Phys 4C Fall 2019

Chapter 1 Solutions to Exercises

1.35. **Balancing the weight**

Let the desired distance be d. We want the upward electric force $e^2/4\pi\epsilon_0 d^2$ to equal the downward gravitational force mg. Hence,

$$d^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{mg} = \left(9\cdot 10^9\,\frac{\mathrm{kg\,m^3}}{\mathrm{s^2\,C^2}}\right)\frac{(1.6\cdot 10^{-19}\,\mathrm{C})^2}{(9\cdot 10^{-31}\,\mathrm{kg})(9.8\,\mathrm{m/s^2})} = 26\,\mathrm{m}^2, \tag{3}$$

which gives $d = 5.1\,\mathrm{m}$. The non-infinitesimal size of this answer is indicative of the feebleness of the gravitational force compared with the electric force. It takes about $3.6\cdot 10^{51}$ nucleons (that's roughly how many are in the earth) to produce a gravitational force at an effective distance of $6.4\cdot 10^6\,\mathrm{m}$ (the radius of the earth) that cancels the electrical force from *one* proton at a distance of 5 m. The difference in these distances accounts for a factor of only $1.6\cdot 10^{12}$ between the forces (the square of the ratio of the distances). So even if all the earth's mass were somehow located the same distance away from the electron as the single proton is, we would still need about $2\cdot 10^{39}$ nucleons to produce the necessary gravitational force.

1.42. **Potential energy in a 1-D crystal**

Suppose the array has been built inward from the left (that is, from negative infinity) as far as a particular negative ion. To add the next positive ion on the right, the amount of external work required is

$$\frac{1}{4\pi\epsilon_0}\left(-\frac{e^2}{a} + \frac{e^2}{2a} - \frac{e^2}{3a} + \cdots\right) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right). \tag{21}$$

The expansion of $\ln(1+x)$ is $x - x^2/2 + x^3/3 - \cdots$, converging for $-1 < x \le 1$. Evidently the sum in parentheses above is just $\ln 2$, or 0.693. The energy of the infinite chain *per ion* is therefore $-(0.693)e^2/4\pi\epsilon_0 a$. Note that this is an exact result; it does not assume that a is small. After all, it wouldn't make any sense to say that "a is small," because there is no other length scale in the setup that we can compare a with.

The addition of further particles on the right doesn't affect the energy involved in assembling the previous ones, so this result is indeed the energy per ion in the entire infinite (in both directions) chain. The result is negative, which means that it requires energy to move the ions away from each other. This makes sense, because the two nearest neighbors are of the opposite sign.

If the signs of all the ions were the same (instead of alternating), then the sum in Eq. (21) would be $(1+1/2+1/3+1/4+\cdots)$, which diverges. It would take an infinite amount of energy to assemble such a chain.

An alternative solution is to compute the potential energy of a given ion due to the full infinite (in both directions) chain. This is essentially the same calculation as above, except with a factor of 2 due to the ions on each side of the given one. If we then sum over all ions (or a very large number N) to find the total energy of the chain, we have counted each pair twice. So in finding the potential energy per ion, we must divide by 2 (along with N). The factors of 2 and N cancel, and we arrive at the above result.

1.48. **Maximum field from a ring**

At $(0, 0, z)$ the field due to an element of charge dQ on the ring has magnitude $dQ/4\pi\epsilon_0(b^2+z^2)$. But only the z component survives, by symmetry, and this brings in a factor of $z/\sqrt{b^2+z^2}$. Integrating over the entire ring simply turns the dQ into Q, so we have $E_z = Qz/4\pi\epsilon_0(b^2+z^2)^{3/2}$. Setting the derivative equal to zero to find the maximum gives

$$0 = \frac{(b^2+z^2)^{3/2}(1) - z(3/2)(b^2+z^2)^{1/2}(2z)}{(b^2+z^2)^3} = \frac{b^2-2z^2}{(b^2+z^2)^{5/2}} \implies z = \pm\frac{b}{\sqrt{2}}. \quad (31)$$

Since we're looking for a point on the positive z axis, we're concerned with the positive root, $z = b/\sqrt{2}$. Note that we know the field must have a local maximum somewhere between $z = 0$ and $z = \infty$, because the field is zero at both of these points.

1.55. **Field from a finite rod**

In Fig. 13, define the distances: $\ell = 0.05\,\text{m}$, $a = 0.03\,\text{m}$, and $b = 0.05\,\text{m}$. The linear charge density of the rod is $\lambda = (8\cdot10^{-9}\,\text{C})/(0.1\,\text{m}) = 8\cdot10^{-8}\,\text{C/m}$. At point A the field points leftward and has magnitude

$$\begin{aligned}
E_A = \frac{1}{4\pi\epsilon_0}\int_a^{a+2\ell}\frac{\lambda\,dx}{x^2} &= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{a+2\ell}\right) = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{2\ell}{a(a+2\ell)}\right)\\
&= \left(9\cdot10^9\,\frac{\text{kg m}^3}{\text{s}^2\,\text{C}^2}\right)\left(8\cdot10^{-8}\,\frac{\text{C}}{\text{m}}\right)\left(\frac{2(.05\,\text{m})}{(.03\,\text{m})(.13\,\text{m})}\right)\\
&= 1.85\cdot10^4\,\frac{\text{N}}{\text{C}}. \qquad (47)
\end{aligned}$$

As a check, if $a \gg \ell$ this result approaches $(1/4\pi\epsilon_0)(2\ell\lambda/a^2)$, which is correctly the field from a point charge $2\ell\lambda$ at a distance a.

At point B, only the vertical component of the field survives, by symmetry. So the field points downward and has magnitude

$$E_B = \frac{1}{4\pi\epsilon_0}\int_{-\ell}^{\ell}\frac{\lambda\,dx}{b^2+x^2}\cdot\frac{b}{\sqrt{b^2+x^2}}, \quad (48)$$

where the second factor gives the vertical component. This integral can be evaluated with a trig substitution, $x = b\tan\theta \implies dx = b\,d\theta/\cos^2\theta$ (or you can just look it up), which yields

$$\begin{aligned}
E_B &= \frac{1}{4\pi\epsilon_0}\int_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)}\frac{\lambda b^2\,d\theta/\cos^2\theta}{b^3(1+\tan^2\theta)^{3/2}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{b}\int_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)}\cos\theta\,d\theta\\
&= \frac{1}{4\pi\epsilon_0}\frac{\lambda}{b}\sin\theta\bigg|_{-\tan^{-1}(\ell/b)}^{\tan^{-1}(\ell/b)} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{b}\frac{2\ell}{\sqrt{\ell^2+b^2}}\\
&= \left(9\cdot10^9\,\frac{\text{kg m}^3}{\text{s}^2\,\text{C}^2}\right)\frac{(8\cdot10^{-8}\,\text{C/m})}{(.05\,\text{m})}\frac{2(.05\,\text{m})}{\sqrt{(.05\,\text{m})^2+(.05\,\text{m})^2}}\\
&= 2.04\cdot10^4\,\frac{\text{N}}{\text{C}}. \qquad (49)
\end{aligned}$$

The $\int\cos\theta\,d\theta$ integral here is just what you would obtain if you parameterized the rod in terms of θ; see Eq. (1.38).

As a check, if $b \gg \ell$ this result approaches $(1/4\pi\epsilon_0)(2\ell\lambda/b^2)$, which is correctly the field from a point charge $2\ell\lambda$ at a distance b.

1.56. Flux through a cube

(a) The total flux through the cube is q/ϵ_0, by Gauss's law. The flux through every face of the cube is the same, by symmetry. Therefore, over any one of the six faces we have $\int \mathbf{E} \cdot d\mathbf{a} = q/6\epsilon_0$.

(b) Because the field due to q is parallel to the surface of each of the three faces that touch q, the flux through these faces is zero. The total flux through the other three faces must therefore add up to $q/8\epsilon_0$, because our cube is one of eight such cubes surrounding q. Since the three faces are symmetrically located with respect to q, the flux through each must be $(1/3)(q/8\epsilon_0) = q/24\epsilon_0$.

Note: if the charge were a true point charge, and if it were located just inside or just outside the cube, then the field would *not* be parallel to each of the three faces that touch the given corner. The flux would depend critically on the exact location of the point charge. Replacing the point charge with a small sphere, whose center lies at the corner, eliminates this ambiguity.

1.61. Potential energy of a sphere

The charge inside a sphere of radius r (with $r < R$) is $q = (4\pi r^3/3)\rho$. The external field of this sphere is the same as if all of the charge were at the center. So the sphere acts like a point charge, as far as the potential energy of an external object is concerned. The next shell to be added, with thickness dr, contains charge $dq = (4\pi r^2\,dr)\rho$. The work done in bringing in this dq (which is the same as the potential energy of the shell due to the sphere) is therefore

$$dW = \frac{1}{4\pi\epsilon_0}\frac{q \cdot dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{(4\pi\rho)^2}{3}\,r^4\,dr. \tag{52}$$

Building up the whole sphere this way, from $r = 0$ to $r = R$, requires the work:

$$W = \int_0^R \frac{1}{4\pi\epsilon_0}\frac{(4\pi\rho)^2}{3}\,r^4\,dr = \frac{1}{4\pi\epsilon_0}\frac{(4\pi\rho)^2}{3}\frac{R^5}{5}\,. \tag{53}$$

The charge in the complete sphere is $Q = (4\pi R^3/3)\rho$, which gives $4\pi\rho = 3Q/R^3$. Thus the potential energy U, which is the same as the work W, can be written as $U = (3/5)Q^2/4\pi\epsilon_0 R$. Note that $Q^2/4\pi\epsilon_0 R$ has the proper energy dimensions of $(\text{charge})^2/(\epsilon_0 \cdot \text{distance})$. Indeed, we could have predicted that much of the result without any calculation. The only question is what the numerical factor out front is. It happens to be $3/5$.

Note that we don't have to worry about the self energy of each infinitesimally thin shell, because by dimensional analysis this energy is proportional to $(dq)^2$. So it is a second-order small quantity and hence can be ignored.

1.77. Electron jelly

The force on a proton, at radius r, from the electron jelly is effectively due to the jelly that is inside radius r. The force points toward the center of the sphere. If the net force on a proton is zero, the force from the other proton must also point along the line (away) from the center. The two protons must therefore lie on the same diameter. They also must be the same distance r from the center; this is true because they feel the same force (in magnitude) from each other, so they must also feel the same force from the jelly, which implies that they must have the same value of r.

Since volume is proportional to r^3, the negative charge inside radius r equals $-2e(r^3/a^3)$. The field at radius r due to the jelly is therefore

$$-\frac{2e(r^3/a^3)}{4\pi\epsilon_0 r^2} = -\frac{er}{2\pi\epsilon_0 a^3}\,. \tag{85}$$

The field at one proton due to the other is $e/\big(4\pi\epsilon_0(2r)^2\big)$. So the total field at one of the protons will equal zero if

$$\frac{er}{2\pi\epsilon_0 a^3} = \frac{e}{4\pi\epsilon_0(2r)^2} \implies r^3 = \frac{a^3}{8} \implies r = \frac{a}{2}\,. \tag{86}$$

This factor of $1/2$ is reasonably clear in retrospect. If all of the $-2e$ electron charge were located in a point charge at the center, it would provide a force on one of the protons that is 8 times the force due to the other proton (because the other proton is twice as far away and half as big). So the forces will balance if we reduce the effective electron charge by a factor of 8. This is accomplished by reducing the effective radius of the jelly by a factor of 2.

1.82. **Energy of concentric shells**

(a) The field is nonzero only in the region $a < r < b$, where it equals $E = Q/4\pi\epsilon_0 r^2$. The total energy is therefore

$$U = \int_a^b \frac{\epsilon_0 E^2}{2}\,dv = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \int_a^b \frac{1}{r^4}\,4\pi r^2\,dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right). \tag{94}$$

If $b = a$ then $U = 0$, of course, because the shells are right on top of each other, so the charges cancel and we effectively have no charge anywhere in the system. If $b \to \infty$ then $U = Q^2/8\pi\epsilon_0 a$, which is correctly the energy of a single shell with charge Q (see Problem 1.32). If $a \to 0$ then U correctly goes to infinity, because the field diverges (sufficiently quickly) near a point charge. Equivalently, it takes an infinite amount of energy to compress a given amount of charge down to a point.

The result in Eq. (94) can be interpreted as follows. As mentioned above, the energy stored in a system consisting of one spherical shell of radius r is $Q^2/8\pi\epsilon_0 r$. Given this result, consider building up the present two-sphere system from scratch (that is, by bringing charges in from infinity) in two steps. It takes an energy $Q^2/8\pi\epsilon_0 a$ to construct the shell of radius a. Then, with that shell in place, it takes an energy $Q^2/8\pi\epsilon_0 b - Q^2/4\pi\epsilon_0 b$ to construct the outer shell of radius b. The first term comes from the self energy of this outer shell. The second term comes from the potential energy of the negative outer shell due to the positive inner shell already in place (which acts like a point charge at its center). The sum of the energies of these two steps yields the result in Eq. (94).

(b) Now let's imagine starting with two neutral shells and then gradually transferring positive charge from the outer shell to the inner shell. At the start, there is no electric field between the shells, so it takes no work to transfer an initial bit of charge dq. But as more charge is piled onto the inner shell, the field grows, and it takes more work to bring in the successive bits dq.

At a moment when there is charge q on the inner shell, the field between the shells is $q/4\pi\epsilon_0 r^2$, so the force on a little charge dq is $q\,dq/4\pi\epsilon_0 r^2$. The work you must do on this dq is the integral of your force times the displacement, or

$$dW = \int_b^a \frac{-q\,dq}{4\pi\epsilon_0 r^2}\,dr = \frac{q\,dq}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \tag{95}$$

where we have included the minus sign in the force because your force points inward (it is opposite to the electric force). However, you can always put the sign in by hand at the end; you certainly have to do positive work to move the positive charge dq toward the positively charged inner shell.

We must now integrate the above work dW over all the bits dq that we bring in. This gives

$$W = \int_0^Q \frac{q\,dq}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \tag{96}$$

in agreement with the result in part (a). It may seem mysterious that the potential energy of a system can be found by integrating $\epsilon_0 E^2/2$ over the volume. But the agreement of the two above methods, applied to our setup involving two shells, should help convince you that the $\epsilon_0 E^2/2$ method does indeed give the energy.