15-5 Damped Simple Harmonic Motion

- When an external force reduces the motion of an oscillator, its motion is **damped**
- Assume the liquid exerts a **damping force** proportional to velocity (accurate for slow motion)

$$F_d = -bv, \quad \textbf{Eq. (15-39)}$$

- *b* is a damping constant, depends on the vane and the viscosity of the fluid

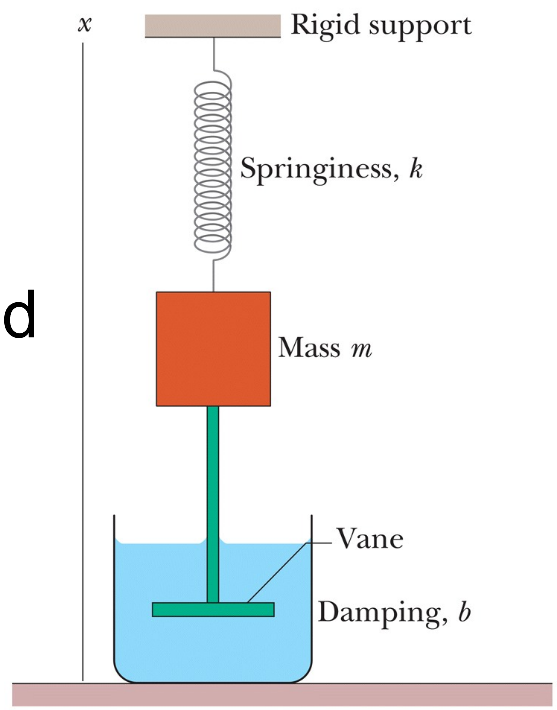


Figure 15-16

15-5 Damped Simple Harmonic Motion

- We use Newton's second law and rearrange to find:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \quad \textbf{Eq. (15-41)}$$

- The solution to this differential equation is:

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi), \quad \textbf{Eq. (15-42)}$$

- With angular frequency:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \quad \textbf{Eq. (15-43)}$$

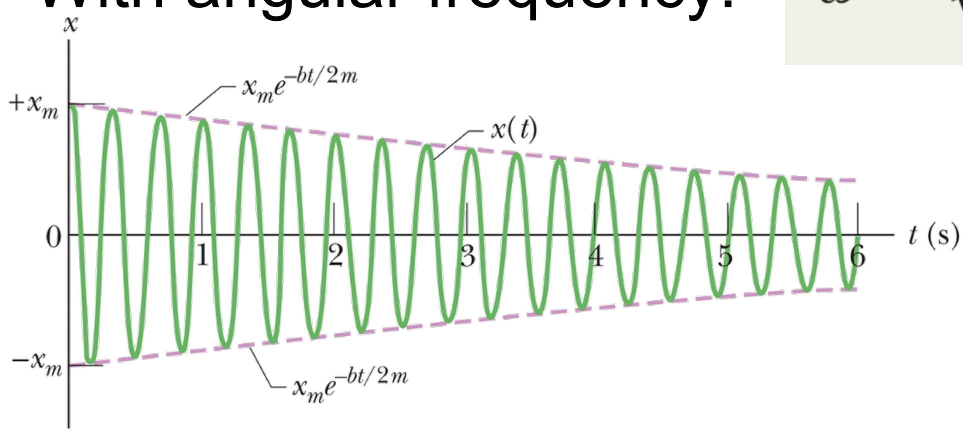


Figure 15-17

$$m\ddot{x} = -kx - b\dot{x} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Complex exponentials

$$x = x_m e^{\Omega t + \phi}; \quad \dot{x} = x_m \Omega e^{\Omega t}; \quad \ddot{x} = x_m \Omega^2 e^{\Omega t}$$

$$m\Omega^2 + b\Omega + k = 0 \Rightarrow \Omega^2 + \frac{b}{m}\Omega + \frac{k}{m} = 0$$

$$\Omega = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2} = -\frac{b}{2m} \pm i\underbrace{\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}}_{\omega'}$$

$$x(t) = x_m e^{-\frac{b}{2m}t} e^{i\omega' t + \phi} \qquad ax^2 + bx + c = 0$$

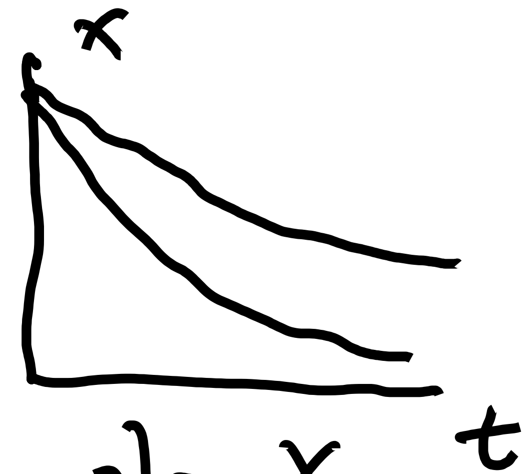
$$\mathrm{Re}\,x(t) = x(t) = x_m e^{-\frac{b}{2m}t} \cos(\omega' t + \phi) \qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$m\ddot{x} = -kx - b$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

Complex exponentials

$$x = x_m e^{\Omega t + \phi};\ \dot{x}$$



$$x(t=0) = X_0$$

$$v(t=0) = V_0$$

$$m\Omega^2 + b\Omega + k = 0 \Rightarrow$$

$$\Omega = -\frac{b}{2m} \pm \sqrt{\underbrace{\left(\frac{b}{2m}\right)^2 - \omega^2}_{>0}} = \Omega_{1,2} < 0$$

$$x(t) = \cancel{x_m e}\ a e^{-|\Omega_1|t} + b e^{-|\Omega_2|t}$$

15-6 Forced Oscillations and Resonance

- This condition is called **resonance**
- This is also approximately when the displacement amplitude is largest
- Resonance has important implications for the stability of structures
- Forced oscillations at resonant frequency may result in rupture or collapse

x_m

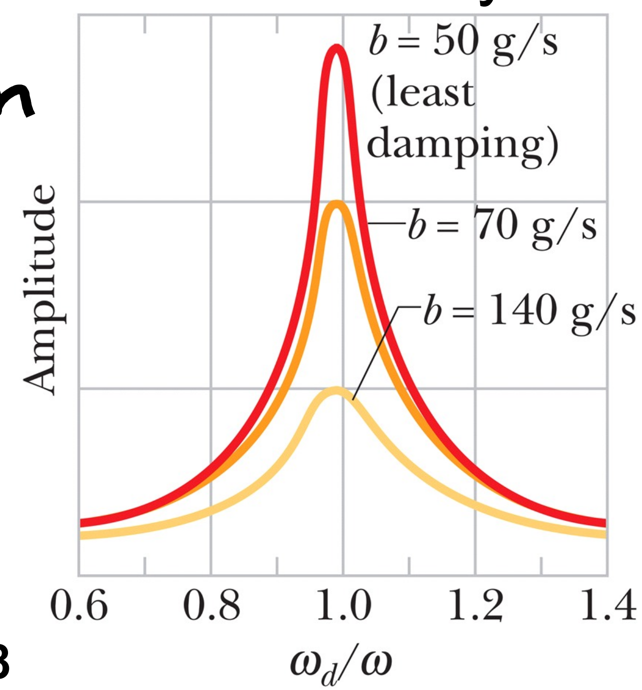


Figure 15-18

$$m\ddot{x} + b\dot{x} + kx = \boxed{f\cos \omega_d t \;\Big|\; x = x_m \cos(\omega_d t + \phi)}$$

$$\underbrace{f\cos \omega_d t}_{f e^{i\omega_d t}}$$

$$x = x_m e^{i(\omega_d t + \phi)}$$

$$\dot{x} = x_m i\omega_d e^{i(\omega_d t + \phi)}$$

$$\ddot{x} = -x_m \omega_d^2 e^{i(\omega_d t + \phi)}$$

$$-m\omega_d^2 x_m e^{i\phi} + ib\,\omega_d x_m e^{i\phi} + k x_m e^{i\phi} = f$$

solve for x_m, ϕ

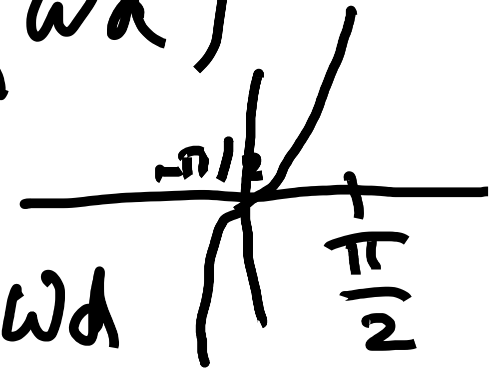
$$x_m\left(\cancel{k}\,\omega^2 + i\frac{b}{m}\omega_d - \cancel{m}\,\omega_d^2\right)e^{i\phi} = f/m \quad \Big|\; \frac{k}{m} = \omega^2$$

$$x_m = \frac{f/m}{\omega^2 - \omega_d^2 + i\frac{b}{m}\omega_d}\,e^{-i\phi}$$

x_m is real
so $x_m = |x_m|$
$|e^{-i\phi}| = 1$

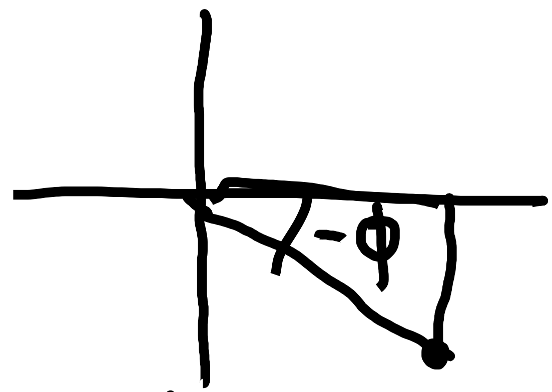
$$\boxed{x_m = \frac{f/m}{\sqrt{(\omega^2 - \omega_d^2)^2 + b^2\omega_d^2/m^2}}}$$

$$e^{-i\phi} = \frac{x_m}{f/m}\left(\omega^2 - \omega_d^2 + i\frac{b}{m}\omega_d\right)$$



$$\tan\phi = -\frac{\mathrm{Im}(e^{-i\phi})}{\mathrm{Re}(e^{-i\phi})} = \frac{-\frac{b}{m}\omega_d}{\omega^2 - \omega_d^2}$$

$$\boxed{\tan\phi = \frac{\frac{b}{m}\omega_d}{\omega_d^2 - \omega^2}}$$



$\omega_d \to \omega$ $\qquad x(t) = x_m \cos(\omega_d t + \phi)$

$\phi \to \pm\frac{\pi}{2}$ $\qquad F_d(t) = f\cos(\omega_d t)$

$$v(t) = \dot{x}(t) = -\omega_d x_m \sin(\omega_d t + \phi)$$

15 Summary

Frequency

- 1 Hz = 1 cycle per second

Period

$$T = \frac{1}{f}. \quad \text{Eq. (15-2)}$$

The Linear Oscillator

$$\omega = \sqrt{\frac{k}{m}} \quad \text{Eq. (15-12)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{Eq. (15-13)}$$

Simple Harmonic Motion

- Find v and a by differentiation

$$x(t) = x_m \cos(\omega t + \phi) \quad \text{Eq. (15-3)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f. \quad \text{Eq. (15-5)}$$

Energy

- Mechanical energy remains constant as K and U change
- K = ½ mv^2, U = ½ kx^2

15 Summary

Pendulums

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$ **Eq. (15-23)**

$$T = 2\pi\sqrt{\frac{L}{g}}$$ **Eq. (15-28)**

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$ **Eq. (15-29)**

Simple Harmonic Motion and Uniform Circular Motion

- SHM is the projection of UCM onto the diameter of the circle in which the UCM occurs

Damped Harmonic Motion

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi),$$

Eq. (15-42)

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$ **Eq. (15-43)**

Forced Oscillations and Resonance

- The velocity amplitude is greatest when the driving force is related to the natural frequency by:

$$\omega_d = \omega$$ **Eq. (15-46)**

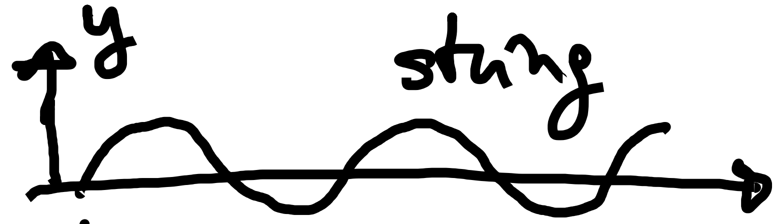
Waves - I

$m\ddot{x} = -kx - b$

$\ddot{x} + b\dot{x} + \frac{k}{m}x \quad \sin(\overbrace{kx - \omega t}^{\text{phase} = \phi})$

Complex exponentials

$\phi x = kx_m e^{\Omega t + \phi}$; $\dot{x}$



transverse wave

longitudinal waves

sound

$$m\Omega^2 + b\Omega + k = 0 \Rightarrow$$

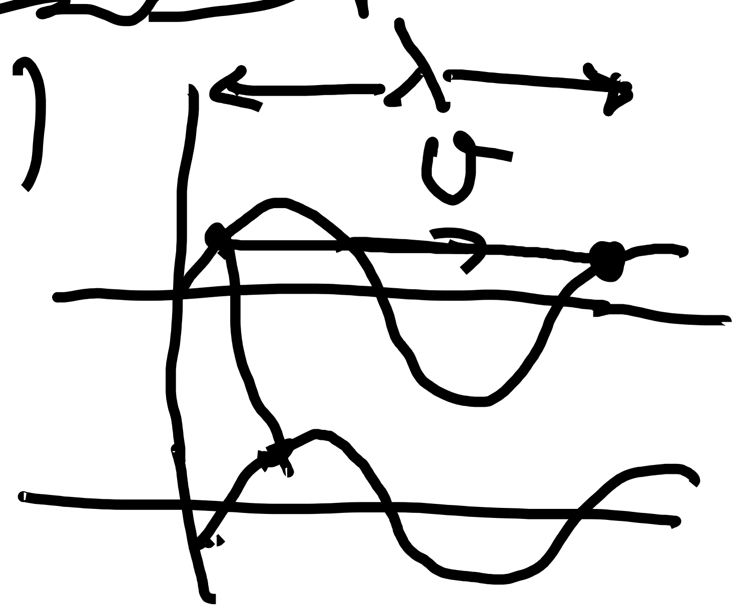
$$\Omega = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2}$$

$$x(t) = x_m e^{-\frac{b}{2m}t} e^{i\omega' t + \phi}$$

$$\mathrm{Re}\, x(t) = x(t) = e^{-\frac{b}{2m}t}\cos(\omega' t + \phi)$$

$$y(x,t) = \underbrace{y_m}_{\text{amplitude}} \sin(\underbrace{kx-\omega t}_{\text{phase}})$$

$$= y_m \sin\left(k\left(x - \frac{\omega}{k}t\right)\right)$$



$v = \frac{\omega}{k}$ $\qquad x = vt + \phi$

wavelength: $\lambda = \frac{2\pi}{k}$ $\rightarrow$ $\underbrace{k}_{\text{wavenumber}} = \frac{2\pi}{\lambda}$

$y(x+\lambda, t) = y(x,t)$

period: $T = 2\pi/\omega$ $\qquad \omega = \frac{2\pi}{T}$

$y(x, t+T) = y(x,t)$

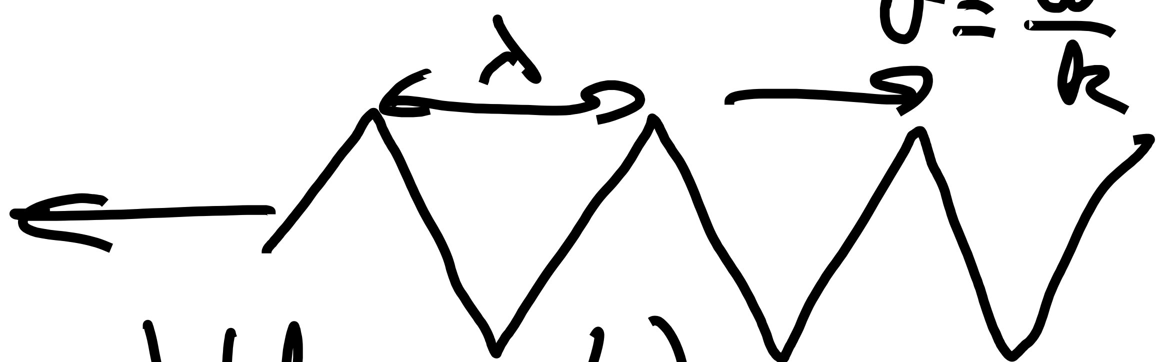
$f = \frac{1}{T} \Rightarrow \omega = 2\pi f$; $v = \frac{\omega}{k} = \lambda \cdot f$

any function

$$y(x,t) = h(kx - \omega t)$$

$v = \frac{\omega}{k}$

wave



$$y(x,t) = h(kx + \omega t)$$

$$h(0.7x + 200t)$$

16-1 Transverse Waves

Types of Waves

1. **Mechanical Waves:** They are governed by Newton's laws, and they can exist only within a material medium, such as water, air, and rock. Examples: water waves, sound waves, and seismic waves.

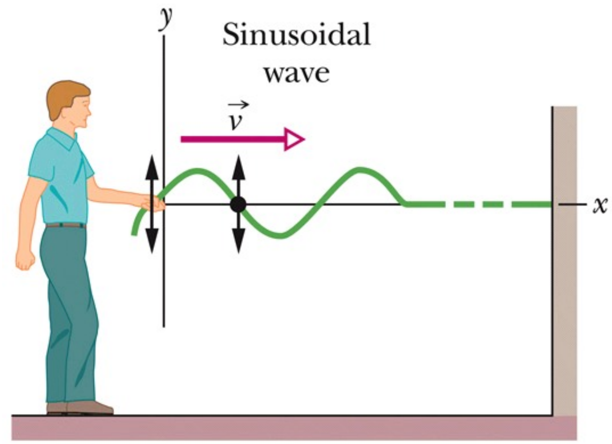
2. **Electromagnetic waves:** These waves require no material medium to exist. Light waves from stars, for example, travel through the vacuum of space to reach us. All electromagnetic waves travel through a vacuum at the same speed c = 299 792 458 m/s.

3. **Matter waves**: These waves are associated with electrons, protons, and other fundamental particles, and even atoms and molecules. Because we commonly think of these particles as constituting matter, such waves are called matter waves.

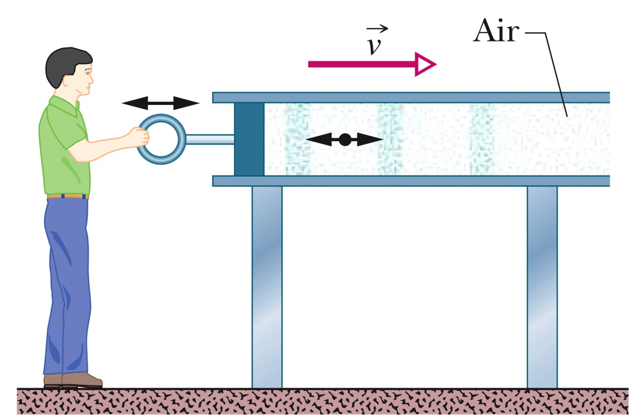
Transverse and Longitudinal Waves

- A sinusoidal wave is sent along the string (Figure (a)). A typical string element moves up and down continuously as the wave passes. This is **transverse wave**.

- A sound wave is set up in an air- filled pipe by moving a piston back and forth (Figure (b)). Because the oscillations of an element of the air (represented by the dot) are parallel to the direction in which the wave travels, the wave is a **longitudinal wave**.



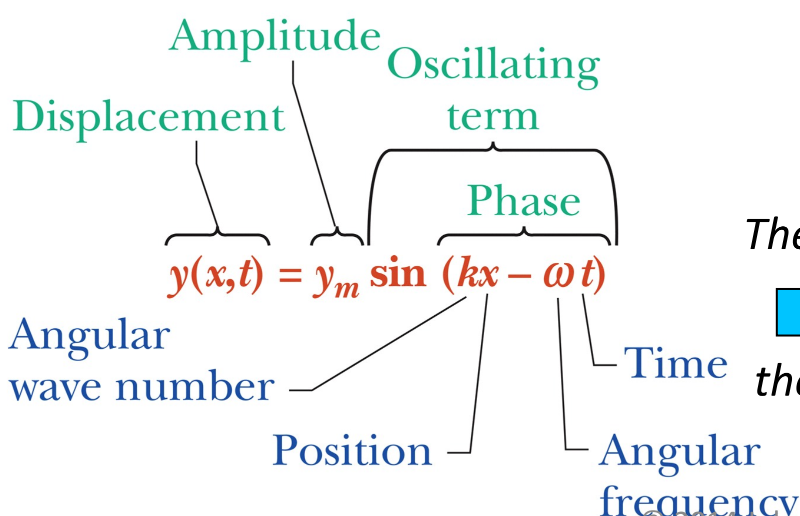
(a) Transverse Wave



(b) Longitudinal Wave

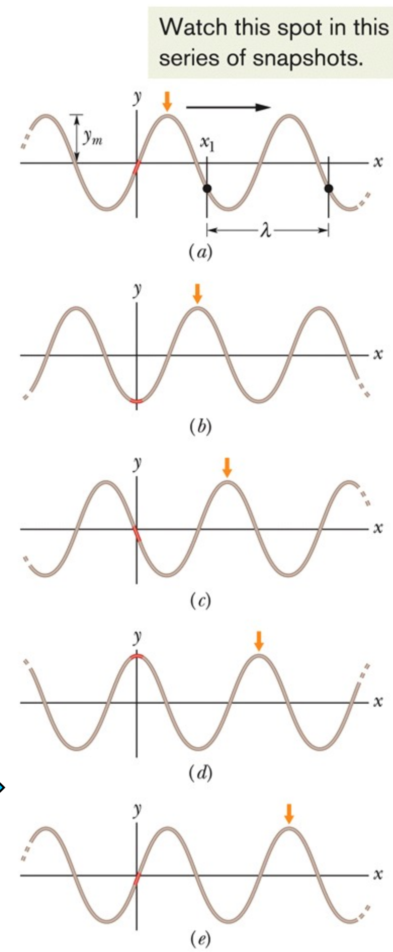
Sinusoidal Function

Five "snapshots" (y *vs* x each at a constant time) of a string wave traveling in the positive direction along an x axis. The amplitude y_m is indicated. A typical wavelength λ, measured from an arbitrary position x_1, is also indicated.



The sine function describes

the shape of the wave

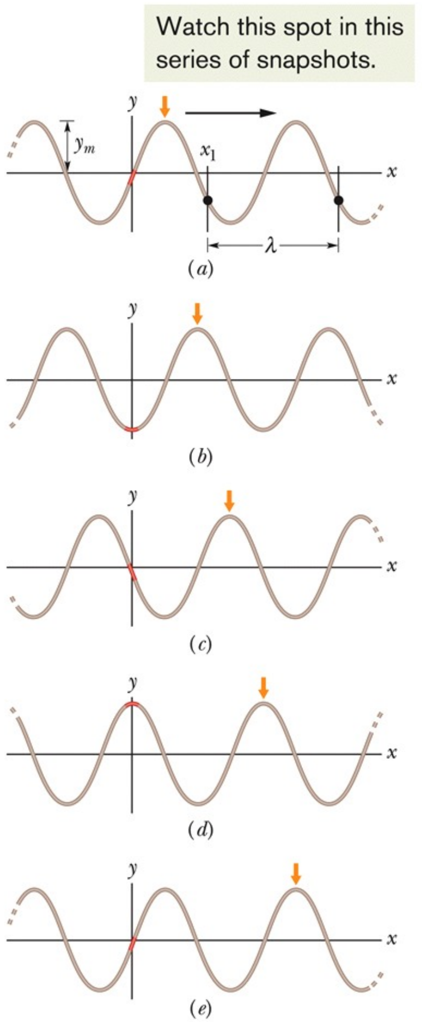


Period, Wave Number, Angular Frequency and Frequency

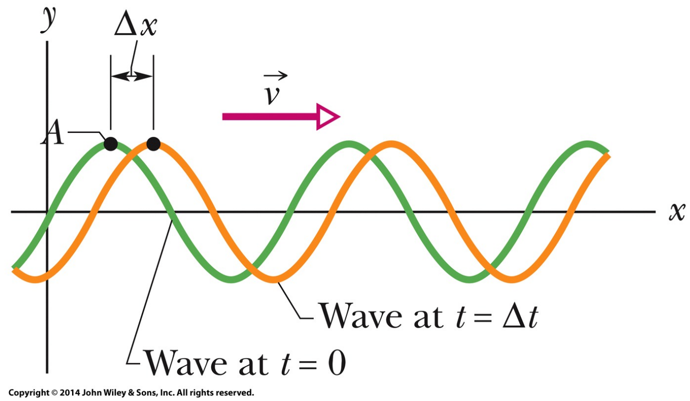
$$k = \frac{2\pi}{\lambda} \quad \text{(angular wave number).}$$

$$\omega = \frac{2\pi}{T} \quad \text{(angular frequency).}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(frequency).}$$



The Speed of a Traveling Wave



Two snapshots of the wave: at time *t=0,* and then at time *t=Δt*. As the wave moves to the right at velocity *v,* the entire curve shifts a distance *Δx* during *Δt.*

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \quad \text{(wave speed)}.$$

$$y(x,t) = y_m \sin(kx + \omega t).$$

16-2 Wave Speed on a Stretched String

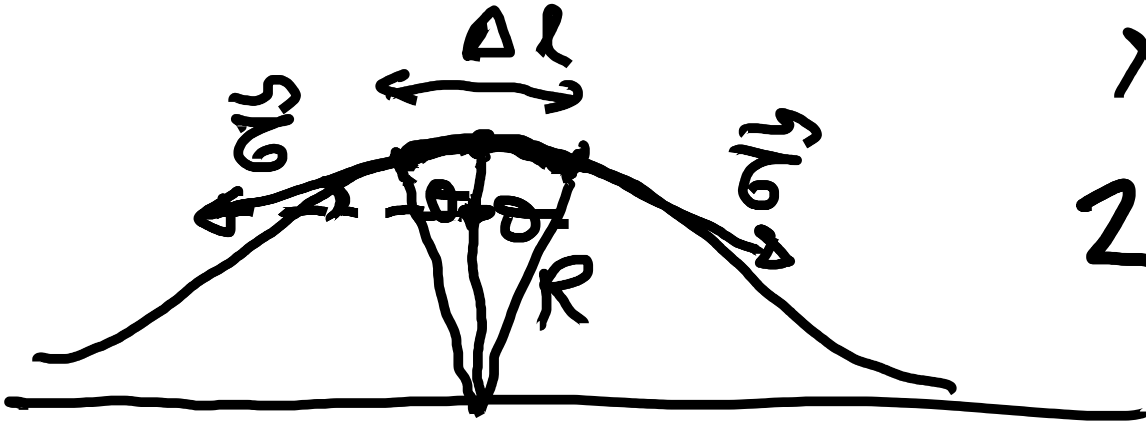
Learning Objectives

The speed of a wave on a stretched string is set by properties of the string (i.e. linear density), not properties of the wave such as frequency or amplitude. Tau is the tension (in N) in the string.

$$\mu = \frac{m}{l} \qquad \text{(linear density)}$$

$$v = \sqrt{\frac{\tau}{\mu}} \quad \text{(speed),}$$

$$\sqrt{\frac{\cancel{kg}\,m}{s^2}\,\frac{m}{\cancel{kg}}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$$



$$\mu = \Delta m / \Delta l$$

$$2\theta = \frac{\Delta l}{R}$$

vertical force

$$F = 2\tau \sin\theta \approx \underbrace{2\tau\cdot\theta} = \overbrace{\mu\,\Delta l}^{\text{mass}}\;\overbrace{\frac{v^2}{R}}^{\text{accel}}$$

$$\Delta m = \mu\,\Delta l$$

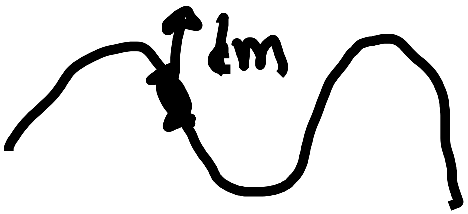
centripetal acceleration $= \dfrac{v^2}{R}$

$$\frac{\cancel{\Delta l}}{\cancel{R}}\,\tau = \mu\,\cancel{\Delta l}\,\frac{v^2}{\cancel{R}} \Rightarrow v^2 = \frac{\tau}{\mu}$$

$$v = \sqrt{\frac{\tau}{\mu}}$$

$$\left.\frac{dU}{dt}\right)_{av} = \left.\frac{dK}{dt}\right)_{av}$$

16-3 Energy and Power of a Wave Traveling along a String



$$y = y_m \sin(kx - \omega t)$$

Learning Objective

$$u = \dot{y} = -\omega y_m \cos(kx - \omega t)$$

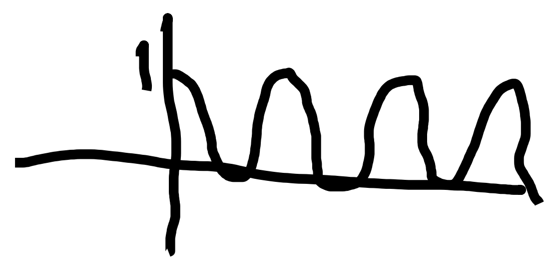
- **16.16** Calculate the average rate at which energy is transported by a transverse wave.

$$dK = \frac{1}{2}\, dm\, u^2 = \frac{1}{2}(\mu\, dx)\, \omega^2 y_m^2 \cos^2(kx - \omega t)$$

$$\frac{dK}{dt} = \frac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$$

$$\left(\cos^2(kx - \omega t)\right)_{av} = \frac{1}{2}$$

$$\left.\frac{dK}{dt}\right)_{av} = \frac{1}{4}\mu v \omega^2 y_m^2$$

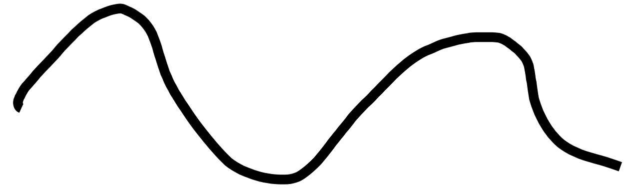


$$\boxed{P_{av} = 2\left.\frac{dK}{dt}\right)_{av} = \frac{1}{2}\mu v \omega^2 y_m^2}$$

string: $\mu = 525\ g/m$

$\tau = 45\ N$

$y_m = 8.5\ mm$



$f = 120\ Hz$

speed ?

$$v = \sqrt{\frac{\tau}{\mu}} = 9.3\ m/s$$

P_{av} ?

$$P_{av} = \frac{1}{2}\mu v \omega^2 y_m^2 = 100\ \frac{J}{s} = 100\ \text{Watt}$$

$\omega = 2\pi f$

Wave equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$y(x,t) =$

$= y_m \sin(kx - \omega t)$

$$\frac{\partial^2 y}{\partial x^2} = -k^2\, y_m \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2\, y_m \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{v = \frac{\omega}{k}}$$