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How rapidly is a passive scalar mixed within closed streamlines?

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The homogenization of a passive 'tracer' in a flow with closed mean streamlines occurs in two stages: first, a rapid phase dominated by shear-augmented diffusion over a time $\approx P^{\frac{1}{3}}(L/U)$, where the Péclet number $P = LU/\kappa$ (L, U and κ are lengthscale, velocity scale and diffusivity), in which initial values of the tracer are replaced by their (generalized) average about a streamline; second, a slow phase requiring the full diffusion time $\approx L^2/\kappa$. The diffusion problem for the second phase, where tracer isopleths are held to streamlines by shear diffusion, involves a generalized diffusivity which is proportional to κ, but exceeds it if the streamlines are not circular. Expressions are also given for flow fields that are oscillatory rather than steady.

1. Introduction

The process of expulsion of the gradient of a conservative scalar field θ (which may be the flux function for a vector field $\boldsymbol{B}$) from a closed two-dimensional streamline pattern is well known from the work of Batchelor (1956). Although he was primarily concerned with the problem where θ is the vorticity, his results apply to any conservative tracer advected by a steady two-dimensional flow field and subject to weak diffusion. Application of these ideas in other contexts has led to diverse theories of magnetic field expulsion from two-dimensional convection cells (Weiss 1966; Moffatt 1978), the resolution of the critical-layer region of a nonlinear wave (Benney & Bergeron 1969; Redekopp 1980) and the three-dimensional circulation of a quasi-geostrophic ocean or atmosphere (Rhines & Young 1982*a*, *b*; Young & Rhines 1982). Because expulsion must compete with other processes of forcing or relaxation, it is important to determine how rapidly it occurs; Batchelor's argument is concerned with the ultimate steady state and gives no clue. Understanding of the rapidity of the process may help with more complex flows, for example when the streamlines pass through a diffusive boundary layer.

In this note we will discuss the initial-value problem

$$\theta_t + J(\psi, \theta) = \kappa \nabla^2 \theta, \tag{1.1a}$$

$$\theta(x, y, 0) = \theta_0(x, y) \tag{1.1b}$$

using several different models for the stream function $\psi(x, y, t)$. Our goal is to understand physically how the passive scalar θ is homogenized by the interaction of the velocity field $\hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi$ and the diffusivity κ. We are especially interested in the timescale of this process because there is some controversy on this point. Note that in this article we confine our attention to problems where θ is not dynamically active. Thus our analysis does not directly apply to the problem considered by Batchelor (1956). Nonetheless, the kinematic problem considered here is an important model whose solution helps in understanding dynamic problems.

Before summarizing the various results that have been previously suggested we will introduce the different timescales and the one non-dimensional parameter suggested by scale analysis.

Let T_e denote the turnover time:

$$T_e = L/U, \tag{1.2}$$

where U is the velocity scale and L is the lengthscale of the flow. Let T_d denote the diffusive timescale:

$$T_d = L^2/\kappa. \tag{1.3}$$

From (1.2) and (1.3) we construct the Péclet number:

$$P \equiv LU/\kappa \tag{1.4a}$$

$$= T_d/T_e. \tag{1.4b}$$

A further timescale is the 'averaging time' T_a:

$$T_a = T_a P^{+\frac{1}{3}} \tag{1.5a}$$

$$= T_d P^{-\frac{2}{3}} \tag{1.5b}$$

(this nomenclature is suggested by the analysis below). Since P must be very large for expulsion to occur

$$T_d \gg T_a \gg T_e \tag{1.6}$$

in the problems that concern us.

Now, from numerical simulations, Weiss (1966) suggested that expulsion occurs on the timescale T_a. A simple theoretical argument given by Moffatt (1978) suggested that the timescale is $P^{-\frac{1}{2}}T_e$. Moffatt & Kamkar (1982) recently showed that this latter argument is incorrect; instead they argued that Weiss's estimate is appropriate. Here we argue that in the most general case the correct timescale for expulsion of gradients is in fact T_d – the diffusive timescale! Weiss's calculations and Moffatt & Kamkar's theoretical analysis involve initial conditions that are 'special' in a sense which will become clear after the discussion below.

We will show that expulsion is best described as shear-augmented dispersion *along* the streamlines ψ = constant. This process replaces the initial condition θ_0 with its (generalized) streamline average $\bar{\theta}_0$. The time taken to do this is T_a. Detailed theory, given below, clarifies this rapid phase of the evolution, and shows the anomalous nature of flow fields involving solid-body rotation in the closed streamline region (an example described by Parker 1966).

The field after the rapid averaging time is given by

$$\bar{\theta}_0(\psi) = \frac{\oint \theta_0(x, y, 0)\,(\mathrm{d}s/|\nabla\psi|)}{\oint (\mathrm{d}s/|\nabla\psi|)}. \tag{1.7}$$

$\bar{\theta}_0(\psi)$ is the initial condition $\theta_0(x, y, 0)$ averaged about streamlines (arclength $\mathrm{d}s$), with allowance made for the non-constant separation between adjacent streamlines. The contour integral in (1.7) is around a closed streamline. The field is uniform in space only if the initial conditions have

$$\frac{\partial \bar{\theta}_0}{\partial \psi} = 0. \tag{1.8}$$

This would be the case for a uniform gradient of tracer, say $\theta_0 = \sin \phi$ and circular streamlines $\psi = \psi(r)$, (r, ϕ) being polar coordinates, but not generally. In all the examples considered by Moffat & Kamkar (1982) and Weiss (1966) (1.8) was satisfied. In general, however, $\bar{\theta}_0$ will not be constant, even if θ_0 has a much larger lengthscale than ψ.

The averaged field $\bar{\theta}(\psi, t)$ now approaches the homogenized state very slowly, on the diffusive timescale T_{d}. Advection is important throughout this process: it is so strong that it 'locks' the θ-field and the ψ-contours. It is this process which finally homogenizes $\bar{\theta}$.

2. Shear dispersion

Our physical understanding of expulsion is based on several simple examples, all of which have a common theme: the interaction of velocity gradients with weak diffusivity produces an accelerated spatial flux of the tracer θ. This discovery was first made by Taylor (1953) in his investigation of shear-flow dispersion in a pipe. This spatial flux is accompanied by an accelerated *spectral* flux of tracer whose Fourier components move from small wavenumbers to large. We will now discuss four different examples which illustrate different aspects of the above process. The first two examples are the most important, and the reader may skip to §3 after these without loss of continuity.

(i) *In a steady plane shear-flow* $\boldsymbol{u} = (U, 0)$, $U = U_0 \cos my$. The tracer equation

$$\theta_t + \boldsymbol{u}\cdot\nabla\theta = \kappa\nabla^2\theta \tag{2.1}$$

becomes

$$\theta_t + U\theta_x = \kappa\nabla^2\theta.$$

Enhanced diffusion occurs along lines $y =$ constant ((x, y) are spatial coordinates). The equations for the moments $\langle x^n\theta\rangle \equiv \int_{-\infty}^{\infty} x^n\theta\,\mathrm{d}x$ are closed, and in particular

$$\langle x^2\theta\rangle_t - \kappa\langle x^2\theta\rangle_{yy} = 2\langle\theta\rangle\frac{u_0^2}{km^2}\cos^2 my + 2\kappa\langle\theta\rangle$$

(see Aris 1956; Saffman 1962; Young, Rhines & Garrett 1982).

For an initial y-independent tracer distribution the solution is

$$\langle x^2\theta\rangle = \langle x^2\theta\rangle_{t=0} + At + B(1 - \mathrm{e}^{-4m^2\kappa t})\cos 2my, \tag{2.2}$$

where

$$A = \left[2\kappa + \frac{U_0^2}{\kappa m^2}\right]\langle\theta\rangle,$$

$$B = \frac{U_0^2}{4\kappa^2 m^4}\langle\theta\rangle.$$

This is essentially Taylor's (1953) result with a diffusivity enhancement $U_0^2/\kappa m^2$, *inversely* proportional to molecular diffusivity, describing the increase in tracer variance along streamlines. The process acts like a simple diffusivity along streamlines only after the cross-stream molecular diffusion distance $(\kappa t)^{\frac{1}{2}}$ exceeds the scale of the flow m^{-1}. The inverse dependence of effective diffusivity on κ holds only in this sense, and it might be termed the strongly diffusive case. Of more interest here is the limit, (ii), *for small κ where the enhanced rate of dispersion along streamlines varies directly with κ* (Okubo 1967; Young *et al.* 1982). In order to illustrate this process it is illuminating to solve (2.1) using a different method. We assume that $U = \alpha y$: this is a local expansion about a point which is adequate provided one is not near points where the velocity gradient vanishes: using the moment method one can show that the θ-field decays more slowly in the neighbourhood of these points. As an initial condition we suppose that

$$\theta(x, y, 0) = \cos kx.$$

More general initial conditions can be produced by Fourier superposition. If $\kappa = 0$ the solution of (2.1) with this initial condition is

$$\theta = \cos k(x - \alpha y t).$$

In order to obtain the solution with $\kappa \neq 0$ one substitutes the ansatz

$$\theta = A(t) \cos k(x - \alpha y t)$$

into (2.1) and obtains an ordinary differential equation for A:

$$A_t + \kappa(1 + \alpha^2 t^2) k^2 A = 0. \tag{2.3}$$

The complete solution is

$$\theta = \exp\left[-\kappa k^2 t - \tfrac{1}{3}\kappa k^2 \alpha^2 t^3\right] \cos k(x - \alpha y t). \tag{2.4}$$

Advection causes the y-wavenumber $(\alpha t) k$ to increase linearly with time as stripes of tracer are tipped over and extended. At a time

$$t_* \approx (\kappa k^2 \alpha^2)^{-\frac{1}{3}} \approx \frac{L}{U} P^{-\frac{1}{3}},$$

where $P \equiv U/k\kappa$ = Péclet number, diffusion takes hold and the subsequent destruction of the Fourier component is sudden, like e^{-t^3}. The cross-over from advection to diffusion is particularly clear if one calculates the tracer dissipation $\kappa\overline{|\nabla \theta|^2}$ (figure 1):

$$\kappa\overline{|\nabla \theta|^2} = \tfrac{1}{2}\kappa k^2 (1 + \alpha^2 t^2) \exp\left[-2\kappa k^2 t - \tfrac{2}{3}\kappa \alpha^2 k^2 t^3\right].$$

Unlike Taylor's limit, in which mixing across streamlines to confining walls (or lines of symmetry) is rapid, the present case does not lead to a simple 'effective' diffusivity. A Gaussian stripe

$$\theta_0 = \frac{\pi}{2a} \exp\left(\frac{-x^2}{4a^2}\right)$$

leads to the solution

$$\theta = \frac{\pi}{2\tilde{a}} \exp\left[\frac{-\tilde{x}^2}{4\tilde{a}^2}\right],$$

$$\tilde{x} = x - \alpha y t,$$

$$\tilde{a}^2 = a^2 + \kappa t + \tfrac{1}{3}\kappa \alpha^2 t^3.$$

So the 'diffusivity' is time-dependent:

$$\frac{d}{dt}\tilde{a}^2 = \kappa + \kappa \alpha^2 t^2,$$

the time-dependence of tracer spreading is the same as in Richardson's famous $L^{\frac{4}{3}}$

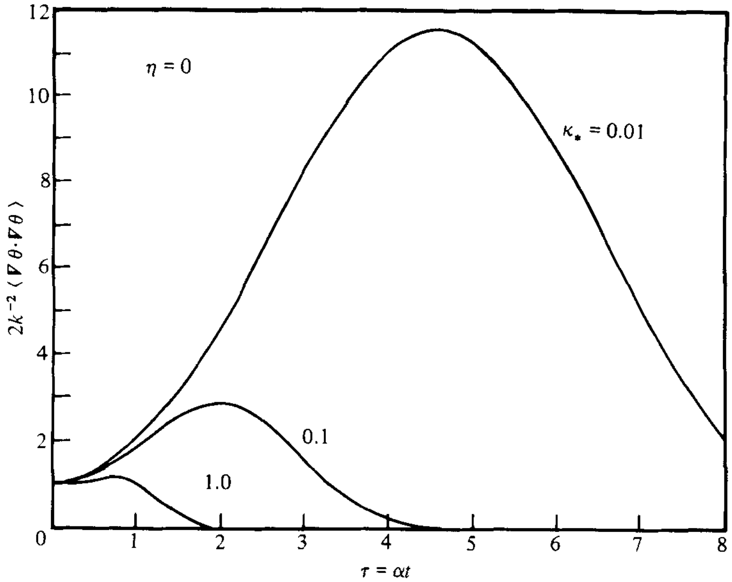

FIGURE 1. The mean-square tracer gradient as a function of time, for various $\kappa_* \equiv \kappa k^2/\alpha$. The ordinate is proportional to the dissipation rate for variance of θ.

law, but for very different reasons than envisaged by him. The stripe expands in width (along the x-axis) like $t^{\frac{3}{2}}$, faster even than with pure advection. This is of course due to diffusion across streamlines from a region of faster or slower flow.

The Fourier analysis may be continued to consider an initial 'spot' of dye. This is the form originally considered by Carter & Okubo (1965). They find

$$\theta(x,y,t) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left[\frac{(x-\frac{1}{2}\alpha yt)^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right], \tag{2.5}$$

where

$$\sigma_y^2 = 2\kappa t,$$

$$\sigma_x^2 = \sigma_y^2(1+\tfrac{1}{12}\alpha^2 t^2),$$

$$\to \tfrac{1}{6}\kappa\alpha^2 t^3 \quad \text{as} \quad t \to \infty.$$

The spot expands diffusively at first. Then the shear takes hold, working upon the diffusively growing cross-streamline width. The effects are multiplicative, yielding the t^3 dependence of σ_x^2.

(iii) *If the shearing velocity field is oscillatory in time, say* $U = \alpha y \cos\omega t$, then just the same procedure as for (ii) yields

$$\theta = \exp\left[-\kappa k^2\left(t - \frac{1}{2}\left(\frac{\alpha}{\omega}\right)^2\left(t - \frac{\sin 2\omega t}{2\omega}\right)\right)\right]\cos k(x - \alpha yt)$$

in place of (2.4). At small time ordinary diffusion dominates, at intermediate time $\alpha^{-1} \ll t \ll \omega^{-1}$ we recover (2.4), and at large time θ spreads along streamlines like a diffusive process with enhanced diffusivity:

$$\kappa + \tfrac{1}{2}\kappa\frac{\alpha^2}{\omega^2}.$$

The time it takes for tracer to become uniform along streamlines is thus

$$t_* \sim \frac{L}{U} P\left(\frac{\omega}{\alpha}\right)^2,$$

which may be much quicker than with simple diffusion if the shear α exceeds the frequency ω.

(iv) *Suppose the strain and vorticity take the general form (subject to the lengthscales of tracer variance being small relative to L, the lengthscale of the velocity)*

$$u_i = u_i(0,0) + x_j \frac{\partial u_i}{\partial x_j}$$

or

$$\psi = ax^2 + bxy + cy^2.$$

(subscripts denote two-dimensional vectors, ψ is the stream function). Equation (2.1) is linear in the tracer θ, so that analysis of a single Fourier component

$$\theta = \mathrm{e}^{\mathrm{i}k_i x_i}$$

will suffice.

The evolution equation (Kraichnan 1974) is

$$\dot{k}_i = -k_j \frac{\partial u_j}{\partial x_i}.$$

In the non-diffusive limit $\kappa = 0$ the eigenvectors $\boldsymbol{k} = k_1 + \alpha k_2$ have exponential behaviour $\propto \mathrm{e}^{\lambda t}$, where λ_1, λ_2 are the eigenvalues of $\partial u_i/\partial x_j$. The eigenvalues are real or imaginary, depending upon the sign of the discriminant $J \equiv b^2 - 4ac$.

If $J > 0$ then locally vorticity dominates strain and the streamlines are closed. The λ are pure imaginary and the tracer pattern rotates with a periodic distortion but no systematic stretching (radial lines remain radial).

If $J < 0$ then strain dominates and the streamlines are open. The λ are real and the tracer pattern is stretched, resulting in an exponential increase in the wavenumber (Batchelor 1959). The case of plane shear, $U = \alpha y$, is anomalous: $J = 0$ and the time dependence is algebraic.

In real turbulent flows, a, b and c are functions of time, and the sign of J changes randomly. Nonetheless, straining events still occur sufficiently often to produce an exponential increase in the wavenumber (Kraichnan 1974; Salmon 1980; Rhines 1983). Furthermore, in an interesting study of two-dimensional turbulence, Weiss (1981) has emphasized the role of the dynamical quantity corresponding to J in driving the enstrophy cascade.

3. Expulsion: the rapid stage

The simplest closed streamline pattern of interest is a steady circular flow analogous to plane shear, case (ii) above. Using polar coordinates (r, ϕ),

$$r^{-1}\psi_r = \Omega(r) \quad (r \leqslant 1),$$
$$= 0 \quad (r > 1),$$

with an initial tracer field

$$\theta = \theta_0(r, \phi).$$

The purely advective solution for $\kappa = 0$ is

$$\theta_{\mathrm{A}} = \theta_0(r, \phi - \Omega t).$$

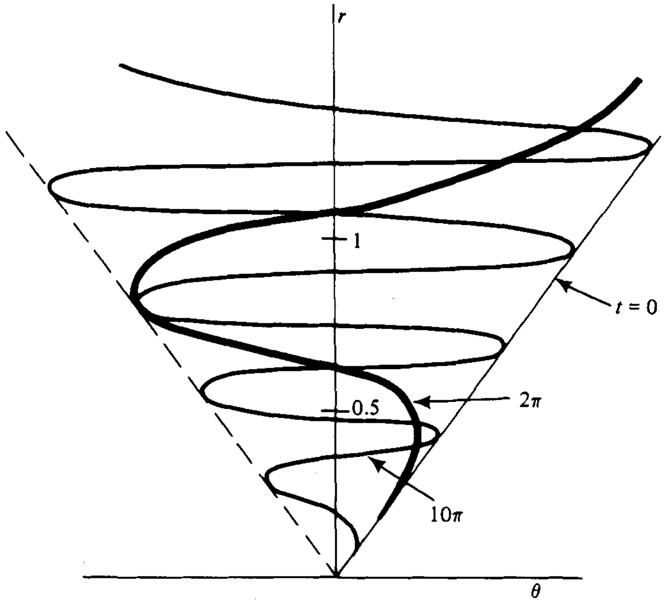

FIGURE 2. The dependence of the advected tracer concentration θ_A (at $\phi = 0$) on radius for the angular velocity field $\Omega(r) = \exp(-r^2)$, with $\theta(t=0) = r\cos\phi$. The wavelength of the concentration decreases like $t^{-1}\Omega_r$.

The radial gradient of θ grows as the isolines of θ are wound up:

$$\theta_{A,r} = \theta_{0,r} - t\Omega_r \theta_{0,\phi}.$$

The process of expulsion is simply the diffusion across adjacent stripes of tracer in the radial direction (figure 2). The width (in r) of the stripes decreases initially like t^{-1}.

The advection–diffusion equation is

$$\theta_t + \Omega(r)\,\theta_\phi = \kappa[r^{-1}(r\theta_r)_r + r^{-2}\theta_{\phi\phi}].$$

Guided by the advective solution, we look at one azimuthal component

$$\theta = A(r,t)\exp[in\tilde{\phi}],$$

where $\tilde{\phi} = \phi - \Omega(r)\,t$. Substituting,

$$A_t = \kappa\{[A_{rr} + r^{-1}A_r - n^2r^{-2}A] + [2in\tilde{\phi}_r A_r + inr^{-1}\tilde{\phi}_r A + in\tilde{\phi}_{rr}A] - n^2\tilde{\phi}_r^2 A\}.$$

The final term stands in ratio Ωt to all the terms in the second brackets, provided that $\Omega_r \sim \Omega/R$, $n \sim 1$ and $L/R \sim 1$, where L is the lengthscale of A, and R the scale of the closed flow pattern. Similarly the final term stands in ratio $(\Omega t)^2$ to the first bracketed set of terms. If diffusion is negligible during the first revolution of the flow ($P = \Omega R^2/\kappa \gg 1$), this shows that

$$A_t + \kappa n^2 t^2 \Omega_r^2 A = O(P,\ (\Omega t)^{-1}).$$

Thus

$$A = \exp[-\tfrac{1}{3}\kappa n^2 \Omega_r^2 t^3],$$

just as in §2. Examining this solution, we find that L/R decreases with time, but slowly enough that A vanishes well before $L/R \ll 1$. And, for a simple large-scale initial tracer field

$$\theta_0 = y = r\sin\phi,$$

the solution is, for $\Omega t \gg 1$,

$$\theta = \exp\left[-\tfrac{1}{3}\kappa\Omega_r^2 t^3\right] r \sin(\phi - \Omega t),$$

in agreement with Moffatt & Kamkar (1982). However, in general let

$$\theta_0 = \sum_0^\infty T_n(r)\, e^{in\phi}.$$

The 'spin-up' solution is then

$$\theta = T_0(r) + \sum_1^\infty \exp\left[-\tfrac{1}{3}\kappa\Omega_r^2 n^2 t^3\right] e^{in\phi}\, T_n(r),$$

which reaches

$$\theta = T_0 = \tfrac{1}{2}\pi \int_0^{2\pi} \theta_0(r, \phi)\, d\phi$$

quickly, by a time $t \sim (\kappa\Omega_r^2 n^2)^{-\frac{1}{3}} \sim (L/U)\,(P)^{+\frac{1}{3}}$ (n is here the lowest significant azimuthal wavenumber present and $\Omega_r \sim \Omega/L$. Notice the failure of 'rapid' averaging if $\Omega_r = 0$. This solid-body rotation is often used as a prototype. Parker (1966) indeed reports a rather lengthy adjustment time for such a velocity field. This rapid phase of expulsion might have been anticipated by equating the streamwise extent of a growing 'spot', (2.5), ($\sim (\frac{1}{12}\kappa\alpha^2 t^3)^{\frac{1}{2}}$), to the circumference of a streamline ($\sim 2\pi R$), to yield: $t \sim P^{\frac{1}{3}}R/U$.

Subsequent to the shear-dispersion stage we are left with a field

$$\theta = \bar{\theta}_0(r)$$

which must relax by pure diffusion, over a much longer time $\approx L^2/\kappa$. Despite the presence of enhanced gradients at the rim of the flow, say $r = 1$, there is no way this process can be accelerated by the shear. The presence of well-mixed regions in a large-scale pattern of tracer θ acts to augment the down-gradient diffusion across the fluid. The added flux is produced by the purely diffusive outer solution ($r > 1$) in response to the modified boundary condition produced at $r = 1$ by the shear.

Collecting the results of §3, we find that the initial averaging process occurs over a time T_a given by:

(i) $T_a \sim L/U)P$ or equivalently L^2/κ for pure diffusion (say with solid-body rotation $\Omega_r = 0$);

(ii) $T_a \sim L/U)P^{\frac{1}{3}}$ or equivalently $(L^2/\kappa)\ P^{\frac{2}{3}}$ for $P \gg 1$, steady shear (this phase being thus much longer than the rotation time L/U but much shorter than the diffusion time L^2/κ);

(iii) $T_a \sim L/U)\,P(\omega L/U)^2$ or equivalently $(L^2/\kappa)\ (\omega L/U)^2$ for oscillatory shear of frequency ω and magnitude $\alpha = U/L$; these estimates will change in obvious ways if the lengthscale of the shear differs greatly from the length of a typical closed streamline. It would be interesting to look at the role of regions of pure strain (iv) in accelerating the expulsion process.

4. Expulsion: the slow stage

In §3 we discussed the fast process by which the initial θ-distribution is shear-dispersed round streamlines in a time T_a. Our discussion focused on circular streamlines, but we mentioned that for an arbitrary ψ the fast process replaces θ_0 by

$$\bar{\theta}_0 = \frac{\oint \theta(ds/|\nabla\psi|)}{\oint (ds/|\nabla\psi|)} \tag{4.1}$$

in a time of order T_a. In (4.1) the integrals are contour integrals around closed streamlines.

We will now discuss this result more carefully and also derive an equation which describes the subsequent evolution of $\bar{\theta}(\psi, t)$. Before beginning we will present some geometric results which will be used in the course of the discussion.

Suppose there is a set of nested, closed, simply connected streamlines in some region of the (x, y)-plane. These curves are given by

$$\psi = \text{constant}.$$

Now, given some scalar function $F(x, y)$, one can construct a function of ψ alone by defining

$$I(\psi) \equiv \iint F(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{4.2}$$

where the area integral is over the area enclosed by a particular ψ-contour. It is straightforward to show that

$$\frac{\mathrm{d}I}{\mathrm{d}\psi} = \oint F \frac{\mathrm{d}s}{|\nabla\psi|} \tag{4.3}$$

(Young 1981), where the line integral above is around the streamline.† Now, return to the advection–diffusion problem

$$\theta_t + J(\psi, \theta) = \kappa\nabla^2\theta \tag{4.4}$$

and define

$$\bar{\theta}(\psi, t) \equiv \frac{\oint \theta(\mathrm{d}s/|\nabla\psi|)}{\oint (\mathrm{d}s/|\nabla\psi|)}, \tag{4.5}$$

where the line integrals are around the streamline ψ. From (4.4) and (4.5) it is easily seen that

$$\bar{\theta}_t = \frac{\kappa\oint \nabla^2\theta(\mathrm{d}s/|\nabla\psi|)}{\oint(\mathrm{d}s/|\nabla\psi|)}, \tag{4.6}$$

since

$$\oint J(\psi, \theta)\,(\mathrm{d}s/|\nabla\psi|) = \oint \nabla\theta\cdot\mathrm{d}\boldsymbol{s}$$
$$= 0,$$

where $\mathrm{d}\boldsymbol{s}$ is the vector line element round the streamline.

Now during the rapid process the right-hand side of (4.6) is negligible because:

$$T_a = P^{-\frac{2}{3}}T_d \ll T_d.$$

Physically this means that the amount of θ between two streamlines changes very little on the timescale T_a. This is because shear dispersion enhances mixing along, but not across, streamlines. The upshot of the rapid-averaging process is then

$$\theta \rightarrow \bar{\theta}_0 = \frac{\oint \theta_0(\mathrm{d}s/|\nabla\psi|)}{\oint (\mathrm{d}s/|\nabla\psi|)} \tag{4.7}$$

† Thus, for example, the circulation $\Gamma = \int \nabla\psi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s$ about a streamline and the kinetic energy $E = \frac{1}{2}\int \nabla\psi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y$ contained within it are related by $\Gamma = 2\,\partial E/\partial\psi$.

provided that

$$T_{\mathrm{a}} \ll t \ll T_{\mathrm{d}}. \tag{4.8}$$

Our goal is now to find an evolution equation for θ when

$$t \approx T_{\mathrm{d}}.$$

After the averaging process is complete we expect that the second term in (4.4) is much larger than the other two. Thus

$$\theta = \bar{\theta}(\psi, t), \tag{4.9}$$

because the advection is now much stronger than the perturbations due to diffusion. An evolution equation for $\bar{\theta}$ is found by substituting (4.9) into the exact result (4.6) and using (4.3). Thus

$$\int \nabla^2\theta(\mathrm{d}s/|\nabla\psi|) = \frac{\partial}{\partial\psi}\iint \nabla^2\theta \,\mathrm{d}x\,\mathrm{d}y \tag{4.10a}$$

$$= \frac{\partial}{\partial\psi}\iint \nabla^2\bar{\theta} \,\mathrm{d}x\,\mathrm{d}y \tag{4.10b}$$

$$= \frac{\partial}{\partial\psi}\oint \nabla\bar{\theta}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s \tag{4.10c}$$

$$= \frac{\partial}{\partial\psi}\bar{\theta}_\psi \oint \nabla\psi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s, \tag{4.10d}$$

where we have used

$$\nabla\bar{\theta} = \bar{\theta}_\psi \nabla\psi$$

to go from (4.10c) to (4.10d). Since the contour of integration is a streamline, $\bar{\theta}_\psi$ can be taken outside the integral. Substituting (4.10d) into (4.6) gives the evolution equation for $\bar{\theta}$:

$$\bar{\theta}_t = \frac{\kappa(\partial/\partial\psi)\,[\zeta(\psi)\,\partial\bar{\theta}/\partial\psi]}{\oint(\mathrm{d}s/|\nabla\psi|)}, \tag{4.11}$$

where

$$\zeta(\psi) = \int \nabla\psi\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s$$

$$= \int \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{s}$$

$$= \text{circulation round the streamline.}$$

Equation (4.11) is the desired evolution equation which describes the slow evolution of the θ-field after the rapid process. It is roughly a diffusion equation in (ψ, t)-space. This analogy can be strengthened by using the area enclosed by the ψ-contour as a new independent variable. Thus

$$A(\psi) = \iint \mathrm{d}x\,\mathrm{d}y,$$

and from (4.5)

$$\frac{\mathrm{d}A}{\mathrm{d}\psi} = \oint\left(\frac{\mathrm{d}s}{|\nabla\psi|}\right),$$

so that (4.11) is

$$\bar{\theta}_t = \kappa\frac{\partial}{\partial A}\left[D(A)\frac{\partial\bar{\theta}}{\partial A}\right], \tag{4.12}$$

where

$$D(A) = \zeta(\psi(A))\frac{\mathrm{d}A}{\mathrm{d}\psi}.$$

Equation (4.12) is precisely a diffusion equation in (A, t)-space where the diffusivity depends on 'position' (i.e. A).

As an example, consider a flow with steady uniform vorticity in an elliptical container with semi-axes a and b (Batchelor 1967, p. 539). (There is one difficulty here: the rapid process discussed in §3 does not occur for this 'solid-body' motion. Instead we imagine that the initial condition is contrived so that θ_0 and ψ contours coincide.) The stream function is

$$\psi = \psi_0\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right],$$

and the area inside a ψ-contour is

$$A(\psi) = \pi\,(\text{major axis})\,(\text{minor axis}) = ab\frac{\psi}{\psi_0},$$

while the circulation is

$$\zeta(\psi) = 2\psi_0\left[\frac{a^2+b^2}{a^2b^2}\right]A(\psi).$$

It follows that

$$D(A) = 2\pi\left[\frac{a^2+b^2}{ab}\right]A,$$

so that (4.14) is just the familiar radially symmetric diffusion equation with A playing the role of the radial coordinate. Of course the θ-pattern is not radially symmetric: the θ contours coincide with the elliptical streamlines. The 'effective diffusivity' κ_e, which is responsible for the slow migration of ν-contours across ψ-contours and the eventual homogenization of θ, is

$$\kappa_e = \left[\frac{a^2+b^2}{2ab}\right]\kappa \tag{4.13a}$$

$$\geqslant \kappa \tag{4.13b}$$

(note how κ_e is equal to κ if the streamlines are circular). The result (4.13) shows that, as the aspect ratio of the ellipse becomes more extreme, θ homogeneizes more rapidly. It is easy to see that the theory presented above is irrelevant when

$$\frac{b}{a} = O(P^{-\frac{2}{3}}),$$

since for these extreme aspect ratios the homogenization time is comparable to T_a.

As a general result we may show that

$$D(A) \geqslant D^0(A),$$

where D^0 is the effective diffusivity of a circular pattern of streamlines with the same enclosed area as the actual pattern. This follows from

$$\left(\int F^2\,\mathrm{d}s\right)\left(\int G^2\,\mathrm{d}s\right) \geqslant \left(\int FG\,\mathrm{d}s\right)^2,$$

where $F^2 = |\nabla\psi|, G^2 = 1/|\nabla\psi|$; hence

$$D(A) = \oint|\nabla\psi|\,\mathrm{d}s\oint\left(\frac{\mathrm{d}s}{|\nabla\psi|}\right) \geqslant \left(\oint\mathrm{d}s\right)^2 \geqslant 4\pi A$$

$$\equiv D^0(A).$$

5. Conclusions

We have given several variants of shear-augmented tracer dispersion, in order to emphasize the diversity of results that govern expulsion of gradients in realistic velocity fields. Application to unsteady wave- and turbulence fields, in which closed streamlines appear and disappear, is particularly interesting. Kraichnan (1974) and Salmon (1980) have produced some useful results for stochastic velocity fields. It is intriguing to wonder whether, say, two-dimensional turbulence has enough persistence in its flow pattern for some signs of homogenization to develop. Because a tracer homogenizes on the diffusive timescale, even slow changes in the stream function may be important.

In geophysical fluids there are persistent lone eddies and gyres within which homogeneization of tracers (and even dynamically active quantities such as potential vorticity) is likely to occur.

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