mean?

$$\frac{\partial N}{\partial t} + (\underline{v}_s + \underline{v}) \cdot \nabla N - \frac{\partial (\omega_p^2 n)}{\partial x} \cdot \frac{\partial N}{\partial k} = 0$$

$$\frac{\partial \langle N \rangle}{\partial t} - \frac{\partial}{\partial k} \cdot \left\langle \left( \frac{\partial \omega_p^2 n}{\partial x} \right) \cdot \tilde{N} \right\rangle = 0$$

but  $\epsilon = \omega_p N$

↳ can recognize,

$$\int dk \omega_p \frac{\partial \langle N \rangle}{\partial t} = \int dk \frac{\partial \langle \epsilon \rangle}{\partial t} = \int dk \left[ -v_{gr}(k) \cdot \left\langle \frac{\partial \omega_p^2 n}{\partial x} \tilde{N} \right\rangle \right] dk$$

and

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial k} \cdot \underline{D} \cdot \frac{\partial \langle N \rangle}{\partial k}$$

$$\underline{D} = \omega_p^2 \sum_{\underline{z}, \Omega} \underline{z} \underline{z} |\rho n_{\underline{z}}|^2 \frac{1}{\Omega} \delta(\Omega - \underline{z} \cdot \underline{v}_{gr})$$

induced diffusion of plasma waves by density perturbations (acoustic waves)!



∴ induced diffusion of plasmons generates plasmon  $k$ -space flux

$$\frac{\partial N}{\partial t} + \frac{\partial \cdot \Gamma}{\partial k} = 0$$

$$\Gamma_k = -D \cdot \frac{\partial \langle N \rangle}{\partial k}$$

then  $\frac{dE}{dt} = -v_{gr} \cdot D \cdot \frac{\partial \langle N \rangle}{\partial k}$

∴ flux of certain sign will relax energy  
 $\left( \frac{v_{gr} \cdot \partial \langle N \rangle}{\partial k} > 0 \right)$

- what is origin of irreversibility? - what prevents back cascade forward?

→ ray chaos! ie overlap of  $\Omega$ :  
 $\Omega = \omega \cdot v_{gr}(k)$   
 resonances

- estimate of  $D$  induced?

$$D \sim \langle \omega^2 \omega_n^2 \rangle \tau_{oc}, \quad \tau_{oc} \sim (\Delta / \Omega = \omega \cdot v_{gr})^{-1}$$

→ Conventional Approach: Modulational Instability

→ is gas plasma waves unstable to density/ion-acoustic perturbation?

→ if so, under what conditions?

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla \cdot \tilde{\mathbf{v}}$$

plasmon radiation  
↓

$$\rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\nabla \tilde{p} - \nabla \int dk \omega_p \tilde{N}(k, \mathbf{x}, t)$$

$$\frac{\partial \tilde{N}}{\partial t} + \mathbf{v}_{gr} \cdot \nabla \tilde{N} = \frac{\partial}{\partial \mathbf{x}} \omega_p \frac{\partial \tilde{p}}{\partial t} \cdot \frac{\partial \langle N \rangle}{\partial k}$$

acoustic ( $\underline{q}, \Omega$ )

$$\left( \frac{\partial \tilde{p}}{\partial t} = \frac{\partial \tilde{p}}{\partial t} \right)$$

linear crack ⇒

$$\Omega^2 = \underline{q}^2 c_s^2 - \underline{q}^2 \frac{\omega_p^2}{m_i} \int dk \frac{\underline{q} \cdot \partial \langle N \rangle / \partial k}{(\Omega - \underline{q} \cdot \mathbf{v}_{gr})}$$

usual acoustic term

radiation pressure

proceeding as in P.T.:

$$\Omega = \Omega_{\underline{k}}^{\prime} + i\gamma_{\underline{k}}$$

$$\Omega^2 = \Omega_{\underline{k}}^{\prime 2} + 2i\Omega_{\underline{k}}^{\prime}\gamma_{\underline{k}} - \gamma_{\underline{k}}^2$$

$$\Omega_{\underline{k}}^{\prime 2} + 2i\Omega_{\underline{k}}^{\prime}\gamma_{\underline{k}} = \underline{g}^2 c_s^2 - \frac{\underline{g}^2 \omega_{pe}^2}{m_i} \int d\underline{y} \underline{g} \cdot \frac{\partial \langle \underline{N} \rangle}{\partial \underline{y}} \left\{ \frac{1}{\Omega - \underline{g} \cdot \underline{v}_{gr}} - i\pi \delta(\Omega - \underline{g} \cdot \underline{v}_{gr}) \right\}$$

$$\left\{ \begin{aligned} \Omega_{\underline{k}}^{\prime 2} &= \underline{g}^2 c_s^2 \\ 2i(\frac{1}{2}c_s)\gamma_{\underline{k}} &= i\pi \frac{\underline{g}^2 \omega_{pe}^2}{m_i} \int d\underline{y} \underline{g} \cdot \frac{\partial \langle \underline{N} \rangle}{\partial \underline{y}} \delta(\Omega - \underline{g} \cdot \underline{v}_{gr}) \end{aligned} \right.$$

⇒ instability of ion acoustic wave if  $\frac{\partial \langle \underline{N} \rangle}{\partial \underline{y}} > 0$  as before  
reson

→ analogy to Landau resonance instability obvious.

Can note:

- population inversion needed for modulational instability

↔ if  $\frac{\partial \text{Re}(\omega)}{\partial k} < 0 \rightarrow$  damping

- resonance is just induced diffusion limit of 3 wave resonance, i.e.

$$\begin{aligned} |k| &< |k'| \\ |\omega_k| &< |\omega_{k'}| \end{aligned}$$

$$\begin{aligned} \frac{c}{\omega_{k+k'} - \omega_k - \omega_{k'}} &= \frac{c}{\left( \omega_{k'} + k \cdot \frac{\partial \omega}{\partial k'} - \omega_k - \omega_{k'} \right)} \\ &= \frac{c}{\left( k \cdot \frac{\partial \omega}{\partial k} - \omega_k \right)} \end{aligned}$$

in consistent notation

$$= \frac{c}{\left( \frac{\partial \omega}{\partial k} - \omega \right)}$$

i.e.  $\left. \begin{array}{l} \text{V}_{\text{phase}} \\ \text{long waves} \end{array} \right\} \sim \left. \begin{array}{l} \text{V}_g \\ \text{short waves} \end{array} \right\}$

→ Also ask: how does modulational instability saturate?

- quasilinear flattening of  $N$

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial k} D_k \frac{\partial \langle N \rangle}{\partial k}$$

if  $\frac{\partial \langle N \rangle}{\partial k} \Big|_{\text{reson}}$  → do  $\gamma_E \rightarrow 0$  ie. source/drive depletion

- exploiting analogy of  $\begin{cases} \text{plasma wave} \leftrightarrow \text{"particle"} \\ \text{ion-acoustic wave} \leftrightarrow \text{"wave"} \end{cases}$

can explore:

- trapping of ray in acoustic mode field (local plateau)

- nonlinear ray-ion-acoustic interaction

⇒ analogous to nonlinear Landau damping

ongoing research topic.

→ Revisiting Langmuir Turbulence -  $\left\{ \begin{array}{l} \text{Collapse} \\ \text{singularity} \\ \text{Formation} \\ \text{etc.} \end{array} \right.$

Recall:

① the  $\infty$  question of turbulence (N.S.):

prove  $\nabla V \rightarrow \infty$  as  $\nu \rightarrow 0$ , so  $\epsilon$  finite as  $\nu \rightarrow 0$

i.e.  $\langle (\nabla V)^2 \rangle \sim \epsilon / \nu$

i.e. singularity formation lies at the heart of turbulence physics.

② Zakharov Equations - Langmuir Turbulence

$$i \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2} \omega_p N^2 \nabla^2 \mathcal{E} = \frac{\omega_p}{2n_0} \nabla n \mathcal{E}$$

Aloma wave envelope

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta n = \frac{1}{4\pi M_i} \nabla^2 |\mathcal{E}|^2$$

con-acoustic wave

and in subsonic case  $\rightarrow$  NLS

$$i \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2} \omega_p N^2 \nabla^2 \mathcal{E} = \frac{-e^2}{4\pi m_e \omega_p^2 \epsilon_0} |\mathcal{E}|^2 \mathcal{E}$$

$\left\{ \begin{array}{l} \text{hereafter} \\ \text{the} \\ \text{focus...} \end{array} \right.$

③ linearized stability analysis  $\Rightarrow$  do! EM self-focusing

$$\Rightarrow \text{instability for } \frac{\Sigma_0}{nT} \rightarrow (k \lambda_D)^2 \quad \left\{ \begin{array}{l} \text{intensity} \\ \text{vs.} \\ \text{diffraction} \end{array} \right.$$

$\downarrow$   
plasma  
curved

④ in wave kinetics, interactions tend to remove population inversion  $\Rightarrow$  form low-k condensate

Now nonlinear evolution  $\Rightarrow$  is there a singularity?  
(akin to focus)

- assume spherical symmetry, for simplicity  $\mu = m_e/m_i$

$$r = \frac{3}{\sqrt{2\mu}} \rho, \quad t = \frac{3}{\mu} \frac{r}{\omega_p}, \quad \ln = \frac{2}{3} n_0 \mu V$$

$$E = \pi (2\pi n_0 T \mu)^{1/2} \phi$$

$$\left\{ \begin{array}{l} i \frac{\partial \phi}{\partial t} + \frac{\partial}{\partial \rho} \left( \frac{1}{\rho^2} \frac{\partial \rho^2 \phi}{\partial \rho} \right) + |\phi|^2 \phi = 0 \end{array} \right. \quad (\otimes)$$

Now,  $\Phi$  has two integrals of motion:

$$1) I_1 = \int_0^\infty \rho^2 |\Phi|^2 d\rho \rightarrow \text{plasmon \#}$$

$$2) I_2 = \int_0^\infty \left\{ \frac{\rho(\rho\Phi)^2}{\rho} + 2|\Phi|^2 - \frac{1}{2}\rho^2|\Phi|^4 \right\} d\rho \rightarrow \text{plasmon energy / Hamiltonian}$$

if define variance of plasmon distribution:

$$A = \int d^3x (\Delta x)^2 / |\Phi|^2$$

$$= \int_0^\infty d\rho \rho^4 / |\Phi|^2 \geq 0$$

then straightforward algebra  $\Rightarrow$

$$\frac{d^2 A}{dt^2} = 6I_2 - 2 \int_0^\infty \frac{\rho(\rho\Phi)^2}{\rho} d\rho - 4 \int_0^\infty \rho^2 |\Phi|^4 d\rho < 6I_2$$

so  $A < 3I_2 t^2 + C_1 t + C_2$

$\therefore I_2 < 0 \Rightarrow A \geq 0$  for only finite times

$\Rightarrow$  singularity (i.e. solution breaks down) at finite time.



$\Rightarrow$  Collapse /  $\lambda$ , in finite time  $\Leftrightarrow$  "finite time singularity"

$\therefore$  "cavitation" formed  $\Rightarrow$  can access any dissipation

Now:

$$- I_2 < 0 \Rightarrow \frac{\epsilon_0}{NT} > k^2 \lambda^2$$

- NLS has self similar solution

$$\phi = \exp\left\{i\lambda^2 \int \frac{dt}{F(t)^2}\right\} \frac{1}{\lambda F(t)} R\left(\rho / \lambda F(t)\right)$$

$$\rho / \lambda F(t) = \Sigma, \quad F(t) = \beta (t_0 - t)^{1/2}$$

$\uparrow$   
 singularity

- collapse  $\Rightarrow$  another self-similar route to dissipation

contrast:  $k^4$  cascade