Phys 4C Fall 2019

Chapter 2&3 Solutions to Exercises

2.50. **Dividing the charge**

The potential is constant over the surface of a given sphere, so we can pull the ϕ outside the integral in Eq. (2.32) and write the potential energy of a sphere as $U = (\phi/2)\int \rho\, dv = \phi q/2$. So if the spheres of radii R_1 and R_2 have charge q and $Q-q$, respectively, the sum of the two potential energies is

$$U = \frac{q}{4\pi\epsilon_0 R_1}\cdot\frac{q}{2} + \frac{Q-q}{4\pi\epsilon_0 R_2}\cdot\frac{Q-q}{2} = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{2R_1} + \frac{(Q-q)^2}{2R_2}\right). \tag{145}$$

Minimizing this by setting the derivative with respect to q equal to zero yields

$$0 = \frac{dU}{dq} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} - \frac{(Q-q)}{R_2}\right). \tag{146}$$

Solving for q gives $q = QR_1/(R_1+R_2)$. So there is charge $QR_1/(R_1+R_2)$ on the first sphere and charge $QR_2/(R_1+R_2)$ on the second sphere.

The two terms in Eq. (146) (without the minus sign in front of the second term) are simply the potentials of the two spheres. So the condition of minimum energy is equivalent to the condition of equal potentials. Note that the second derivative, $d^2U/d^2q = 1/R_1 + 1/R_2$, is positive, so the extremum is indeed a minimum of U, not a maximum. This is consistent with the special case where $R_1 = R_2$; equal division of the charge involves half as much total energy as piling all of Q on one sphere, from Eq. (145).

3.52. **Aluminum capacitor**

The capacitance is

$$C = \frac{\epsilon_0 A}{s} = \frac{\left(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\right)\left(\pi(0.075\,\text{m})^2\right)}{4\cdot 10^{-5}\,\text{m}} = 3.910\cdot 10^{-9}\,\text{F} = 3910\ \text{pF}. \tag{260}$$

3.54. **Dividing the surface charge**

If σ_1 is the surface density on the top face of the inner plate, and if σ_2 is the density on the bottom face, then the magnitudes of the electric fields in the top and bottom regions are $E_1 = \sigma_1/\epsilon_0$ and $E_2 = \sigma_2/\epsilon_0$. These follow from using Gauss's law with surfaces that pass through the interior of the middle plate where the field is zero. The difference in potential between the middle and top plates is $E_1(0.05\,\text{m})$, and the difference in potential between the middle and bottom plates is $E_2(0.08\,\text{m})$. Since the top and bottom plates are at the same potential, we must have $5E_1 = 8E_2 \implies 5\sigma_1 = 8\sigma_2$. Combining this with the given fact that $\sigma_1 + \sigma_2 = \sigma$, we quickly find $\sigma_1 = (8/13)\sigma$ and $\sigma_2 = (5/13)\sigma$.

REMARK: From similar reasoning involving Gaussian surfaces with one side lying inside a conductor, it follows that the density on the bottom face of the top plate is $-\sigma_1$, and the density on the top face of the bottom plate is $-\sigma_2$. Assuming that there is zero net charge on the outer two plates, this leaves at total of $\sigma_1 + \sigma_2 = \sigma$ for the outer surfaces of these plates. It must get divided evenly, because otherwise these two surfaces would create a nonzero field between them, which would change the above fields and make the outer plates not be at the same potential. If any additional charge is dumped on the outer plates, it simply gets divided evenly between their two outer surfaces.

3.56. **Field just outside a capacitor**

If the disks were infinitely large, the desired field would be zero. But with a finite R, the repulsive field from the positive disk (which acts like an infinite plane, for points infinitesimally close to it) is slightly larger than the attractive field from the negative disk, which does't quite act like an infinite plane.

Let's find the field due to a disk with radius R and surface density σ, at a general point a distance z from the center of the disk along the axis. This can be found by slicing up the disk into rings and finding the z component of the field due to the charge in each ring. We obtain (the $z/\sqrt{r^2+z^2}$ factor here gives the z component):

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\int_0^R \frac{(2\pi r\,dr)\sigma}{r^2+z^2}\cdot\frac{z}{\sqrt{r^2+z^2}} = \frac{\sigma z}{2\epsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} \\ &= -\frac{\sigma z}{2\epsilon_0\sqrt{r^2+z^2}}\bigg|_0^R = \frac{\sigma}{2\epsilon_0} - \frac{\sigma z}{2\epsilon_0\sqrt{R^2+z^2}}\,. \end{aligned} \tag{263}$$

As expected, if $z \ll R$ we obtain the standard $\sigma/2\epsilon_0$ field from an infinite plane. In the case of the negative disk in this problem, z equals the separation s. So the difference in the magnitudes of the (oppositely pointing) fields from the two disks, at a point just outside the positive disk, is $\sigma s/2\epsilon_0\sqrt{R^2+s^2}$. The net field therefore has this magnitude and is directed away from the positive disk. In the (usual) case at hand where $s \ll R$, the net field is essentially equal to $\sigma s/2\epsilon_0 R$, which is s/R times the $\sigma/2\epsilon_0$ field from an infinite plane.

3.59. Coaxial capacitor

Neglecting end effects, we can assume that the charge $\pm Q$ is uniformly distributed along each cylinder. The field between the cylinders is that of a line charge with density $\lambda = Q/L$, so $E = \lambda/2\pi\epsilon_0 r = Q/2\pi\epsilon_0 Lr$. The magnitude of the potential difference between the cylinders is then

$$|\Delta\phi| = \int_b^a E\,dr = \int_b^a \frac{Q\,dr}{2\pi\epsilon_0 Lr} = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right). \tag{265}$$

Since $C = Q/|\Delta\phi|$, the capacitance is given by $C = 2\pi\epsilon_0 L/\ln(a/b)$. If $a - b \ll b$, then we can use the Taylor series $\ln(1+\epsilon) \approx \epsilon$ to write

$$\ln\left(\frac{a}{b}\right) = \ln\left(1 + \frac{a-b}{b}\right) \approx \frac{a-b}{b}. \tag{266}$$

So the capacitance becomes $C \approx 2\pi\epsilon_0 bL/(a-b)$. But $2\pi bL$ is the area A of the inner cylinder, and $a - b$ is the separation s between the cylinders. So the capacitance can be written as $C = \epsilon_0 A/s$, which agrees with the standard result for the parallel-plate capacitor.

3.66. Adding a capacitor

Let the two capacitors be labeled 1 and 2. If the initial charge on capacitor 1 is Q, then

$$Q = C_1 V_\text{i}, \tag{287}$$

where $C_1 = 100$ pF and $V_\text{i} = 100$ volts. So $Q = (10^{-10}\,\text{F})(100\,\text{V}) = 10^{-8}\,\text{C}$. When capacitor 2 is connected in parallel, the charge Q is shared between the two capacitors, that is, $Q = Q_1 + Q_2$. But the voltages across the two capacitors are equal because they are connected in parallel. This voltage is $V_\text{f} = 30$ volts. So we have $Q_1 = C_1 V_\text{f}$ and $Q_2 = C_2 V_\text{f}$. Adding these relations gives

$$Q = (C_1 + C_2)V_\text{f}, \tag{288}$$

which is the statement that capacitances in parallel simply add. Equating the right-hand sides of Eqs. (287) and (288) gives

$$C_1(100\,\text{V}) = (C_1 + C_2)(30\,\text{V}) \implies C_2 = C_1 \cdot \frac{70}{30} = 233\text{ pF}. \tag{289}$$

The initial energy stored is

$$\frac{1}{2}QV_\text{i} = \frac{1}{2}(10^{-8}\,\text{C})(100\,\text{V}) = 5\cdot 10^{-7}\,\text{J}. \tag{290}$$

The final energy stored is

$$\frac{1}{2}Q_1V_\text{f} + \frac{1}{2}Q_2V_\text{f} = \frac{1}{2}QV_\text{f} = \frac{1}{2}(10^{-8}\,\text{C})(30\,\text{V}) = 1.5\cdot 10^{-7}\,\text{J}. \tag{291}$$

(The final energy is smaller than the initial energy by the factor V_f/V_i.) Therefore, $3.5\cdot 10^{-7}$ J of energy is lost. This much energy has to go *somewhere* before the system can settle down to static equilibrium. If it is not stored anywhere else (for instance, in a weight lifted by a motor driven by the current from C_1 to C_2) it will eventually be dissipated in circuit resistance, no matter how small that resistance may be. (If the circuit is superconducting, the current will keep sloshing back and forth. We'll talk about LC circuits in Chapter 8.)

3.70. **Force and energy for two plates**

From Eq. (1.49), the force per unit area on one of the plates is σ times the average of the fields on either side of the plate. (Equivalently, it is σ times the field from the other plate.) This average field is $E/2$, where E is the field between the plates. But E equals σ/ϵ_0, so $\sigma = \epsilon_0 E$ (it will be more useful to write the field in terms of E than σ). The force per unit area is therefore

$$\frac{F}{A} = \sigma\frac{E}{2} = (\epsilon_0 E)\frac{E}{2} \quad\Longrightarrow\quad F = A\frac{\epsilon_0 E^2}{2}. \tag{305}$$

Since E is given by ϕ/s, we can write F in terms of the potential as

$$F = \frac{A\epsilon_0\phi^2}{2s^2} = \frac{(0.2\,\text{m})^2\big(8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}\big)(10\,\text{V})^2}{2(0.03\,\text{m})^2} = 2.0\cdot 10^{-8}\,\text{N}. \tag{306}$$

If the charge is held constant as the plates come together, then the electric field is independent of the separation, so we see from Eq. (305) that the force is also independent of the separation. (Equivalently, ϕ is proportional to s in Eq. (306), so F is independent of s.) The total work done by the electric force (which could be used to lift an external object, etc.) is then $W = F\cdot s = (2.0\cdot 10^{-8}\,\text{N})(0.03\,\text{m}) = 6\cdot 10^{-10}$ J. Note that the work can be written symbolically as

$$W = F\cdot s = \frac{A\epsilon_0 E^2}{2}\cdot s = (As)\frac{\epsilon_0 E^2}{2} = (\text{volume})\frac{\epsilon_0 E^2}{2}. \tag{307}$$

Since $\epsilon_0 E^2/2$ is the energy density, the work does indeed equal the energy initially stored in the field. Alternatively, the work can be written in terms of ϕ as (using $C = \epsilon_0 A/s$ for a parallel-plate capacitor)

$$W = F\cdot s = \frac{A\epsilon_0\phi^2}{2s^2}\cdot s = \frac{1}{2}\frac{\epsilon_0 A}{s}\phi^2 = \frac{1}{2}C\phi^2, \tag{308}$$

which is the energy stored in the capacitor.

What is the work done if the plates remain connected to the 10 volt battery? In this case, since ϕ is constant, the force of $A\epsilon_0\phi^2/2s^2$ in Eq. (306) grows like $1/s^2$ as s goes to zero. The integral of this diverges near zero, so the work is theoretically infinite. However, eventually the battery won't be able to supply the necessary charge to the plates, so ϕ will inevitably decrease.